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LONG TIME ASYMPTOTIC BEHAVIOR OF SOLUTION OF IMPLICIT DIFFERENCE SCHEME FOR A SEMI-LINEAR PARABOLIC EQUATION *1)

Zhi-zhong Sun
(*Department of Mathematics, Southeast University, Nanjing 210096, China*)

Long-Jun Shen
(*Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*)

Abstract

In this paper, the solution of back-Euler implicit difference scheme for a semi-linear parabolic equation is proved to converge to the solution of difference scheme for the corresponding semi-linear elliptic equation as t tends to infinity. The long asymptotic behavior of its discrete solution is obtained which is analogous to that of its continuous solution. At last, a few results are also presented for Crank-Nicolson scheme.

Key words: Asymptotic behavior, Implicit difference scheme, Semi-linear parabolic equation, Convergence.

1. Introduction

Consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} = \triangle u - \phi(u) + f(x,y), \quad (x,y,t) \in \Omega \times R_+, \tag{1.1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.1.2}$$

$$u|_{t=0} = u_0(x,y), \quad (x,y) \in \Omega, \tag{1.1.3}$$

where $\triangle$ is Laplac's operator, Ω is a rectangular $[0,l]^2$, $R_+ = (0,\infty)$, $\phi'(u) \geq 0$. As t tends to ∞ and $\phi'(u)$ satisfies some conditions, the solution of (1.1) converges to that of the following semi-linear elliptic boundary value problem

$$\triangle\, u - \phi(u) + f(x,y) = 0, \quad (x,y) \in \Omega, \tag{1.2.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2.2}$$

Comparing to the case of continuous problem, it is very interesting to discuss the asymptotic behavior of discrete solution of difference scheme for (1.1). For one-dimensional problem (1.1) and $\phi(u) = u^3$, Hui Feng and Long-jun Shen proved the solution of backward Euler difference scheme and forward Euler difference scheme converge to the solution of the difference scheme for the relevant nonlinear stationary problem as t tends to infinity and obtained the long time asymptotic behavior of discrete solution in [1] and [2] respectively by energy method.

In this paper, we consider back-Euler implicit difference scheme for (1.1) and give the asymptotic error estimates by using Browder fixed point theorem, maximum principle and energy method. For the Crank-Nicolson difference scheme, some similar results are also given.

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Let $h, \Delta t_n$ be the space step-size and the time step-size respectively, $h = l/J$, where J is an integer. Denote $\Omega_h = \{(x_i, y_j) \mid x_i = ih, y_j = jh, 0 \le i, j \le J\}$ and $H = \{w \mid w = \{w_{ij}\}_{i,j=0}^{J},\ w_{i0} = w_{iJ} = w_{0j} = w_{Jj} = 0, 0 \le i, j \le J\}$.

For $w \in H$, introduce the following notations:

$$\delta_x w_{i+\frac{1}{2},j} = (w_{i+1,j} - w_{ij})/h, \qquad \delta_y w_{i,j+\frac{1}{2}} = (w_{i,j+1} - w_{ij})/h,$$
$$\delta_x^2 w_{ij} = (w_{i+1,j} - 2w_{ij} + w_{i-1,j})/h^2, \qquad \delta_y^2 w_{ij} = (w_{i,j+1} - 2w_{ij} + w_{i,j-1})/h^2,$$
$$\triangle_h w_{ij} = \delta_x^2 w_{ij} + \delta_y^2 w_{ij},$$
$$\|w\|_C = \max_{1 \le i,j \le J-1} |w_{ij}|, \qquad \|w\| = \sqrt{h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} (w_{ij})^2},$$
$$\|\delta_h w\| = \sqrt{h^2 \left[\sum_{i=0}^{J-1} \sum_{j=1}^{J-1} (\delta_x w_{i+\frac{1}{2},j})^2 + \sum_{i=1}^{J-1} \sum_{j=0}^{J-1} (\delta_y w_{i,j+\frac{1}{2}})^2 \right]},$$
$$\|\Delta_h w\| = \sqrt{h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} (\Delta_h w_{ij})^2}.$$

It is easy to know that $\|w\|_C$, $\|w\|$, $\|\delta_h w\|$ and $\| \triangle_h w\|$ are all norms of the space H. In addition, if $v \in H$ and $w \in H$, we define the inner product

$$(v, w) = h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} v_{ij} w_{ij}.$$

It is obvious that

$$\|w\| = \sqrt{(w, w)}.$$

The back-Euler implicit difference scheme for (1.1) we will consider is as follows

$$\frac{u_{ij}^n - u_{ij}^{n-1}}{\Delta t_n} = \triangle_h u_{ij}^n - \phi(u_{ij}^n) + f(x_i, y_j),$$
$$1 \le i, j \le J-1, n = 1, 2, 3, \cdots \tag{1.3.1}$$
$$u_{i0}^n = u_{iJ}^n = u_{0j}^n = u_{Jj}^n = 0, 0 \le i, j \le J, n = 1, 2, 3, \cdots, \tag{1.3.2}$$
$$u_{ij}^0 = u_0(x_i, y_j),\ 0 \le i, j \le J. \tag{1.3.3}$$

For (1.2), we construct the following difference scheme

$$\triangle_h u_{ij}^* - \phi(u_{ij}^*) + f(x_i, y_j) = 0,\ 1 \le i, j \le J-1, \tag{1.4.1}$$
$$u_{i0}^* = u_{iJ}^* = u_{0j}^* = u_{Jj}^* = 0, 0 \le i, j \le J, \tag{1.4.2}$$

In next section, the difference schemes (1.3) and (1.4) are proved to have unique and bounded discrete solutions respectively. In section 3, the asymptotic error estimates are given, from which it is known that the solution of (1.3) converges to the solution of (1.4).

The main result of this paper is Theorem 3.1 proved in section 3.

2. Preliminary Results

For our need, we list following lemmas.

Lemma 2.1. *For any discrete function $w \in H$, we have*

$$\|\delta_h w\|^2 + (w, \triangle_h w) = 0,$$
$$\|w\|^2 \le \frac{l^2}{8}\|\delta_h w\|^2,$$
$$\|\delta_h w\|^2 \le \frac{l^2}{8}\| \triangle_h w\|^2.$$

Lemma 2.2[3−4]**.** *Let $(H,(\cdot,\cdot))$ be a finite dimensional inner product space, $\|\cdot\|$ the associated norm, and $g: H \to H$ be continuous. Assume, moreover, that*

$$\exists \alpha > 0,\ \forall z \in H,\ \|z\| = \alpha,\ (g(z), z) \ge 0,$$

then, there exists an element $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \le \alpha$.

Lemma 2.3. *Let $T_n = \sum_{k=1}^n \Delta t_k$. Suppose the sequence $\{a_n\}$ satisfies*

$$a_n \le \exp(-c_1\Delta t_n)a_{n-1} + c_2 \exp(-c_3 T_n)\Delta t_n,\ n = 1, 2, 3, \cdots,$$

where $a_n \ge 0, n = 0, 1, 2, \cdots, c_i \ge 0, i = 1, 2, 3,$ then

$$a_n \le \left(a_0 + \frac{2c_2}{c_3}\right)\exp(-\delta T_n), n = 1, 2, 3, \cdots,$$

where $\delta = \min\{c_1, \frac{c_3}{2}\}$.

Proof.

$$\begin{aligned}
a_n \le & \exp(-c_1\Delta t_n)a_{n-1} + c_2\exp(-c_3T_n)\Delta t_n \\
\le & \exp(-c_1\Delta t_n)[\exp(-c_1\Delta t_{n-1})a_{n-2} \\
& + c_2\exp(-c_3T_{n-1})\Delta t_{n-1}] + c_2\exp(-c_3T_n)\Delta t_n \\
\le & \exp[-c_1(\Delta t_n + \Delta t_{n-1})]a_{n-2} \\
& + c_2[\exp(-c_1\Delta t_n)\exp(-c_3T_{n-1})\Delta t_{n-1} + \exp(-c_3T_n)\Delta t_n] \\
\le & \cdots \\
\le & \exp[-c_1(\Delta t_n + \Delta t_{n-1} + \cdots + \Delta t_1)]a_0 \\
& + c_2\{\exp[-c_1(\Delta t_n + \Delta t_{n-1} + \cdots + \Delta t_2)]\exp(-c_3T_1)\Delta t_1 \\
& + \exp[-c_1(\Delta t_n + \Delta t_{n-1} + \cdots + \Delta t_3)]\exp(-c_3T_2)\Delta t_2 + \cdots \\
& + \exp(-c_1\Delta t_n)\exp(-c_3T_{n-1})\Delta t_{n-1} + \exp(-c_3T_n)\Delta t_n\} \\
\le & \exp(-c_1T_n)a_0 + c_2\{\exp[-c_1(T_n - T_1)]\exp(-c_3T_1)\Delta t_1 \\
& + \exp[-c_1(T_n - T_2)]\exp(-c_3T_2)\Delta t_2 + \cdots \\
& + \exp[-c_1(T_n - T_{n-1})]\exp(-c_3T_{n-1})\Delta t_{n-1} + \exp(-c_3T_n)\Delta t_n\}.
\end{aligned}$$

Since

$$\exp[-c_1(T_n - T_i)]\exp(-\frac{c_3}{2}T_i) \le \exp[-\delta(T_n - T_i)]\exp(-\delta T_i) \le \exp(-\delta T_n),$$
$$\exp(-c_1T_n) \le \exp(-\delta T_n), \qquad \exp(-\frac{c_3}{2}T_n) \le \exp(-\delta T_n),$$

we have

$$\begin{aligned} a_n &\le \exp(-\delta T_n)\left\{a_0 + c_2\left[\exp(-\frac{1}{2}c_3T_1)\Delta t_1\right.\right. \\ &\left.\left.+\exp(-\frac{1}{2}c_3T_2)\Delta t_2 + \cdots + \exp(-\frac{1}{2}c_3T_n)\Delta t_n\right]\right\} \\ &\le \exp(-\delta T_n)\left[a_0 + c_2\int_0^\infty \exp(-\frac{1}{2}c_3 t)dt\right] \\ &= (a_0 + \frac{2c_2}{c_3})\exp(-\delta T_n). \end{aligned}$$

Lemma 2.4. *The difference scheme (1.3) has a unique solution.*

Proof. In order to prove the lemma by the induction, assume $u^0, u^1, \cdots, u^{n-1}$ exist. Let $g: H \to H$ be defined as follows:

$$g(v)_{ij} = v_{ij} - \Delta t_n[\triangle_h v_{ij} - \phi(v_{ij}) + f(x_i, y_j)] - u_{ij}^{n-1}, 1 \le i, j \le J-1.$$

Then g is clearly continuous. Furthermore, using Cauchy-Schartz inequality and Lemma 2.1, we have

$$\begin{aligned} (g(v), v) &= (v, v) - \Delta t_n[(\triangle_h v, v) - (\phi(v), v) + (f, v)] - (u^{n-1}, v) \\ &= (v, v) + \Delta t_n\|\delta_h v\|^2 + \Delta t_n\ (\phi(v) - \phi(0), v) + \Delta t_n(\phi(0) - f, v) - (u^{n-1}, v) \\ &\ge (v, v) + \Delta t_n(\phi(0) - f, v) - (u^{n-1}, v) \\ &\ge \|v\|^2 - \Delta t_n\|\phi(0) - f\|\ \|v\| - \|u^{n-1}\|\ \|v\| \\ &= \|v\|[\|v\| - (\Delta t_n\|\phi(0) - f\| + \|u^{n-1}\|)]. \end{aligned}$$

When $\|v\| = \Delta t_n\|\phi(0) - f\| + \|u^{n-1}\|$, $(g(v), v) \ge 0$. By Lemma 2.2, there exists an element $v^* \in H$ such that $g(v^*) = 0$ and $\|v^*\| \le \Delta t_n\|\phi(0) - f\| + \|u^{n-1}\|$. This v^* is just the solution u^n of (1.3).

The proof of the uniqueness is easy and we omit it.

Lemma 2.5. *Let*

$$a = \frac{1}{4}(\|f\|_{L_\infty} + |\phi(0)|), \quad b = \|u_0\|_{L_\infty}$$

then the solution u^n of (1.3) satisfies

$$\|u^n\|_C \le \frac{1}{2}l^2 a + b, \quad n = 0, 1, 2, \cdots.$$

Proof. Let $u_{ij}^n = w_{ij}^n + a[x_i(l - x_i) + y_j(l - y_j)]$. Then

$$\frac{w_{ij}^n - w_{ij}^{n-1}}{\Delta t_n} = \triangle_h w_{ij}^n - \phi(u_{ij}^n) + f(x_i, y_j) - 4a, 1 \le i, j \le J-1. \tag{2.1}$$

Since $w_{ij}^n \le u_{ij}^n$, we have $\phi(w_{ij}^n) \le \phi(u_{ij}^n)$. It follows from (2.1) that

$$\begin{aligned} \frac{w_{ij}^n - w_{ij}^{n-1}}{\Delta t_n} &= \triangle_h w_{ij}^n - \phi(u_{ij}^n) + f(x_i, y_j) - 4a \\ &\le \triangle_h w_{ij}^n - [\phi(w_{ij}^n) - \phi(0)] + f(x_i, y_j) - \phi(0) - 4a \\ &= \triangle_h w_{ij}^n - \phi'(\xi_{ij}^n)w_{ij}^n + f(x_i, y_j) - \phi(0) - 4a \\ &\le \triangle_h w_{ij}^n - \phi'(\xi_{ij}^n)w_{ij}^n, \end{aligned} \tag{2.2}$$

where ξ_{ij}^n is between w_{ij}^n and 0. From (2.2), we have

$$[1 + \Delta t_n\phi'(\xi_{ij}^n)]w_{ij}^n \le \Delta t_n \triangle_h w_{ij}^n + w_{ij}^{n-1}, \quad 1 \le i, j \le J-1. \tag{2.3}$$

Let

$$w_{i_0,j_0}^n = \max_{1\le i,j\le J-1} w_{ij}^n.$$

If $w_{i_0,j_0}^n > 0$, then by using (2.3) we have

$$w_{i_0,j_0}^n \le [1+\Delta t_n \phi'(\xi_{i_0,j_0}^n)]w_{i_0,j_0}^n \le \Delta t_n \triangle_h w_{i_0,j_0}^n + w_{i_0,j_0}^{n-1} \le \max_{1\le i,j\le J-1} w_{ij}^{n-1}$$

i.e.,

$$\max_{1\le i,j\le J-1} w_{ij}^n \le \max_{1\le i,j\le J-1} w_{ij}^{n-1}$$

Therefore

$$\max\{0, \max_{1\le i,j\le J-1} w_{ij}^n\} \le \max\{0, \max_{1\le i,j\le J-1} w_{ij}^{n-1}\}, n=1,2,3,\cdots$$

By recursive process, we have

$$\max\{0, \max_{1\le i,j\le J-1} w_{ij}^n\} \le \max\{0, \max_{1\le i,j\le J-1} w_{ij}^0\} \le b, n=1,2,3,\cdots \tag{2.4}$$

from which we obtain

$$\begin{aligned}\max_{1\le i,j\le J-1} u_{ij}^n &\le \max_{1\le i,j\le J-1}\{w_{ij}^n + a[x_i(l-x_i)+y_j(l-y_j)]\}\\ &\le \max_{1\le i,j\le J-1} w_{ij}^n + \frac{1}{2}l^2 a \le \frac{1}{2}l^2 a + b, n=0,1,2,\cdots \end{aligned}\tag{2.5}$$

If $\omega_{i_0,j_0}^n \le 0$, the validity of (2.5) is obvious by the definition of $\{\omega_{ij}\}$.

Similarly, let $u_{ij}^n = \tilde{w}_{ij}^n - a[x_i(l-x_i)+y_j(l-y_j)]$, we can obtain

$$\min_{1\le i,j\le J-1} u_{ij}^n \ge -(\frac{1}{2}l^2 a + b), n=0,1,2,\cdots \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\|u^n\|_C \le \frac{1}{2}l^2 a + b, n=0,1,2,\cdots.$$

Lemma 2.6. *The difference scheme (1.4) has a unique solution.*

Proof. Let $g: H\to H$ be defined as follows

$$g(v)_{ij} = -\triangle_h v_{ij} + \phi(v_{ij}) - f(x_i,y_j), 1\le i,j\le J-1.$$

Then g is clearly continuous. Applying Lemma 2.1 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}(g(v),v) &= -(\triangle_h v, v) + (\phi(v),v) - (f,v)\\ &= \|\delta_h v\|^2 + (\phi(v)-\phi(0),v) + (\phi(0)-f,v)\\ &\ge \|\delta_h v\|^2 - \|\phi(0)-f\|\,\|v\|\\ &\ge \frac{8}{l^2}\|v\|^2 - \|\phi(0)-f\|\,\|v\|\\ &\ge \frac{8}{l^2}\|v\|(\|v\| - \frac{l^2}{8}\|\phi(0)-f\|).\end{aligned}$$

When $\|v\| = \frac{l^2}{8}\|\phi(0)-f\|,(g(v),v)\ge 0$. By Lemma 2.2, there exists a $v^*\in H$ such that $g(v^*)=0$ and $\|v^*\|\le \frac{l^2}{8}\|\phi(0)-f\|$. It is easy to know that this v^* is just the solution of (1.4).

It is not difficult to show the uniqueness of the solution of (1.4).

Lemma 2.7. *The solution of (1.4) satisfies*

$$\|u^*\|_C \le \frac{1}{2}l^2 a,$$

where a is defined in Lemma 2.5.

The proof is similar to that of Lemma 2.5. We omit it.

3. Asymptotic Behavior of Implicit Difference Solutions

We will analyze the difference between the solution of (1.3) and the solution of (1.4). Let

$$v_{ij}^n = u_{ij}^n - u_{ij}^*.$$

Subtracting (1.4) from (1.3), we have

$$\frac{v_{ij}^n - v_{ij}^{n-1}}{\Delta t_n} = \triangle_h v_{ij}^n - \left[\phi(u_{ij}^n) - \phi(u_{ij}^*)\right], \quad 1 \le i,j \le J-1, n=1,2,3,\cdots \tag{3.1.1}$$

$$v_{i0}^n = v_{iJ}^n = v_{0j}^n = v_{Jj}^n = 0, 0 \le i,j \le J, n = 1,2,3,\cdots, \tag{3.1.2}$$

$$v_{ij}^0 = u_0(x_i,y_j) - u_{ij}^*, 0 \le i,j \le J. \tag{3.1.3}$$

Taking the inner product of (3.1.1) with $2v^n$, we obtain

$$\begin{aligned}&\|v^n\|^2 - \|v^{n-1}\|^2 + \|v^n - v^{n-1}\|^2\\ =&2\Delta t_n(v^n, \triangle_h v^n) - 2\Delta t_n(v^n, \phi(u^n) - \phi(u^*))\\ =&-2\Delta t_n\|\delta_h v^n\|^2 - 2\Delta t_n(u^n - u^*, \phi(u^n) - \phi(u^*))\\ \le&-2\Delta t_n\|\delta_h v^n\|^2.\end{aligned}$$

Then,

$$\|v^n\|^2 + 2\Delta t_n\|\delta_h v^n\|^2 \le \|v^{n-1}\|^2, \quad n = 1,2,3,\cdots. \tag{3.2}$$

Using Lemma 2.1, we have

$$(1 + \frac{16}{l^2}\Delta t_n)\|v^n\|^2 \le \|v^{n-1}\|^2, \quad n = 1,2,3,\cdots,$$

or,

$$\|v^n\|^2 \le \frac{1}{(1+\frac{16}{l^2}\Delta t_n)}\|v^{n-1}\|^2 \le \exp\left(-\frac{8}{l^2}\Delta t_n\right)\|v^{n-1}\|^2, \quad n = 1,2,3,\cdots,$$

when $\frac{8}{l^2}\Delta t_n \le 1$, where we have used $1 + 2x \ge e^x \quad (0 \le x \le 1)$. Repeatedly applying above inequality, we obtain

$$\begin{aligned}\|v^n\|^2 \le&\exp\left[-\frac{8}{l^2}(\Delta t_n + \Delta t_{n-1} + \cdots + \Delta t_1)\right]\|v^0\|^2\\ =&\exp\left(-\frac{8}{l^2}T_n\right)\|v^0\|^2, n = 1,2,3,\cdots.\end{aligned} \tag{3.3}$$

Rewriting (3.2) as follows:

$$\|v^k\|^2 + 2\Delta t_k\|\delta_h v^k\|^2 \le \|v^{k-1}\|^2, \quad k = 1,2,3,\cdots.$$

Summing up for k from $n+1$ to m, then letting m tend to ∞ and applying (3.3), we can obtain

$$\sum_{k=n+1}^{\infty}\Delta t_k\|\delta_h v^k\|^2 \le \frac{1}{2}\exp\left(-\frac{8}{l^2}T_n\right)\|v^0\|^2, n = 1,2,3,\cdots \tag{3.4}$$

Taking the inner product of (3.1.1) with $2\triangle_h v^n$, we have

$$2(\triangle_h v^n, v^n - v^{n-1}) = 2\Delta t_n\|\triangle_h v^n\|^2 - 2\Delta t_n(\triangle_h v^n, \phi(u^n) - \phi(u^*)).$$

Since

$$-2(\triangle_h v^n, v^n - v^{n-1}) = \|\delta_h v^n\|^2 - \|\delta_h v^{n-1}\|^2 + \|\delta_h(v^n - v^{n-1})\|^2$$

and

$$2|(\triangle_h v^n, \phi(u^n) - \phi(u^*))| \le \| \triangle_h v^n\|^2 + c^2\|v^n\|^2,$$

where $c = \max_{|u| \le \frac{1}{2}l^2 a + b} \phi'(u)$, we have

$$-(\|\delta_h v^n\|^2 - \|\delta_h v^{n-1}\|^2 + \|\delta_h(v^n - v^{n-1})\|^2) \ge 2\Delta t_n \| \triangle_h v^n\|^2 - \Delta t_n(\| \triangle_h v^n\|^2 + c^2\|v^n\|^2),$$

or,

$$\|\delta_h v^n\|^2 - \|\delta_h v^{n-1}\|^2 + \Delta t_n \| \triangle_h v^n\|^2 \le c^2 \Delta t_n \|v^n\|^2. \tag{3.5}$$

Applying (3.3), we obtain

$$\|\delta_h v^n\|^2 - \|\delta_h v^{n-1}\|^2 + \Delta t_n \| \triangle_h v^n\|^2 \le c^2 \Delta t_n \exp\big(-\frac{8}{l^2} T_n\big)\ \|v^0\|^2, n = 1, 2, 3, \cdots. \tag{3.6}$$

And using Lemma 2.1, we have

$$(1 + \frac{8}{l^2}\Delta t_n)\|\delta_h v^n\|^2 \le \|\delta_h v^{n-1}\|^2 + c^2 \Delta t_n \exp\big(-\frac{8}{l^2} T_n\big)\|v^0\|^2, n = 1, 2, 3, \cdots,$$

or,

$$\|\delta_h v^n\|^2 \le \exp\big(-\frac{4}{l^2}\Delta t_n\big)\|\delta_h v^{n-1}\|^2 + c^2 \Delta t_n \exp\big(-\frac{8}{l^2} T_n\big)\|v^0\|^2, n = 1, 2, 3, \cdots,$$

when $\frac{4}{l^2}\Delta t_n \le 1$. By Lemma 2.3,

$$\|\delta_h v^n\|^2 \le (\|\delta_h v^0\|^2 + \frac{1}{4}c^2 l^2 \|v^0\|^2) \exp\big(-\frac{4}{l^2} T_n\big), n = 1, 2, 3, \cdots. \tag{3.7}$$

Rrewriting (3.6) as follows:

$$\|\delta_h v^k\|^2 - \|\delta_h v^{k-1}\|^2 + \Delta t_k \| \triangle_h v^k\|^2 \le c^2 \Delta t_k \exp\big(-\frac{8}{l^2} T_k\big)\ \|v^0\|^2, k = 1, 2, 3, \cdots.$$

Summing up for k from $n+1$ to m, then letting m tend to ∞ and using (3.7), we can obtain

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \Delta t_k \| \triangle_h v^k\|^2 \\
\le& \|\delta_h v^n\|^2 + c^2\|v^0\|^2 \sum_{k=n+1}^{\infty} \Delta t_k \exp\big(-\frac{8}{l^2} T_k\big) \\
\le& \|\delta_h v^n\|^2 + c^2\|v^0\|^2 \int_{T_n}^{\infty} \exp\big(-\frac{8}{l^2} t\big) dt \\
\le& \|\delta_h v^n\|^2 + \frac{1}{8}c^2 l^2 \|v^0\|^2 \exp\big(-\frac{8}{l^2} T_n\big) \\
\le& (\|\delta_h v^0\|^2 + \frac{3}{8}c^2 l^2 \|v^0\|^2) \exp\big(-\frac{4}{l^2} T_n\big), n = 0, 1, 2, \cdots.
\end{aligned} \tag{3.8}$$

Summarizing above results, we have

Theorem 3.1. *Let $\{u_{ij}^n\}$ and $\{u_{ij}^*\}$ be the solutions of difference schemes (1.3) and (1.4) respectively and denote $v_{ij}^n = u_{ij}^n - u_{ij}^*$. If*

$$\Delta t_n \le l^2/8, \quad n = 1, 2, 3, \cdots,$$

then, we have

$$\|v^n\|^2 \le \exp\big(-\frac{8}{l^2}T_n\big)\|v^0\|^2, \qquad n=1,2,3,\cdots;$$
$$\sum_{k=n+1}^{\infty} \Delta t_k\|\delta_h v^k\|^2 \le \frac{1}{2}\exp\big(-\frac{8}{l^2}T_n\big)\|v^0\|^2, \qquad n=1,2,3,\cdots;$$
$$\|\delta_h v^n\|^2 \le (\|\delta_h v^0\|^2+\frac{1}{4}c^2l^2\|v^0\|^2)\exp\big(-\frac{4}{l^2}T_n\big), \qquad n=1,2,3,\cdots;$$
$$\sum_{k=n+1}^{\infty} \Delta t_k\|\triangle_h v^k\|^2 \le (\|\delta_h v^0\|^2+\frac{3}{8}c^2l^2\|v^0\|^2)\exp\big(-\frac{4}{l^2}T_n\big), \qquad n=1,2,3,\cdots,$$

where $T_n=\sum_{k=1}^n \Delta t_k$, $c=\max_{|u|\le\frac{1}{2}l^2a+b}\phi'(u)$.

4. Crank-Nicolson Scheme

Consider Crank-Nicolson scheme for (1.1) as follows:

$$\frac{u_{ij}^n-u_{ij}^{n-1}}{\Delta t_n}=\triangle_h u_{ij}^{n-\frac{1}{2}}-\phi(u_{ij}^{n-\frac{1}{2}})+f(x_i,y_j),$$
$$1\le i,j\le J-1, n=1,2,3,\cdots \tag{4.1.1}$$
$$u_{i0}^n=u_{iJ}^n=u_{0j}^n=u_{Jj}^n=0,\ 0\le i,j\le J,\ n=1,2,3,\cdots, \tag{4.1.2}$$
$$u_{ij}^0=u_0(x_i,y_j),\ 0\le i,j\le J. \tag{4.1.3}$$

where $u_{ij}^{n-\frac{1}{2}}=\frac{1}{2}(u_{ij}^n+u_{ij}^{n-1})$. (4.1.1) can be rewritten as

$$\frac{2(u_{ij}^{n-\frac{1}{2}}-u_{ij}^{n-1})}{\Delta t_n}=\triangle_h u_{ij}^{n-\frac{1}{2}}-\phi(u_{ij}^{n-\frac{1}{2}})+f(x_i,y_j),\ 1\le i,j\le J-1, n=1,2,3,\cdots.$$

Theorem 4.1. *The difference scheme (4.1) has a unique solution.*

Proof. We will prove the theorem by induction. Assume $u^0,u^1,\cdots,u^{n-1}$ exist. Let $g:H\to H$ be defined as

$$g(v)_{ij}=2v_{ij}-\Delta t_n[\triangle_h v_{ij}-\phi(v_{ij})+f(x_i,y_j)]-2u_{ij}^{n-1}, 1\le i,j\le J-1.$$

Obviously, $g(v)$ is continuous. In addition,

$$\begin{aligned}(g(v),v)&=2(v,v)-\Delta t_n[(\triangle_h v,v)-(\phi(v),v)+(f,v)]-2(u^{n-1},v)\\&=2\|v\|^2+\Delta t_n\left[\|\delta_h v\|^2+(\phi(v)-\phi(0),v)+(\phi(0)-f,v)\right]-2(u^{n-1},v)\\&\ge 2\|v\|^2-\Delta t_n\|\phi(0)-f\|\,\|v\|-2\|u^{n-1}\|\,\|v\|\\&=2\|v\|\left[\|v\|-(\frac{1}{2}\Delta t_n\|\phi(0)-f\|+\|u^{n-1}\|)\right].\end{aligned}$$

When $\|v\|=\frac{1}{2}\Delta t_n\|\phi(0)-f\|+\|u^{n-1}\|$, $(g(v),v)\ge 0$. Thus, by Lemma 2.2, there exists a $v^*\in H$ such that $g(v^*)=0$ and $\|v^*\|\le\frac{1}{2}\Delta t_n\|\phi(0)-f\|+\|u^{n-1}\|$. Observe that $2v^*-u^{n-1}$ is just the solution u^n of difference scheme (4.1).

The proof of the uniqueness is simple, which we omit.

Next we will discuss the asymptotic behavior of the difference scheme (4.1). We have the following results.

Theorem 4.2. *Suppose* $\{u_{ij}^n\}$ *and* $\{u_{ij}^*\}$ *are the solution of (4.1) and (1.4) respectively and* $v_{ij}^n=u_{ij}^n-u_{ij}^*$. *Then, we have*

$$\|v^n\|^2+2\Delta t_n\|\delta_h v^{n-\frac{1}{2}}\|^2\le\|v^{n-1}\|^2, \quad n=1,2,3,\cdots. \tag{4.2}$$

In addition, if $\{\Delta t_n\}_{n=1}^{\infty}$ has a positive lower bound, then

$$\lim_{n\to\infty} \|v^n\| = 0. \tag{4.3}$$

Proof. (a) Subtracting (1.4) from (4.1), we obtain the error equations

$$\frac{v_{ij}^n - v_{ij}^{n-1}}{\Delta t_n} = \triangle_h v_{ij}^{n-\frac{1}{2}} - \phi'(\eta_{ij}^{n-\frac{1}{2}}) v_{ij}^{n-\frac{1}{2}}, \quad 1 \le i,j \le J-1, n = 1,2,3,\cdots \tag{4.4.1}$$

$$v_{i0}^n = v_{iJ}^n = v_{0j}^n = v_{Jj}^n = 0,\ 0 \le i,j \le J,\ n = 1,2,3,\cdots, \tag{4.4.2}$$

$$v_{ij}^0 = u_0(x_i, y_j) - u_{ij}^*,\ 0 \le i,j \le J. \tag{4.4.3}$$

where $\eta_{ij}^{n-\frac{1}{2}}$ is between $u_{ij}^{n-\frac{1}{2}}$ and u_{ij}^*.

Multiplying (4.4.1) by $v_{ij}^n + v_{ij}^{n-1}$ and summing up for i and j, we have

$$\begin{aligned}\frac{1}{\Delta t_n}\left(\|v^n\|^2 - \|v^{n-1}\|^2\right) &= 2\left(v^{n-\frac{1}{2}}, \triangle_h v^{n-\frac{1}{2}}\right) - 2h^2 \sum_{i,j=1}^{J-1} \phi'(\eta_{ij}^{n-\frac{1}{2}}) \left(v_{ij}^{n-\frac{1}{2}}\right)^2 \\ &\le 2\left(v^{n-\frac{1}{2}}, \triangle_h v^{n-\frac{1}{2}}\right) = -2\|\delta_h v^{n-\frac{1}{2}}\|^2,\end{aligned}$$

or,

$$\|v^n\|^2 + 2\Delta t_n \|\delta_h v^{n-\frac{1}{2}}\|^2 \le \|v^{n-1}\|^2, \quad n = 1,2,3,\cdots.$$

This is (4.2).

(b) It follows from (4.2) that the sequence $\{\|v^n\|\}_{n=0}^{\infty}$ is a decreasing one and has a lower bound 0. So $\{\|v^n\|\}_{n=0}^{\infty}$ has a limit as n tends to ∞. Denote

$$\lim_{n\to\infty} \|v^n\| = c. \tag{4.5}$$

Taking the limit in the two sides of (4.2) and using above equality, we have

$$\lim_{n\to\infty} \Delta t_n \|\delta_h v^{n-\frac{1}{2}}\|^2 = 0.$$

Since $\{\Delta t_n\}_{n=1}^{\infty}$ has a positive lower bound, we have

$$\lim_{n\to\infty} \|\delta_h v^{n-\frac{1}{2}}\|^2 = 0. \tag{4.6}$$

Lemma 2.1 gives

$$\lim_{n\to\infty} \|v^{n-\frac{1}{2}}\|^2 = 0,$$

or,

$$\lim_{n\to\infty} v_{ij}^{n-\frac{1}{2}} = 0, \quad 1 \le i,j \le J-1. \tag{4.7}$$

It follows that

$$lim_{n\to\infty} u_{ij}^{n-\frac{1}{2}} = u_{ij}^* \quad \text{and} \quad lim_{n\to\infty} \eta_{ij}^{n-\frac{1}{2}} = u_{ij}^*.$$

Taking the limit for n in (4.4.1) and applying (4.7), we have

$$\lim_{n\to\infty} \left(v_{ij}^n - v_{ij}^{n-1}\right) = 0, \quad 1 \le i,j \le J-1. \tag{4.8}$$

Now, taking the limit for n in the equality

$$2\left[(v_{ij}^n)^2 + (v_{ij}^{n-1})^2\right] = \left(v_{ij}^n + v_{ij}^{n-1}\right)^2 + \left(v_{ij}^n - v_{ij}^{n-1}\right)^2$$

and using (4.7) and (4.8), we get

$$\lim_{n\to\infty} \left[(v_{ij}^n)^2 + (v_{ij}^{n-1})^2\right] = 0, \quad 1 \le i,j \le J-1.$$

Summing up above equality for i and j, we obtain

$$\lim_{n\to\infty}\left(\|v^n\|^2+\|v^{n-1}\|^2\right)=0.$$

Inserting (4.5) into the left hand side of above equality, we have $c=0$. Thus,

$$\lim_{n\to\infty}\|v^n\|=0.$$

This completes the proof.

It is hoped that some estimates similar to those listed in Theorem 3.1 can be obtained, but we have not succeeded.

Using the results of this paper and following the idea of the paper [5], we can obtain the error estimates, uniformly in time, of the difference scheme (1.3).

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