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N DIMENSIONAL FINITE WAVELET FILTERS *1)

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Abstract

In this paper, a large class of n dimensional orthogonal and biorthognal wavelet filters (lowpass and highpass) are presented in explicit expression. We also characterize orthogonal filters with linear phase in this case. Some examples are also given, including no$n$ separable orhogonal and biorthogonal filters with linear phase.

Key words: n Dimension, Linear phase, Wavelet filters.

1. Introduction

In [1], I. Daubechies constructed orthogonal and biorthogonal wavelet filters in one dimension which have been proved to be very useful in signal and image processing. But except some short filters have explicit solution, almost all the orthogonal filters given in Daubechies' book are numerical results. In some applications, people need filters with high precision, in this case, one need to compute the filters himself. In this paper, we give a class of n dimensional orthogonal and biorthogonal wavelet filters in explicit expression. With these parameterized filters, we can easily realize the adaptive selection of filters in many applications.

Recently, many researchers are working on nonseparable wavelets(see [2], [3], [5], [6] and the references therein) because of the shortcoming of separable filters pointed out in [2]. Using the same method in [5], we can construct n dimensional wavelet filters. It is interesting that among these filters, we can find many nonseparable filters with linear phase, which can not be obtained by using the tensor product of one dimensional wavelet filters.

Our main results and their proofs are proposed in next section. By using the method in section II, some examples are given in section III, including one dimension case and two dimension case with linear phase.

2. Main Results

We will discuss orthogonal in detail first, then using similar method, we give the expression of biorthogonal filters.

2.1 Orthogonal case

The well known method to construct wavelet is MRA. The definition of n dimensional orthogonal MRA is as follows.

Definition. *A sequence of subspaces* $\{V_j\}_{j\in \mathbb{Z}}$ *of* $L^2(R^n)$ *is called a MRA if it satisfies the following properties:*

(a). $\cap V_j = \{0\}, \quad \overline{\cup V_j} = L^2(R^n)$;

(b). $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$, *for all* $j \in \mathbb{Z}, x \in R^n$;

(c). *There exists a function* $\varphi(x) \in V_0$ *such that* $\{\varphi(x-k)\}_{k\in \mathbb{Z}^n}$ *is an orthonormal basis of* V_0.

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Remark.

1). if $x = (x_1, \cdots, x_n)$, then $2x = (2x_1, \cdots, 2x_2)$.

2). $\mathbb{Z}^n$ is the set of all n dimensional integers.

Let $\{V_j\}$ be a n dimensional MRA, then there exists a function $m(\xi)(\xi \in R^n)$ such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi),$$

where $\hat{\varphi}$ is the Fourier transform of φ, and m is called Symbol Function of the scaling function φ. The orthogonality of $\{\varphi(x-k)\}_{k\in\mathbb{Z}^n}$ implies that $m(\xi)$ satisfies

$$\sum_{\nu\in E^n} |m(\xi+\nu\pi)|^2 = 1, \tag{2.1}$$

where E^n denotes the set of all vertexes of n dimensional unit square box.

The construction of φ can be reduced to construct $m(\xi)$. We want to solve (2.1) in some general cases. In this paper, we assume that $m(\xi)$ is a polynomial of $e^{i\xi}$ with the constant term does not equal to 0.

Let $\xi = (\xi_1, \cdots, \xi_n)$. Rewrite $m(\xi)$ in its polyphase form as

$$m(\xi) = \sum_{\nu\in E^n} x^\nu f_\nu(x^2), \tag{2.2}$$

where $x = e^{i\xi}$, $x^\nu = x_1^{\nu_1} \cdot \cdots \cdot x_n^{\nu_n}$, $x_k = e^{i\xi_k}$, $k = 1, \cdots, n$, $\nu = (\nu_1, \cdots, \nu_n) \in E^n$.

It is easy to see that (2.1) is equivalent to

$$\sum_{\nu\in E^n} |f_\nu(e^{i\xi})|^2 = \frac{1}{2^n}. \tag{2.3}$$

To solve (2.3), a theorem is needed as follows.

Theorem 1. *Suppose that $\{f_{e_k}, e_k \in E^n, k = 1, \cdots, 2^n\}$ satisfies (2.3), define*

$$(F_{e_1}, \cdots, F_{e_{2^n}})^T = UD(f_{e_1}, \cdots, f_{e_{2^n}})^T \tag{2.4}$$

where U is any real unitary matrix of size $2^n \times 2^n$, and $D = diag(x^{e_1}, \cdots, x^{e_{2^n}})$, $E^n = \{e_k, k = 1, \cdots, 2^n\}$. Then $\{F_\nu, \nu \in E^n\}$ also satisfy (2.3).

Proof. Since U and D are both unitary matrices, the proof is immediately.

Define

$$\widetilde{m}(\xi) = \sum_{\nu\in E^n} x^\nu F_\nu(x^2), \tag{2.5}$$

then $\widetilde{m}(\xi)$ is a trigonometric polynomial which satisfies (2.1).

Denote the set of all real unitary matrices with size $2^n \times 2^n$ by $\mathcal{U}_n$ and the set of all 2^n dimensional real unit column vectors by $\mathcal{V}_n$. Define

$$(f_{e_1}, \cdots, f_{e_{2^n}})^T = 2^{-\frac{n}{2}}(\otimes_{k=1}^N U_k D_k)V \tag{2.6}$$

where $U_k \in \mathcal{U}_n$, $V \in \mathcal{V}_n$, $D_k = \mathrm{diag}(x^{e_1}, \cdots, x^{e_{2^n}})$, for $k = 1, \cdots, N$.

Let

$$\mathcal{F}_{N,n} = \{f | f = 2^{-\frac{n}{2}}(\otimes_{k=1}^N U_k D_k)V, U_k \in \mathcal{U}_n, k = 1, \cdots, 2^n, V \in \mathcal{V}_n\}. \tag{2.7}$$

Then we have the following theorem:

Theorem 2. *For all $f \in \mathcal{F}_{N,n}$, then the set $\{f_{e_k}, e_k \in E^n, k = 1, \cdots, 2^n\}$ satisfies equation (2.3).*

Proof. The proof is immediately.

Denote $X_E = (x^{e_1}, \cdots, x^{e_{2^n}})$.

Define

$$m(\xi) = X_E \cdot f(x^2) \tag{2.8}$$

where $f \in \mathcal{F}_{N,n}$, $\cdot$ is the matrix multiply operator. Then $m(\xi)$ satisfies (2.1).

Define

$$\mathcal{S}_{N,n} = \{m(\xi)|m(\xi) = X_E \cdot f(x^2), f \in \mathcal{F}_{N,n}\} \tag{2.9}$$

and

$$\mathcal{L}_{N,n} = \{m(\xi)|m(\xi) \in \mathcal{S}_{N,n}, m(0) = 1\} \tag{2.10}$$

then each member of $\mathcal{L}_{N,n}$ is a lowpass filter. It is necessary to find corresponding highpass filters. The following theorem present a set of highpass filters.

Theorem 3. *Assume that $m(\xi) \in \mathcal{L}_{N,n}$ be defined as in (2.8). Let $U \in \mathcal{U}_n$, $U = (V_1, \cdots, V_{2^n})$, where V_k is k^{th} column of U, and V_1 equals to V defined in (2.6) corresponding to $m(\xi)$. Then for $k = 2, 3, \cdots, 2^n$, define*

$$m_k(\xi) = X_E \cdot f^k(x^2) \tag{2.11}$$

where

$$f^k = 2^{-\frac{n}{2}}(\otimes_{\mu=1}^{N} U_\mu D_\mu)V_k, \tag{2.12}$$

then $m_k(\xi)$, $k = 2, 3, \cdots, 2^n$, are the corresponding highpass wavelet filters.

Proof. Denote $M = 2^{\frac{n}{2}}(f, f^2, \cdots, f^{2^n})$ is matrix of size $2^n \times 2^n$. Since $U \in \mathcal{U}_n$, then M is a unitary matrix. Therefore we can see that the conclusion is true.

Theorem 3 shows that if we have lowpass filter defined as (2.8), then the construction of highpass filter is reduced to find a constant unitary matrix with the one column is known. It also shows that the highpass filters corresponding to one lowpass filters in high dimension case are not unique.

In the following, we characterize the set $\mathcal{L}_{N,n}$, Theorem 4 give the representation of $\mathcal{L}_{N,n}$.

Theorem 4. *For $m(\xi) \in \mathcal{L}_{N,n}$, if and only if*

$$m(\xi) = 2^{-n} X_E U_N D^2 \cdots U_1 D^2 U_1^T \cdots U_N^T V_0,$$

where $V_0 = (1 \cdots 1)^T$ is a column vector with all entries equal to 1.

Proof. Let $m(\xi) \in \mathcal{L}_{N,n}$, then

$$m(\xi) = 2^{-\frac{n}{2}} X_E U_N D^2 \cdots U_1 D^2 V.$$

and $m(0) = 2^{-\frac{n}{2}} V_0^T \widetilde{V}$, where $\widetilde{V} = U_N \cdots U_1 V$ is a unit vector. It is easy to see that $m(0) = 1$ if and only if $\widetilde{V} = 2^{-\frac{n}{2}} V_0$. Then $V = 2^{-\frac{n}{2}} U_1^T \cdots U_N^T V_0$. The proof is completed.

Up to now, by using Theorem 3 and Theorem 4, we can freely construct n dimensional wavelet filters for any given positive integer n.

In previous theorems, we have propose some solutions of (2.3), but we can not prove that the solution is complete, but in one dimension case, we can prove this conclusion.

Theorem 5. *When $n = 1$, for any trigonometric polynomial $m(\xi)$ with the constant term is not 0, and $m(\xi)$ satisfies (2.1), then there exist a positive integer N such that $m(\xi) \in \mathcal{S}_{N,1}$.*

To prove the theorem, the following two lemmas are needed when the dimension $n = 1$.

Lemma 1. *Given a trigonometric polynomial $m(\xi)$ which satisfies (2.1), and the constant term is not 0, then the degree of $m(\xi)$ as the polynomial of $e^{i\xi}$ must be odd.*

The proof of the lemma is obvious.

Lemma 2. *Given a trigonometric polynomial $m(\xi)$ which satisfies (2.1) with degree $2N + 1$, rewrite $m(\xi)$ into its polyphase form as $m(\xi) = f_0(x^2) + x f_1(x^2)$, then there exists a real unitary matrix U, such that $\widetilde{m}(\xi) = F_0(x^2) + x F_1(x^2)$ satisfies (2.1), where*

$$(F_0(x), F_1(x))^T = diag(1, x^{-1}) U^T (f_0(x), f_1(x))^T. \tag{2.13}$$

Proof. It is easy to find a real unitary matrix U such that both F_0 and F_1 defined by (2.13) are polynomials of x. We also can see that $\widetilde{m}(\xi) = F_0(x^2) + x F_1(x^2)$ has degree $2N$ at most.

From Lemma 1, we know that $\widetilde{m}(\xi)$ must be a polynomial of $e^{i\xi}$ with degree $2N-1$. Because $m(\xi)$ satisfies (2.1), then $\widetilde{m}(\xi)$ defined above must satisfy (2.1).

By using Lemma 1 and Lemma 2, we can see that Theorem 4 is valid.

In some applications, it is better to use a filter with linear phase than a filter with nonlinear phase. But it is well known that in one dimension case, there does not exist a orthogonal filter with linear phase except Haar filter. But in high dimension case, we can find many filters with linear phase.

Before constructing filters with linear phase, we define a matrix operator. For a matrix A of size $m \times m$, define $A^S := H_m A H_m$, where $H_m = (h_{kl})_{k,l=1}^m$ is a matrix of size $m \times m$, with $h_{kl} = 1$ when $k + l = m + 1$ and 0 otherwise. It is easy to check that $(A^S)^T = (A^T)^S$, and $(AB)^S = A^S B^S$ for any two square matrix A and B of the same size.

Denote

$$J_n := \{U | U \in \mathcal{U}_n, U^S = \pm U\}. \tag{2.14}$$

Definition. *Given a trigonometric polynomial $m(\xi)$, $\xi := (\xi_1 \cdots \xi_n) \in R^n$, if*

$$\overline{m(\xi)} = \pm e^{-iM_1\xi_1} \cdots e^{-iM_n\xi_n} m(\xi), \tag{2.15}$$

where M_k is positive integer for $k = 1, \cdots, n$, then we say $m(\xi)$ has linear phase.

Denote $W_{N,n} := \{m(\xi) | m(\xi) \in \mathcal{L}_{N,n}, \overline{m(\xi)} = \pm e^{-i(2N+1)(\xi_1+\cdots+\xi_n)} m(\xi)\}$. It is clear that each member in $W_{N,n}$ is a square lowpass filter with linear phase. We only consider $W_{N,n}$ in high dimension case because they are square filters.

Theorem 6. *Let $m(\xi) = 2^{-n} X_E U_N D^2 \cdots U_1 D^2 U_1^T \cdots U_N^T V_0$, if $U_k \in J_n$, $k = 1, \cdots, N$, then $m(\xi) \in W_{N,n}$.*

Proof. It is easy to see that

$$e^{i(2N+1)(\xi_1+\cdots+\xi_n)} \overline{m(\xi)} = 2^{-n} X_E U_N^S D^2 \cdots U_1^S D^2 (U_1^S)^T \cdots (U_N^S)^T V_0.$$

If $U_k \in J_n$, $k = 1, \cdots, N$, then $m(\xi) = \pm e^{i(2N+1)(\xi_1+\cdots+\xi_n)} \overline{m(\xi)}$, that is, $m(\xi) \in W_{N,n}$.

It is interesting to investigate the member of $W_{N,n}$. It is well known that $W_{N,1}$ only include Haar filter, the following theorem give a simple proof of the conclusion.

Theorem 7. *Assume that $m(\xi) \in W_{N,1}$, then $m(\xi) = \frac{1}{2}(1 + e^{i(2N+1)\xi})$.*

Proof. Let $m(\xi) \in W_{N,1}$. From Theorem 5,

$$m(\xi) = 2^{-1} X_E U_N D^2 \cdots U_1 D^2 U_1^T \cdots U_N^T V_0,$$

and from the definition of $W_{N,1}$,

$$m(\xi) = 2^{-1} X_E U_N^S D^2 \cdots U_1^S D^2 (U_1^T)^S \cdots (U_N^T)^S V_0.$$

Under the assumption that the constant term of $m(\xi)$ is not 0, we can prove that $U_N^T U_N^S = \widetilde{D}$, where $\widetilde{D}$ is a diagonal matrix. It is easy to see that $\widetilde{D}^S = \widetilde{D}$, therefore $\widetilde{D} = \pm I_1$, where I_1 is the identity matrix of size 2×2. Hence we can prove that $U_k^T U_k^S = \pm I_1$, for $k = 1, \cdots, N$. For any $U \in \mathcal{U}_1$, $U^T U^S = \pm I_1$ if and only if U has one of the following form: $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. Under the assumption that the constant term of $m(\xi)$ is not zero and $m(0) = 1$, we can see that $U_k = \pm I_1$. The theorem is proved.

¿From Theorem 6, we know that if we want to construct a filter with linear phase, we must investigate the necessary set J_n.

Denote

$$\mathcal{D}_n = \{\widetilde{D} | \widetilde{D} = (d_{kl})_{k,l=1}^{2^n} \in \mathcal{U}_n, d_{kl} = -1, 0 \text{ or } 1, \widetilde{D} = \widetilde{D}^T = \pm \widetilde{D}^S\}.$$

For J_n, we have the following theorem.

Theorem 8. *$U \in J_n$, if and only if there exists a $\widetilde{D} \in \mathcal{D}_n$, such that $\widetilde{D} U \in J_n$.*

Proof. Suppose $U \in J_n$, then $U^S = \pm U$. For any $\widetilde{D} \in \mathcal{D}_n$,

$$(\widetilde{D}U)^S = \widetilde{D}^S U^S = \pm \widetilde{D}U,$$

that is $\widetilde{D}U \in J_n$.

On the other hand, if for certain $\widetilde{D} \in \mathcal{D}_n$, such that $\widetilde{D}U \in J_n$, then $U^S = (\widetilde{D}\widetilde{D}U)^S = \widetilde{D}^S(\widetilde{D}U)^S = \pm\widetilde{D}\widetilde{D}U = \pm U$, that is $U \in J_n$.

The proof is completed.

It is complicated to write out the explicit expression of J_n, but for $n = 2$, we can write J_2 easily.

Denote

$$B(\alpha_1, \alpha_2) = \begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & -d & -a & b \\ d & c & b & a \end{pmatrix} \tag{2.16}$$

where $a = \cos\alpha_1 \cos\alpha_2$, $b = \cos\alpha_1 \sin\alpha_2$, $c = \sin\alpha_1 \cos\alpha_2$, $d = -\sin\alpha_1 \sin\alpha_2$.

Then we have the following theorem whose proof can be found in [5].

Theorem 9.

$$J_2 = \{U | U = B(\alpha_1, \alpha_2)\widetilde{D}, \alpha_1, \alpha_2 \in R, \widetilde{D} \in \mathcal{D}_2\} \tag{2.17}$$

2.2 Biorthogonal wavelet filters

Using the similar method of orthogonal case, we can construct biorthogonal wavelet filters.

Define

$$m_N(\xi) = 2^{-n} X_E \cdot U_N D^2 \cdots U_1 D^2 U_1^{-1} \cdots U_N^{-1} V_0 \tag{2.18}$$

where V_0 is a column vector of size 2^n with all its entries are 1, and U_k is a nonsingular matrix for $k = 1, \cdots, N$.

According to $m_N(\xi)$, define another polynomial

$$\widetilde{m_N}(\xi) = 2^{-n} X_E \cdot \widetilde{U_N} D^2 \cdots \widetilde{U_1} D^2 \widetilde{U_1}^{-1} \cdots \widetilde{U_N}^{-1} V_0 \tag{2.19}$$

where $\widetilde{U_k}^T = U_k^{-1}$, $k = 1, \cdots, N$. Then we have the following theorem.

Theorem 10. *m_N and $\widetilde{m_N}$ satisfy the following equation:*

$$\sum_{\nu \in E^n} m_N(\xi + \nu\pi)\overline{\widetilde{m_N}(\xi + \nu\pi)} = 1 \tag{2.20}$$

Proof. Let

$$f = U_N D \cdots U_1 D U_1^{-1} \cdots U_N^{-1} V_0$$

and

$$\widetilde{f} = \widetilde{U_N} D \cdots \widetilde{U_1} D \widetilde{U_1}^{-1} \cdots \widetilde{U_N}^{-1} V_0,$$

then (2.20) is equivalent to $\widetilde{f}^{\,T}(\xi) \cdot f(\xi) = 2^n$. From the definition of $\widetilde{f}$, we can see that the conclusion is valid.

¿From the well known definition of biorthogonal wavelet theory, we know that m_N and $\widetilde{m_N}$ are a pair of lowpass wavelet filters.

Given lowpass filters $m_N(\xi)$ and $\widetilde{m}_N(\xi)$ are defined as (2.18) and (2.19). Let U be any real unitary matrix of size $2^n \times 2^n$ such that the first column of U equals to V_0. Suppose

$$U = (V_0 \; V_1 \; \cdots \; V_{N-1}),$$

then for $k = 1, \cdots, 2^n - 1$, define

$$g_N^k(\xi) = 2^{-n} X_E \cdot U_N D^2 \cdots U_1 D^2 U_1^{-1} \cdots U_N^{-1} V_k \tag{2.21}$$

and

$$\widetilde{g}_N^k(\xi) = 2^{-n} X_E \cdot \widetilde{U_N} D^2 \cdots \widetilde{U_1} D^2 \widetilde{U_1}^{-1} \cdots \widetilde{U_N}^{-1} V_k. \tag{2.22}$$

The following theorem prove that g_N^k and $\widetilde{g}_N^k$ are the corresponding highpass filters.

Theorem 11. *Let $m_N(\xi)$ and $\widetilde{m}_N(\xi)$ are the lowpass filters defined as in (2.18) and (2.19) respectively, then for $k = 1, \cdots, 2^n - 1$, $g_N^k(\xi)$ and $\widetilde{g}_N^k(\xi)$ defined in (2.21) and (2.22) respectively are the corresponding highpass filters.*

Proof. Since U is unitary matrices, from the theory of wavelets ([1]), we know that the conclusion of the theorem is valid.

It is also useful to construct symmetric biorthogonal wavelet filters. The following theorem present a simple method.

Theorem 12. *Let $m(\xi) = 2^{-n} X_E U_N D^2 \cdots U_1 D^2 U_1^{-1} \cdots U_N^{-1} V_0$, if $U_k \in J_n$, $k = 1, \cdots, N$, then $m(\xi)$ has linear phase.*

Proof. It is easy to see that

$$e^{i(2N+1)(\xi_1+\cdots+\xi_n)}\overline{m(\xi)} = 2^{-n} X_E U_N^S D^2 \cdots U_1^S D^2 (U_1^S)^{-1} \cdots (U_N^S)^{-1} V_0.$$

If $U_k \in J_n$, $k = 1, \cdots, N$, then $m(\xi) = \pm e^{i(2N+1)(\xi_1+\cdots+\xi_n)}\overline{m(\xi)}$, that is, $m(\xi)$ has linear phase.

3. Examples

In this section, we will give some examples.

n=1.

Orthogonal case:

Denote $A(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $D := \mathrm{diag}(1 \quad x^2)$, then define

$$m_N(\xi) = \frac{1}{\sqrt{2}}(1 \quad x) A(\alpha_1) D \cdots A(\alpha_N) D (\cos\alpha_0 \quad \sin\alpha_0)^T \tag{3.1}$$

then it is easy to see that $m_N(\xi)$ satisfies (2.1) for all $\alpha_0, \cdots, \alpha_N$.

For a lowpass filter, $m_N(0) = 1$ is a necessary condition. We have

$$m_N(0) = \frac{1}{\sqrt{2}}(\cos(\alpha_0 + \cdots \alpha_N) + \sin(\alpha_0 + \cdots \alpha_N))$$

such that $m_N(0) = 1$ holds if and only if $\alpha_0 + \cdots \alpha_N = \frac{\pi}{4}$. As it is pointed out in Theorem 5, all one dimensional orthogonal wavelet filters are included in this form.

Examples associate this form with Daubechies wavelets can be found in [6].

Biorthogonal case:

The biorthogonal pair of lowpass filters are:

$$m(x) = \frac{1}{8}(1+x)^3 = \frac{1}{16}(1 \ x) \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\widetilde{m}(x) = \frac{1}{4}(-1 + 3x + 3x^2 - x^3) = \frac{1}{16}(1 \ x) \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

n=2

Orthogonal case:

These examples can also be found in [5].

The lowpass filter is:

$$\begin{pmatrix} 0.1188 & 0.0142 & 0.0259 & -0.0081 \\ 0.0794 & -0.0944 & -0.0236 & 0.0115 \\ 0.0077 & 0.1562 & 0.0975 & 0.0876 \\ -0.0487 & -0.0077 & 0.2430 & 0.3405 \end{pmatrix}$$

The three highpass filters are

$$\begin{pmatrix} 0.2800 & 0.0336 & -0.0053 & 0.0016 \\ 0.1871 & -0.2223 & 0.0048 & -0.0023 \\ -0.0136 & -0.2752 & -0.0111 & -0.0100 \\ 0.0858 & 0.0135 & -0.0277 & -0.0388 \end{pmatrix}$$

$$\begin{pmatrix} 0.1489 & 0.0179 & 0.1357 & -0.0424 \\ 0.0995 & -0.1183 & -0.1238 & 0.0604 \\ 0.0159 & 0.3199 & -0.0505 & -0.0454 \\ -0.0998 & -0.0157 & -0.1259 & -0.1765 \end{pmatrix}$$

$$\begin{pmatrix} -0.0696 & -0.0083 & 0.3137 & -0.0979 \\ -0.0465 & 0.0552 & -0.2861 & 0.1395 \\ -0.0077 & -0.1560 & 0.0136 & 0.0122 \\ 0.0486 & 0.0076 & 0.0340 & 0.0476 \end{pmatrix}.$$

We also compute some symmetric filters. From Theorem 9 in section II, we can construct many filters with linear phase. For example, we have the following four filters.

The lowpass filter is:

$$\frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

The three highpass filters are

$$\frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix},$$

$$\frac{1}{8}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{pmatrix},$$

$$\frac{1}{8}\begin{pmatrix} -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{pmatrix}.$$

Biorthogonal case:

Decomposition lowpass filter:

$$\frac{1}{20}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix};$$

Decomposition highpass filters:

$$-\frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix};$$

$$-\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix};$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix};$$

Reconstruction lowpass filter:

$$-\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -4 & -4 & 1 \\ 1 & -4 & -4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix};$$

Reconstruction highpass filters:

$$\frac{1}{20}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -4 & 4 & -1 \\ 1 & -4 & 4 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix};$$

$$\frac{1}{20}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -4 & -4 & 1 \\ -1 & 4 & 4 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix};$$

$$-\frac{1}{20}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -4 & 4 & -1 \\ -1 & 4 & -4 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

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