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A V-CYCLE MUTIGRID FOR QUADRILATERAL ROTATED Q_1 ELEMENT WITH NUMERICAL INTEGRATION*[1)]

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Abstract

In this paper, a V-cycle multigrid method is presented for quadrilateral rotated Q_1 elements with numerical integration.

Key words: Multigrid, Rotated Q_1 elements, Numerical integration.

1. Introduction

The rotated Q_1 nonconforming element first proposed and used to solve the Stokes problem by Rannacher and Turek in [12]. Klouček, Li and Luskin have implemented it to simulate the martensitic crystals with microstructures [9], [10]. Recently, Shi and Ming [14] gave a detailed mathematics analysis for this element under the bi-section condition for mesh subdivisions, which was first introduced by Shi [13] for analyzing the quadrilateral Wilson element. Meanwhile they also proposed some effective numerical quadrature schemes for this element[14]. Moreover, they have succeeded in using this element for the Mindlin-Reissner plate problem [11]. Quasi-optimal maximum norm estimations for the quadrilateral rotated Q_1 element approximation of Navier-Stokes equations were established in [17].

In this paper, we will investigate multigrid methods for solving discrete algebraic equations obtained by use of the quadrilateral rotated Q_1 elements. An effective V-cycle multigrid algorithm is presented with numerical integrations. A uniform convergence factor is obtained. A similar idea has been exploited for the Wilson nonconforming element [15] and the TRUNC plate element [16]. We also mention that some nonconforming multigrid algorithms for the second order problem are studied in early papers, see [1], [6] for P_1 nonconforming element, and [8] for the rectangular rotated Q_1 element.

The outline of the paper is as follows. In section 2, we introduce the quadrilateral rotated Q_1 element. In the last section an effective V-cycle multigrid algorithm is presented.

2. Quadrilateral Rotated Q_1 Elements

We consider the following general 2-order elliptic boundary value problem over a convex polygonal domain in R^2:

$$\begin{aligned} \pounds u = -(\partial_x(a_{11}\partial_x u) + \partial_y(a_{12}\partial_x u) + \partial_x(a_{12}\partial_y u) + \partial_y(a_{22}\partial_y u)) + au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the coefficients $a_{11}, a_{12}, a_{22},\ a \in W^{1,\infty}(\Omega)$, and $a \geq 0$, the right hand term $f \in W^{1,q}(\Omega), q \geq 2$, $W^{1,\infty}(\Omega)$ and $W^{1,q}(\Omega)$ are the usual Sobolev spaces.

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We assume that the differential operator $\mathcal{L}$ is uniformly elliptic, i.e. there exists a positive constant c such that

$$c^{-1}(\xi_1^2+\xi_2^2) \le \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \le c(\xi_1^2+\xi_2^2)$$

for all points $(x,y) \in \bar{\Omega}$ and real vectors (ξ_1, ξ_2).

The weak form of this problem is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v) = \int_\Omega [a_{11}\partial_x u\partial_x v + a_{12}(\partial_x u\partial_y v + \partial_y u\partial_x v) + a_{22}\partial_y u\partial_y v + auv]dxdy.$$

Let Γ_h be a partition of the convex polygonal $\overline{\Omega}$ by convex quadrilaterals. Denote $\Gamma = \partial\Omega$. We define by P_k the space of polynomials of degrees no more than k, and by Q_k the space of polynomials of degrees no more than k in each variable. Let the diameter of K be h_K and assume that $h_K \le h$. As in Figure 1, we denote the four vertices of K by $P_i(x_i, y_i), 1 \le i \le 4$, and the sub-triangle of K with vertices P_{i-1}, P_i, and P_{i+1} by T_i (the index of P_i is modulo 4). Define $\rho_K = \max_{1\le i\le 4}$ (diameter of the circles inscribed in T_i). It is assumed that the partition satisfies the assumption: there exists a constant $\sigma > 2$ independent of h such that

$$h_K \le \sigma\rho_K. \tag{2.2}$$

Note that this assumption is equivalent to the usual regularity condition for quadrilateral partitions (see [7], pp. 247). Let $\hat{K} = [-1,1] \times [-1,1]$ be the reference square having the vertices $\hat{P}_i (1 \le i \le 4)$, then there exists a unique mapping $F_K \in Q_1(\hat{K})$ given by

$$x^K = \sum_{i=1}^{4} x_i N_i(\xi,\eta), \quad y^K = \sum_{i=1}^{4} y_i N_i(\xi,\eta),$$

where

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

such that $F_K(\hat{p}_i) = p_i, 1 \le i \le 4, \quad F_K(\hat{K}) = K$. We also denote $e_1 = \overline{P_4P_1}, e_2 = \overline{P_1P_2}, e_3 = \overline{P_2P_3}, e_4 = \overline{P_3P_4}$.

To each function $\hat{v}(\xi,\eta)$ defined on $\hat{K}$, we associate a function v on K such that $\hat{v} = v \circ F_K$.

In the following, we list some geometric properties of an arbitrary quadrilateral mesh:

$$\begin{array}{ll}
x^K = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta, & y^K = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta. \\
4a_0 = x_1 + x_2 + x_3 + x_4, & 4b_0 = y_1 + y_2 + y_3 + y_4. \\
4a_1 = -x_1 + x_2 + x_3 - x_4, & 4b_1 = -y_1 + y_2 + y_3 - y_4. \\
4a_2 = -x_1 - x_2 + x_3 + x_4, & 4b_2 = -y_1 - y_2 + y_3 + y_4. \\
4a_{12} = x_1 - x_2 + x_3 - x_4, & 4b_{12} = y_1 - y_2 + y_3 - y_4.
\end{array}$$

$$DF_K(\xi,\eta) = \begin{pmatrix} a_1 + a_{12}\eta & a_2 + a_{12}\xi \\ b_1 + b_{12}\eta & b_2 + b_{12}\xi \end{pmatrix}$$

and the Jacobi of F_K is $J_K(\xi,\eta) = det(DF_K) = J_0^K + J_1^K\xi + J_2^K\eta$, where, $J_0^K = a_1b_2 - a_2b_1, J_1^K = a_1b_{12} - a_{12}b_1, J_2^K = a_{12}b_1 - a_2b_{12}$. Denote the inverse of F_K by F_K^{-1}, then

$$(DF_K)^{-1}(\xi,\eta) = \frac{1}{J_K(\xi,\eta)} \begin{pmatrix} b_2 + b_{12}\xi & -a_2 - a_{12}\xi \\ -b_1 - b_{12}\eta & a_1 + a_{12}\eta \end{pmatrix}$$

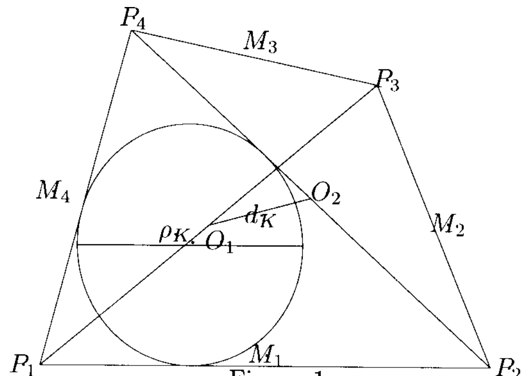


Figure 1

We state a condition on the mesh subdivision which appeared in [13].

Condition A. The distance d_K between the midpoints of the diagonals of $K \in \Gamma_h$ is of order $O(h_K^2)$ for any K as $h \to 0$.

Two kinds of the quadrilateral rotated Q_1 finite element space can be defined as follows:

(1). Let $B^p_{\hat{K}} = Span\{\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3, \hat{\phi}_4\}$. Then $B^p_K = Span\{\hat{\phi}_1 \circ F_K^{-1}, \hat{\phi}_2 \circ F_K^{-1}, \hat{\phi}_3 \circ F_K^{-1}, \hat{\phi}_4 \circ F_K^{-1}\}$.

$$\hat{\phi}_1(\xi,\eta) = \tfrac{1}{4}(\xi^2-\eta^2) - \tfrac{1}{2}\xi + \tfrac{1}{4}, \qquad \hat{\phi}_2(\xi,\eta) = \tfrac{1}{4}(\eta^2-\xi^2) - \tfrac{1}{2}\eta + \tfrac{1}{4},$$
$$\hat{\phi}_3(\xi,\eta) = \tfrac{1}{4}(\xi^2-\eta^2) + \tfrac{1}{2}\xi + \tfrac{1}{4}, \qquad \hat{\phi}_2(\xi,\eta) = \tfrac{1}{4}(\eta^2-\xi^2) + \tfrac{1}{2}\eta + \tfrac{1}{4}.$$

$$V_h^p = \{v \in L^2(\Omega) \mid \hat{v} \in B^p_{\hat{K}} \ \ \forall K \in \Gamma_h, \ \ v|_{K_1}(c_{\mathcal{M}}) = v|_{K_2}(c_{\mathcal{M}}),$$
$$c_{\mathcal{M}} \text{ is the middle point of } E_{12} = K_1 \cap K_2\},$$

the shape function $\hat{v}(\xi,\eta) = \sum_{i=1}^4 v_i\hat{\phi}_i(\xi,\eta), \ \ v_i = v(c_{M_i}), \ 1 \le i \le 4$.

(2). Let $B^a_{\hat{K}} = Span\{\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4\}$. Then $B^a_K = Span\{\hat{\psi}_1 \circ F_K^{-1}, \hat{\psi}_2 \circ F_K^{-1}, \hat{\psi}_3 \circ F_K^{-1}, \hat{\psi}_4 \circ F_K^{-1}\}$.

$$\hat{\psi}_1(\xi,\eta) = \tfrac{3}{8}(\xi^2-\eta^2) - \tfrac{1}{2}\xi + \tfrac{1}{4}, \qquad \hat{\psi}_2(\xi,\eta) = \tfrac{3}{8}(\eta^2-\xi^2) - \tfrac{1}{2}\eta + \tfrac{1}{4},$$
$$\hat{\psi}_3(\xi,\eta) = \tfrac{3}{8}(\xi^2-\eta^2) + \tfrac{1}{2}\xi + \tfrac{1}{4}, \qquad \hat{\psi}_2(\xi,\eta) = \tfrac{3}{8}(\eta^2-\xi^2) + \tfrac{1}{2}\eta + \tfrac{1}{4}.$$

$$V_h^a = \{v \in L^2(\Omega) \mid \hat{v} \in B^a_{\hat{K}} \ \ \forall K \in \Gamma_h, \int_{e_{12}} v|_{K_1} = \int_{e_{12}} v|_{K_2}, \ e_{12} = K_1 \cap K_2\},$$

the shape function $\hat{v}(\xi,\eta) = \sum_{i=1}^4 v_i^k\hat{\psi}_i(\xi,\eta), \ \ v_i^k = \frac{1}{e_i}\int_{e_i} v^k, \ 1 \le i \le 4$. We denote B_K be B_K^p and B_K^a.

To solve the Dirichlet problem (2.1), we introduce the associated homogeneous spaces:

$$V^p_{0,h} = \{v_h \in V_h^p, v_h = 0 \text{ at the middle point of edges lying on the boundary } \partial\Omega\},$$

$$V^a_{0,h} = \{v_h \in V_h^a, \int_e v_h = 0, e = \partial K \cap \partial\Omega\},$$

and define

$$\|v\|_h^2 = \sum_{K\in\Gamma_h} \|v\|^2_{1,K}, \qquad |v|_h^2 = \sum_{K\in\Gamma_h} |v|^2_{1,K}.$$

It is obvious that $|\cdot|_h$ is a norm on $V^p_{0,h}$ or $V^a_{0,h}$.

We need some interpolation results. Define the Lagrangian interpolation operator π_h : $C(\bar{\Omega}) \to V_h$ to be either $\pi_h^p : C(\bar{\Omega}) \to V_h^p$ or $\pi_h^a : L^2(\Omega) \to V_h^a$ as follows:

$$\forall v \in C(\bar{\Omega}), \ \pi_h^p v \in V_h^p : \ \pi_h^p v(c_{\mathcal{M}}) = v(c_{\mathcal{M}}) \ \ \forall c_{\mathcal{M}},$$

where $c_{\mathcal{M}}$ is the midpoint of the edge $\mathcal{F} \in \partial K$, $K \in \Gamma_h$, and

$$\forall v \in C(\bar{\Omega}),\ \pi_h^a v \in V_h^a: \quad \int_e \pi_h^a v ds = \int_e v ds \quad \forall e \in \partial K,\ \forall K \in \Gamma_h.$$

The following lemma concerns the interpolation error of the finite element space V_h^a and V_h^p.

Lemma 2.1. *[12] For $v \in H^2(K)$, and if the*

Condition A. *holds, then*

$$\|v - \pi_h v\|_{k,K} \le Ch^{2-k}|v|_{2,K}, \quad k = 0, 1, 2.$$

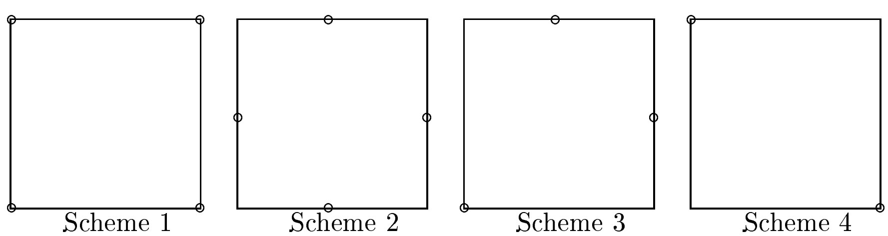
Define the quadrature scheme on the reference element $\hat{K}$ as follows:

$$\int_{\hat{K}} \hat{\phi}(\xi, \eta) d\xi d\eta \approx \sum_{i=1}^{I} \hat{\omega}_i \hat{\phi}(\hat{Q}_i), \qquad \hat{\phi} \in C(\hat{K}),$$

where the weight $\hat{\omega}_i > 0$, the quadrature point $\hat{Q}_i = (\xi_i, \eta_i) \in \hat{K}$, $i = 1, \cdots, I$. Let $\hat{Q} = Span\{1, \xi, \eta, \xi^2 - \eta^2\}$, we assume that the quadrature is exact on $\hat{Q}$, hence it is also exact on P_1. The following four schemes will be considered:

$$\begin{array}{lll}
\text{Scheme1}: I = 4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1), (1,-1), (1,1), (-1,1), \\
\text{Scheme2}: I = 4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,0), (0,-1), (1,0), (0,1), \\
\text{Scheme3}: I = 3, & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1), (1,0), (0,1), \\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,-1), (-1,0), (0,1), \\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,1), (-1,0), (0,-1), \\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,1), (1,0), (0,-1). \\
\text{Scheme4}: I = 2, & \hat{\omega}_i = 2, & \{\hat{Q}_i\}_{i=1}^2 = (-1,-1), (1,1), \text{ or } (1,-1), (-1,1).
\end{array}$$

Figure 2



In the above figure, we only draw one case of *Scheme 3* and *Scheme 4*, the other cases can be obtained symmetrically.

Remark 2.1. In fact, there are some other possibilities for the numerical quadrature. For example, in the scheme 1, if we denote the weights $\hat{\omega}_i$ in the counterclockwise manner, then the following choices are also possible:

1. $\hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\omega}_4 = 4$;
2. $\hat{\omega}_1 = \hat{\omega}_3$ and $\hat{\omega}_2 = \hat{\omega}_4$.

The quadrature on K is given by

$$\int_K \phi dx \approx \sum_{i=1}^{I} \omega_{i,K} \phi(Q_{i,K}) \equiv Q_K(\phi),$$

where $\phi(x) = \hat{\phi}(\hat{x}),\ \omega_{i,K} = \hat{\omega}_i J_K(\hat{Q}_i),\ Q_{i,K} = F_K(\hat{Q}_i)$. Now we apply the quadrature scheme Q_K to the finite element equation (2.1). Define

$$a_h(u,v) \equiv \sum_{K\in\Gamma_h} Q_K[a_{11}\partial_x u\partial_x v + a_{12}(\partial_x u\partial_y v + \partial_y u\partial_x v) + a_{22}\partial_y u\partial_y v + auv],$$

and $(f,v)_h \equiv \sum_{K\in\Gamma_h} Q_K(fv)$, we solve the following equation:

$$a_h(u_h, v) = (f,v)_h \quad \forall v \in V_{0,h}. \tag{2.3}$$

From now on, we always assume that the **Condition A** holds.

Theorem 2.1. *[14] Suppose $a_{ij}, a \in W^{1,\infty}(\Omega), f \in W^{1,q}(\Omega), q > 2$, and $u, u_h \in V_{0,h}^a$ are the solution of (2.1), (2.3), respectively, then*

$$|u-u_h|_h \le Ch[(\sum_{i,j=1}^{2}(\|a_{ij}\|_{1,\infty} + \|a\|_{1,\infty})\|u\|_2 + |u|_2 + \|f\|_{1,q}].$$

3. Multigrid Implementation

It is known that the condition number of stiff matrix of (2.3) is of order $O(h^{-2})$, which results in a slow convergence rate in real computations. The multigrid method is a useful tool to solve such kind of systems. In this section an effective V-cycle multigrid algorithm is presented for the quadrilaterial rotated Q_1 element. We use the isoparametric conforming bilinear element space as the coarse-grid correction space on all coarse levels $l = 1, \dots, L-1$. It is shown that this V-cycle multigrid requires only one smoothing step on all coarse level $l < L$, while on the last level L a sufficient number of smoothing steps is needed. A similar idea has been exploited for the Wilson nonconforming element in [15] and for the TRUNC plate element in [16].

Define the operator $A_h : V_h \to V_h$ as follows:

$$(A_h u, v) = a_h(u,v) \quad \forall u, v \in V_h.$$

Then (2.3) can be represented as:

$$A_h u_h = f_h, \tag{3.1}$$

where $f_h \in V_h$, $(f_h, v)_h = (f,v)_h, \quad v \in V_h$.

Let $\{\Gamma_l\}_{l=1}^{L}$ be a sequence of quadrilateral partitions of Ω. Assume that Γ_l is obtained by connecting the midpoint of two opposite sides of $K \in \Gamma_{l-1}$. Moreover, we assume $\Gamma_L = \Gamma_h$. In order to construct a multigrid algorithm for (3.1), we define the isoparametric conforming bilinear finite element space $S_l \subset H_0^1(\Omega)$ on the grid $\Gamma_l,\ l < L$. It is obvious that

$$S_1 \subset S_2 \subset \dots \subset S_{L-1} \not\subset V_h.$$

Because $S_{L-1} \not\subset V_h$, we must define a suitable intergrid transfer operator $I_h : S_{L-1} \to V_h$. Note that $S_{L-1} \subset C(\bar{\Omega})$, we simply choose the interpolation operator π_h in Lemma 2.1 as I_h, i.e.

$$\int_e I_h v ds = \int_e v ds \quad \forall v \in S_{L-1}, \tag{3.2}$$

where e is an edge of $K \in \Gamma_h$.

Let $t_{L-1} : C(\bar{\Omega}) \to S_{L-1}$ be the isoparametric bilinear interpolation operator, then

Lemma 3.1. *For the operator I_h, t_{L-1}, we have*

$$(1).\|I_h v - v\|_0 \le Ch|v|_1 \quad |I_h v|_h \le C|v|_1 \quad \forall v \in S_{L-1}.$$
$$(2).\|t_{L-1}\xi - I_h t_{L-1}\xi\|_0 \le Ch^2|\xi|_2 \quad \forall \xi \in H^2(\Omega) \cap H_0^1(\Omega).$$

Proof. Lemma 2.1 gives

$$\|I_h v - v\|_0 \le Ch^2 (\sum_{K\in\Gamma_{L-1}} |v|_{2,K}^2)^{\frac{1}{2}} \quad \forall v \in S_{L-1}. \tag{3.3}$$

Then by the inverse inequality, we can see that the first inequality of Lemma 3.1 is valid.

On the other hand, Lemma 2.1 and the estimate of the interpolation operator t_{L-1} [7] yield

$$\begin{aligned}\|t_{L-1}\xi - I_h t_{L-1}\xi\|_0 &\le Ch^2 (\sum_{K\in\Gamma_{L-1}} |t_{L-1}\xi|_{2,K}^2)^{\frac{1}{2}} \\ &\le Ch^2|\xi|_2 \quad \forall \xi \in H^2(\Omega)\cap H_0^1(\Omega).\end{aligned}$$

We complete the proof.

By the similar technique in [14], we can show that

Lemma 3.2.

$$|a_{L-1}(u,v) - \bar{a}_{L-1}(u,v)| \le Ch_{L-1}|u|_1 (\sum_{K\in\Gamma_{L-1}} |v|_{2,K}^2)^{\frac{1}{2}} \quad \forall u,\ v \in S_{L-1},$$

where $a_{L-1}(\cdot,\cdot)$ and $\bar{a}_{L-1}(\cdot,\cdot)$ denote the bilinear form with and without numerical quadrature on the coarse level $L-1$, respectively.

Define the operators $A_{S_l} : S_l \to S_l$ and $Q_{S_l} : S_{L-1} \to S_l,\ l = 1,\dots,L-1$ as follows:

$$(A_{S_l}u, v) = a_{L-1}(u,v) \quad \forall u,\ v \in S_l,$$

$$(Q_{S_l}u, v) = (u,v) \quad \forall u \in S_{L-1},\ v \in S_l.$$

Noting that here we apply the quadrature scheme on the level $L-1$ to all other coarse levels $(l = 1,\dots,L-2)$ as in [3]. Moreover, define the projection operators Q_{L-1}, $P_{L-1} : V_h \to S_{L-1}$ as follows:

$$(Q_{L-1}u, v) = (u, I_h v) \quad \forall u \in V_h,\ v \in S_{L-1}, \tag{3.4}$$

$$a_{L-1}(P_{L-1}u, v) = a_h(u, I_h v) \quad \forall u \in V_h,\ v \in S_{L-1}. \tag{3.5}$$

It is easy to check that

$$|P_{L-1}v|_1 \le C|v|_h. \tag{3.6}$$

Using the similar technique in [2], we can construct certain smoothing operator $R_h : V_h \to V_h$ such that

$$C\frac{1}{\lambda_h}(v,v) \le (R_h v, v) \quad \forall v \in V_h, \tag{3.7}$$

$$a_h(R_h A_h v, v) \le \theta a_h(v,v) \quad \forall v \in V_h, \tag{3.8}$$

where λ_h is the largest eigenvalue of A_h and $\theta \in (0,2)$. By(3.7), (3.8) and a similar argument of Theorem 3.6, 5.1 in [2], we have

Lemma 3.3. *For any $v \in V_h$, it holds*

$$c\frac{\|A_h K_h^m v\|_0^2}{\lambda_h} \le a_h((I-K_h^2)K_h^m v, K_h^m v) \le C\frac{1}{m}a_h(v,v),$$

where $K_h = I - R_h A_h$, and m is the number of smoothing steps.

Similarly, on the coarse space S_l $(l = 1, ..., L-1)$, the smoothing operator $R_{S_l} : S_l \to S_l$ satisfies

$$(1). C\frac{1}{\lambda_l}(v,v) \le (R_{S_l}v, v) \quad \forall V \in S_l, \tag{3.9}$$

$$(2).a_{L-1}(R_{S_l}A_{S_l}v,v) \le \theta a_{L-1}(v,v) \quad \forall v \in S_l, \tag{3.10}$$

where λ_l is the largest eigenvalue of A_{S_l} and $\theta \in (0,2)$.

It is known that the Richardson, Jacobi and symmetric Gauss-Seidel iteration satisfy the above conditions. (cf. [2] for details)

Now we define the V-cycle multigrid algorithm as follows.

Multigrid Algorithm

Given $g \in V_h$, define $B_L g$ by

(1). Set $x_0 = 0, \quad x^n = x^{n-1} + R_h(g - A_h x^{n-1}), \quad n = 1, ..., m.$

(2). Define $x^{m+1} = x^m + I_h q$, where

$$q = M_{L-1}Q_{L-1}(g - A_h x^m).$$

(3). Set $y_0 = x^{m+1}$ and

$$y^n = y^{n-1} + R_h(g - A_h y^{n-1}), \quad n = 1, ..., m.$$

(4). Define $B_L g = y^m$.

The operator M_{L-1} in the above algorithm is defined as follows: Let $M_1 = A_{S_1}^{-1}$. For a given $g_l \in S_l$, M_l $(l = 2, ..., L-1)$ is defined by

(i). Set $x_1 = R_l g_l$.

(ii). Define $M_l g_l = x_1 + p$, where $p \in S_{l-1}$ is given by

$$p = M_{l-1}Q_{S_{l-1}}(g_l - A_{S_l}x_1).$$

It is seen that on each coarse grid space S_l, we perform only one smoothing step. It is easy to check that

$$I - B_L A_h = K_h^m(I - I_h P_{L-1} + I_h(I - M_{L-1}A_{S_{L-1}})P_{L-1})K_h^m. \tag{3.11}$$

By a similar argument in [3], we can prove

Lemma 3.4. *For the operator $I - M_{L-1}A_{S_{L-1}}$, we have*

$$|a_{L-1}((I - M_{L-1}A_{S_{L-1}})u,u)| \le \delta_0 a_{L-1}(u,u) \quad \forall u \in S_{L-1},$$

where the constant $\delta_0 \in (0,1)$ is independent of the mesh h and the level L.

Let $\{\lambda_j\}_{j=1}^{N_h}$ and $\{\varphi_j\}_{j=1}^{N_h}$ be the eigenvalues and corresponding normalized eigenfunctions of A_h, i.e.

$$A_h\varphi_j = \lambda_j\varphi_j, \quad j = 1, ..., N_h,$$

and

$$(\varphi_i, \varphi_j) = \delta_{ij},$$

where δ_{ij} is Kronecker symbol.

For any $v \in V_h$, we write $v = \sum_j^{N_h} c_j\varphi_j$. Let $A_h^s v = \sum_j^{N_h} \lambda_j^s c_j\varphi_j$, then we define the following discrete norm on the space V_h:

$$|||v|||_{s,h} := (A_h^s v, v)^{\frac{1}{2}}. \tag{3.12}$$

It is easy to see that

$$|||v|||_{1,h} = a_h(v,v)^{\frac{1}{2}}, \; |||v|||_{0,h} = \|v\|_0. \tag{3.13}$$

Lemma 3.5. *For the operator P_{L-1} defined by (3.5), we have*

$$\|v - P_{L-1}v\|_0 \le Ch|||v|||_{1,h} \quad \forall v \in V_h.$$

Proof. Consider the following auxiliary problem

$$\begin{cases} \pounds\eta &= v - P_{L-1}v \quad in \quad \Omega, \\ \eta &= 0 \quad on \quad \partial\Omega. \end{cases} \tag{3.14}$$

Since Ω is a convex polygon, the elliptic regularity property follows that

$$\|\eta\|_2 \le C\|v - P_{L-1}v\|_0. \tag{3.15}$$

On the other hand,

$$\begin{aligned} \|v - P_{L-1}v\|_0^2 &= (\pounds\eta, v - P_{L-1}v) \\ &= \bar{a}_h(\eta, v) - \bar{a}_{L-1}(\eta, P_{L-1}v) - d_h(\eta, v) \\ &= \bar{a}_h(\pi_h\eta, v) - \bar{a}_{L-1}(t_{L-1}\eta, P_{L-1}v) + \bar{a}_h(\eta - \pi_h\eta, v) \\ &\quad + \bar{a}_{L-1}(t_{L-1}\eta - \eta, P_{L-1}v) - d_h(\eta, v) \\ &= a_h(\pi_h\eta, v) - a_{L-1}(t_{L-1}\eta, P_{L-1}v) \\ &\quad + \bar{a}_h(\eta - \pi_h\eta, v) + \bar{a}_{L-1}(t_{L-1}\eta - \eta, P_{L-1}v) \\ &\quad + [\bar{a}_h(\pi_h\eta, v) - a_h(\pi_h\eta, v)] \\ &\quad + [a_{L-1}(t_{L-1}\eta, P_{L-1}v) - \bar{a}(t_{L-1}\eta, P_{L-1}v)] - d_h(\eta, v) \\ &= a_h(\pi_h\eta - I_h t_{L-1}\eta, v) + \bar{a}_h(\eta - \pi_h\eta, v) \\ &\quad + \bar{a}_{L-1}(t_{L-1}\eta - \eta, P_{L-1}v) \\ &\quad + [\bar{a}_h(\pi_h\eta, v) - a_h(\pi_h\eta, v)] \\ &\quad + [a_{L-1}(t_{L-1}\eta, P_{L-1}v) - \bar{a}(t_{L-1}\eta, P_{L-1}v)] - d_h(\eta, v) \\ &= \sum_{i=1}^{6} J_i. \end{aligned}$$

We estimate the terms $J_i, i = 1, \ldots, 6$ one by one as follows. An application of Lemma 2.1, 3.1 and (3.15) yields

$$|J_1| \le Ch|\eta|_2|v|_h \le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}.$$

By Lemma 2.1 and (3.15), we get

$$|J_2| \le Ch|\eta|_2|v|_h \le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}.$$

By (3.6) and (3.15), we have

$$|J_3| \le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}.$$

By Lemma 3.4 and (3.15), we get

$$\begin{aligned} |J_4| &\le Ch|v|_h\Big(\sum_{K\in\Gamma_h} |\pi_h\eta|_{2,K}^2\Big)^{\frac{1}{2}} \\ &\le Ch|v|_h|\eta|_2 \\ &\le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}. \end{aligned}$$

Similarly, by Lemma 3.2 and (3.15), we have

$$|J_5| \le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}.$$

Finally, applying Lemma 5.3 and (3.15) gives

$$|J_6| \le Ch\|\eta\|_2|v|_h \le Ch\|v - P_{L-1}v\|_0|||v|||_{1,h}.$$

So we get Lemma 3.5.

Lemma 3.6.

$$|||v - I_h P_{L-1} v|||_{1,h} \le Ch|||v|||_{2,h} \quad \forall v \in V_h.$$

Proof. By Lemma 3.1 and Lemma 3.5, we get

$$\begin{aligned} \|v - I_h P_{L-1} v\|_0 &\le \|v - P_{L-1} v\|_0 + \|(I - I_h) P_{L-1} v\|_0 \\ &\le Ch|||v|||_{1,h} + Ch|P_{L-1} v|_1 \\ &\le Ch|||v|||_{1,h}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |||v - I_h P_{L-1} v|||_{1,h} &= \sup_{w \in V_h, |||w|||_{1,h}=1} a_h(v - I_h P_{L-1} v, w) \\ &= \sup_{w \in V_h, |||w|||_{1,h}=1} a_h(v, w - I_h P_{L-1} w) \\ &\le \sup_{w \in V_h, |||w|||_{1,h}=1} |||v|||_{2,h} \|w - I_h P_{L-1} w\|_0 \\ &\le Ch|||v|||_{2,h}. \end{aligned}$$

The proof is completed. Finally, we show the main result of this section.

Theorem 3.1. *For any $\delta \in (\delta_0, 1)$, if m, the number of smoothing steps on the last level L, is large enough, then*

$$|a_h((I - B_L A_h)v, v)| \le \delta a_h(v, v) \quad \forall v \in V_h.$$

Proof. Let $\tilde{v} = K_h^m v$, by Lemma 3.4, we get

$$\begin{aligned} &|a_h((I - B_L A_h)v, v)| \\ &\quad \le |a_h((I - I_h P_{L-1})\tilde{v}, \tilde{v})| + |a((I - M_{L-1} A_{S_{L-1}}) P_{L-1} \tilde{v}, P_{L-1} \tilde{v})| \\ &\quad \le |a_h((I - I_h P_{L-1})\tilde{v}, \tilde{v})| + \delta_0 |a(I_h P_{L-1} \tilde{v}, \tilde{v})| \\ &\quad \le (1 + \delta_0)|a_h((I - I_h P_{L-1})\tilde{v}, \tilde{v})| + \delta_0 |a(\tilde{v}, \tilde{v})|. \end{aligned}$$

On the other hand, Lemma 3.3 and Lemma 3.6 imply

$$\begin{aligned} |a_h((I - I_h P_{L-1})\tilde{v}, \tilde{v})| &\le Ch|||\tilde{v}|||_{2,h} |||\tilde{v}|||_{1,h} \\ &= C\left(\frac{\|A_h \tilde{v}\|_0^2}{\lambda_h}\right)^{\frac{1}{2}} |||\tilde{v}|||_{1,h}^{\frac{1}{2}} \\ &\le C(a_h((I - K_h^2) K_h^m v, K_h^m v))^{\frac{1}{2}} |||v|||_{1,h}^{\frac{1}{2}} \\ &\le C \frac{1}{\sqrt{m}} a_h(v, v). \end{aligned}$$

Then, if m is large enough, we have

$$\begin{aligned} |a_h((I - B_L A_h)v, v)| &\le \left(\frac{C(1 + \delta_0)}{\sqrt{m}} + \delta_0\right) a_h(v, v) \\ &\le \delta a_h(v, v). \end{aligned}$$

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