Journal of Computational Mathematics, Vol.21, No.6, 2003, 733–746.

THE DUAL MIXED METHOD FOR AN UNILATERAL PROBLEM *1)

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Abstract

In this paper, the dual mixed method for an unilateral problem, which is the simplified modelling of scalar function for the friction-free contact problem, is considered. The dual mixed problem is introduced, the existence and uniqeness of the solution of the problem are presented, and error bounds $O(h^{\frac{3}{4}})$ and $O(h^{\frac{3}{2}})$ are obtained for the dual mixed finite element approximations of Raviart-Thomas elements for $k=0$ and $k=1$ respectively.

Key words: Dual Mixed Method, Unilateral Problem.

1. Introduction

In the early work [2], the dual mixed finite element method for the unilateral problem with a lower order term was considered, in which the divergence constraint is naturally incorporated into the unilateral formulation (the details can be found in the Remark 2.1 in section 2). In this paper , we consider an unilateral problem in the absence of the lower order term and involving the complex boundary conditions, which can be considered as the simplified modelling of scalar function for the friction-free contact problem (c.f.[5], [6] and [7]).

Let Ω be a bounded domain in R^2, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_F \cup \Gamma_C, \Gamma_D \cap \Gamma_F = \phi, \Gamma_F \cap \Gamma_C = \phi, \Sigma = \partial\Omega \setminus \Gamma_D$ and $\bar{\Gamma}_C \subset \Sigma$. For a given $f \in L^2(\Omega), t \in L^2(\Gamma_F), g \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, we consider the following unilateral problem:

$$\begin{cases} find \;\; u \in \mathbf{C}, \;\; such \;\; that \\ (\nabla u, \nabla(v-u)) \geq (f, v-u) + < t, v-u >_{\Gamma_F} \quad \forall \;\; v \in C, \end{cases} \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, $< \cdot,\cdot >$ denotes the duality product, and

$$\mathbf{C} = \{v \in H^1_{\Gamma_D}(\Omega) : v \geq g \;\; on \;\; \Gamma_C\}, \tag{1.2}$$

and

$$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \;\; on \;\; \Gamma_D\}. \tag{1.3}$$

It can be easily seen that the variational inequality (1.1) is equivalent to the following minimization problem:

$$\begin{cases} find \;\; u \in \mathbf{C}, \;\; such \;\; that \\ F(u) = \min F(v) \;\; \forall \;\; v \in \mathbf{C}, \end{cases} \tag{1.4}$$

where

$$F(v) = \frac{1}{2}(\nabla v, \nabla v) - (f, v) - < t, v >_{\Gamma_F} . \tag{1.5}$$

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* Received July 4, 2001.
1) The project was supported by National Natural Sciences Foundation of China.

By the general variational calculus, problem (1.1) (or (1.4)) is equivalent to the following differential problem:

$$\begin{cases} -\Delta u = f \quad in \quad L^2(\Omega), \\ u = 0 \quad on \quad \Gamma_D \quad in \quad H^{\frac{1}{2}}(\Gamma_D), \partial_\nu u = t \quad on \quad \Gamma_F \quad in \quad H^{-\frac{1}{2}}(\Gamma_F), \\ u - g \geq 0 \quad on \quad \Gamma_C \quad in \quad H_{00}^{\frac{1}{2}}(\Gamma_C), \partial_\nu u \geq 0 \quad on \quad \Gamma_C, \\ < \partial_\nu u, u - g >_{\Gamma_C} = 0. \end{cases} \tag{1.6}$$

Here and what follows, the general notations of Sobolev spaces (c.f. [1], [8]) are used. And we introduce the following notations for sequence use:

$$H_{div}(\Omega) = \{\mathbf{q} \in (L^2(\Omega))^2 = \mathbf{L}^2(\Omega) : div\mathbf{q} \in L^2(\Omega)\}, \tag{1.7}$$

with norm

$$\|\mathbf{q}\|_{div} = \{\|\mathbf{q}\|_{0,\Omega}^2 + \|div\mathbf{q}\|_{0,\Omega}^2\}^{\frac{1}{2}}. \tag{1.8}$$

The paper is organized as follows: In Sect.2 we derive a saddle-point problem with Lagrangian multiplier to relax the constraint: $div\mathbf{p} + f = 0$ in Ω, which is refered to as the dual mixed problem; And in Sect.3 the existence and uniqueness of the solution of the dual mixed problem are presented. Finally, in Sect.4, Raviart-Thomas finite element approximation ($k \geq 0$) to the dual mixed problem is considered, and the error bounds $O(h^{\frac{3}{4}})$ (for $k = 0$) and $O(h^{\frac{3}{2}})$ (for $k = 1$) are obtained respectively. Additionally, in Appendix we present the equivalence of $\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)}$ and $\inf_{v \in H^1(\Omega), \gamma v|_{\Gamma_C} = \mu} \|v\|_{1,\Omega} \quad \forall \quad \mu \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, which is used in the context.

2. Derivation of the Dual Mixed Problem

Along the lines of [3], we now derive the dual mixed formulation for the unilateral problem (1.1).

Lemma 2.1. $\forall \quad v \in \mathbf{C}$

$$F(v) = \sup_{\mathbf{q} \in K} \{-\frac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}\}, \tag{2.1}$$

where

$$K = \{\mathbf{q} \in Q : q_\nu \geq 0 \quad on \quad \Gamma_C\}, \tag{2.2}$$

$$Q = \{\mathbf{q} \in H_{div}(\Omega) : div\mathbf{q} + f = 0 \quad in \quad \Omega, q_\nu = t \quad on \quad \Gamma_F\}, \tag{2.3}$$

and ν denotes the outer unit normal vector on $\partial\Omega$, and $q_\nu = \mathbf{q} \cdot \nu$ the outer normal component of $\mathbf{q}$ on $\partial\Omega$.

Proof. $\forall \quad v \in \mathbf{C}, \mathbf{q} \in K$, by Green's integration formula and (1.5) we have

$$\begin{aligned} F(v) &= \tfrac{1}{2}(\nabla v, \nabla v) + (div\mathbf{q}, v) - < q_\nu, v >_{\Gamma_F} \\ &= \tfrac{1}{2}(\nabla v, \nabla v) - (\mathbf{q}, \nabla v) + < q_\nu, v >_{\Gamma_C} \\ &\geq \tfrac{1}{2}(\nabla v, \nabla v) - (\mathbf{q}, \nabla v) + < q_\nu, g >_{\Gamma_C} \\ &\geq -\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}, \end{aligned} \tag{2.4}$$

and the equalities hold iff $\mathbf{q} = \nabla v \quad in \quad \Omega$ and $v = g \quad on \quad \Gamma_C$. Thus the lemma is proved.

From (2.1) and the problem (1.1), we have the following dual problem:

$$\begin{aligned} &\inf_{v \in \mathbf{C}} F(v) = \inf_{v \in \mathbf{C}} \sup_{\mathbf{q} \in K} \{-\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}\} \\ &= -\inf_{\mathbf{q} \in K} \{\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) - < q_\nu, g >_{\Gamma_C}\}, \end{aligned} \tag{2.5}$$

which is equivalent to the following problem:

$$\begin{cases} find \quad \mathbf{p} \in K, \quad such \quad that \\ (\mathbf{p}, \mathbf{q} - \mathbf{p}) \geq < g, q_\nu - p_\nu >_{\Gamma_C} \quad \forall \quad \mathbf{q} \in K. \end{cases} \tag{2.6}$$

First we introduce the Lagrangian multiplier as in [3] to relax the constraint $q_\nu \geq 0$ on Γ_C in the convex set K. $\forall \xi \in H^{-\frac{1}{2}}(\Gamma_C)$, let

$$\delta^+(\xi|0) = \begin{cases} 0, & \xi \geq 0 \quad on \quad \Gamma_C \\ +\infty, & \xi < 0 \quad on \quad \Gamma_C, \end{cases} \tag{2.7}$$

then

$$\delta^+(\xi|0) = \sup_{\mu \in \Lambda} < \xi, \mu >_{\Gamma_C} \tag{2.8}$$

where

$$\Lambda = \{\mu \in H_{00}^{\frac{1}{2}}(\Gamma_C) : \mu \leq 0 \quad on \quad \Gamma_C\}. \tag{2.9}$$

Thus from (2.7), (2.8) and the problem (2.5), we have

$$\begin{aligned} &\inf_{\mathbf{q} \in K}\{\tfrac{1}{2}(\mathbf{q},\mathbf{q}) - < q_\nu, g >_{\Gamma_C}\} \\ &= \inf_{\mathbf{q} \in Q}\{\tfrac{1}{2}(\mathbf{q},\mathbf{q}) - < q_\nu, g >_{\Gamma_C} + \delta^+(q_\nu|0)\} \\ &= \inf_{\mathbf{q} \in Q} \sup_{\mu \in \Lambda} L(\mathbf{q}, \mu), \end{aligned} \tag{2.10}$$

where

$$L(\mathbf{q},\mu) = \frac{1}{2}(\mathbf{q},\mathbf{q}) + < q_\nu, \mu - g >_{\Gamma_C} . \tag{2.11}$$

It can be seen that the problem (2.10) is equivalent to the following mixed variational inequalities:

$$\begin{cases} find \quad \mathbf{p} \in Q, \lambda \in \Lambda, \quad such \quad that \\ (\mathbf{p},\mathbf{q}) + < q_\nu, \lambda - g >_{\Gamma_C} = 0 \quad \forall \quad \mathbf{q} \in Q_0, \\ < p_\nu, \mu - \lambda >_{\Gamma_C} \leq 0 \quad \forall \quad \mu \in \Lambda, \end{cases} \tag{2.12}$$

where

$$Q_0 = \{\mathbf{q} \in H_{div}(\Omega) : div\mathbf{q} = 0 \quad in \quad \Omega, q_\nu = 0 \quad on \quad \Gamma_F\}. \tag{2.13}$$

In order to relax the constraint: $div\mathbf{q} + f = 0$ in Ω in the problem (2.12) (see the expression of the set Q in (2.3)), we introduce another Lagrangian multiplier as follows: Let

$$Q_t^* = \{\mathbf{q} \in H_{div}(\Omega) : q_\nu = t \quad on \quad \Gamma_F\}, \tag{2.14}$$

$$Q_0^* = \{\mathbf{q} \in H_{div}(\Omega) : q_\nu = 0 \quad on \quad \Gamma_F\}, \tag{2.15}$$

and $\forall \ w \in L^2(\Omega)$ let

$$\delta(w|\{0\}) = \begin{cases} 0 \quad if \quad w = 0, \\ +\infty \quad otherwise \end{cases} = \sup_{v \in L^2(\Omega)} (v, w), \tag{2.16}$$

then the saddle-point problem (2.10) can be written as follows

$$\begin{aligned} &\inf_{\mathbf{q} \in Q} \sup_{\mu \in \Lambda}\{\tfrac{1}{2}(\mathbf{q},\mathbf{q}) + < q_\nu, \mu - g >_{\Gamma_C}\} \\ &= \inf_{\mathbf{q} \in Q_t^*} \sup_{\mu \in \Lambda}\{\tfrac{1}{2}(\mathbf{q},\mathbf{q}) + < q_\nu, \mu - g >_{\Gamma_C} + \delta(div\mathbf{q} + f|\{0\})\} \\ &= \inf_{\mathbf{q} \in Q_t^*} \sup_{[\mu, v] \in \Lambda \times L^2(\Omega)} \tilde{L}(\mathbf{q}; [\mu, v]), \end{aligned} \tag{2.17}$$

where

$$\tilde{L}(\mathbf{q};[\mu,v]) = \frac{1}{2}(\mathbf{q},\mathbf{q}) + (div\mathbf{q} + f, v) + < q_\nu, \mu - g >_{\Gamma_C} . \tag{2.18}$$

It can be seen that the problem (2.17) is equivalent to the following dual mixed problem:

$$\begin{cases} find \quad \mathbf{p} \in Q_t^*, [\lambda, u] \in \Lambda \times L^2(\Omega), \quad such \ that \\ (\mathbf{p},\mathbf{q}) + (div\mathbf{q}, u) + < q_\nu, \lambda - g >_{\Gamma_C} = 0 \quad \forall \quad \mathbf{q} \in Q_0^*, \\ (div\mathbf{p} + f, v - u) + < p_\nu, \mu - \lambda >_{\Gamma_C} \leq 0 \quad \forall \quad v \in L^2(\Omega), \mu \in \Lambda, \end{cases} \tag{2.19}$$

and the second inequality in (2.19)is equivalent to the following relations:

$$(div\mathbf{p} + f, v) = 0 \quad \forall \quad v \in L^2(\Omega), \tag{2.20}$$

and

$$< p_\nu, \mu - \lambda >_{\Gamma_C} \leq 0 \quad \forall \quad \mu \in \Lambda. \tag{2.21}$$

Thus the problem (2.19) can be written as follows:

$$\begin{cases} find \quad \mathbf{p} \in Q_t^*, [\lambda, u] \in \Lambda \times L^2(\Omega), \quad such \ that \\ (\mathbf{p}, \mathbf{q}) + (div\mathbf{p}, div\mathbf{q}) + (div\mathbf{q}, u + f) + < q_\nu, \lambda - g >_{\Gamma_C} = 0 \\ \qquad \forall \quad \mathbf{q} \in Q_0^*, \\ (div\mathbf{p} + f, v - u) + < p_\nu, \mu - \lambda >_{\Gamma_C} \leq 0 \quad \forall \quad [\mu, v] \in \Lambda \times L^2(\Omega). \end{cases} \tag{2.22}$$

Let

$$\begin{cases} a(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q}) + (div\mathbf{p}, div\mathbf{q}), \\ b(\mathbf{q}; [\mu, v]) = (div\mathbf{q}, v) + < q_\nu, \mu >_{\Gamma_C}, \end{cases} \tag{2.23}$$

then the problem (2.22) can be written in the abstract formulation:

$$\begin{cases} find \quad \mathbf{p} \in Q_t^*, [\lambda, u] \in \Lambda \times L^2(\Omega), \quad such \ that \\ a(\mathbf{p}, \mathbf{q}) + b(\mathbf{q}; [\lambda, u]) = -(f, div\mathbf{q}) + < q_\nu, g >_{\Gamma_C} \quad \forall \quad \mathbf{q} \in Q_0^*, \\ b(\mathbf{p}; [\mu, v] - [\lambda, u]) \leq -(f, v - u) \quad \forall \quad [\mu, v] \in \Lambda \times L^2(\Omega). \end{cases} \tag{2.24}$$

Remark 2.1. Here we present the reason why it is not needed to introduce another Lagrangian multiplier leraxing the constraint: $div\mathbf{p} - u + f = 0$ in Ω for the unilateral problem in [2]. The problem in [2] is: Given $f \in L^2(\Omega), g \in H^1(\Omega)$, find $u \in \tilde{C}$ such that

$$(\nabla u, \nabla(v - u)) + (u, v - u) \geq (f, v - u) \quad \forall \quad v \in \tilde{C}, \tag{2.25}$$

where

$$\tilde{C} = \{v \in H^1(\Omega) : v \geq g \quad a.e \quad on \quad \Gamma\}. \tag{2.26}$$

It can be easily seen that the problem (2.25)is equivalent to the following problem

$$\begin{cases} find \quad u \in \tilde{C}, \quad such \quad that \\ J(u) = \min J(v) \quad \forall \quad v \in \tilde{C}, \end{cases} \tag{2.27}$$

where

$$J(v) = \frac{1}{2}\{(\nabla v, \nabla v) + (v, v)\} - (f, v). \tag{2.28}$$

Noting the following inequalities: $\forall \quad v \in \tilde{C}$ and $\mathbf{q} \in \tilde{K} = \{\mathbf{q} \in H_{div}(\Omega) : q_\nu \geq 0 \quad on \quad \Gamma\}$,

$$\begin{aligned} &\tfrac{1}{2}(\nabla v, \nabla v) \geq -\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) + (\nabla v, \mathbf{q}) \\ &\geq -\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) - (div\mathbf{q}, v) + < q_\nu, v >_\Gamma \\ &\geq -\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) - (div\mathbf{q}, v) + < q_\nu, g >_\Gamma, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\tfrac{1}{2}(v, v) - (f, v) \geq -\tfrac{1}{2}(div\mathbf{q} + f, div\mathbf{q} + f) + (div\mathbf{q}, v) \\ &= -\tfrac{1}{2}(div\mathbf{q}, div\mathbf{q}) - (div\mathbf{q}, f) + (div\mathbf{q}, v) - \tfrac{1}{2}(f, f). \end{aligned} \tag{2.30}$$

From (2.28)–(2.30), we have $\forall \quad v \in \tilde{C}, \mathbf{q} \in \tilde{K}$,

$$J(v) \geq -\frac{1}{2}\{(\mathbf{q}, \mathbf{q}) + (div\mathbf{q}, div\mathbf{q})\} - (f, div\mathbf{q}) + < g, q_\nu >_\Gamma -\frac{1}{2}(f, f), \tag{2.31}$$

from which it can be seen that $\forall \quad v \in \tilde{C}$

$$J(v) = \sup_{\mathbf{q} \in \tilde{K}} \{-\frac{1}{2}[\mathbf{q}, \mathbf{q}] - (f, div\mathbf{q}) + < g, q_\nu >_\Gamma\} - \frac{1}{2}(f, f), \tag{2.32}$$

where

$$[\mathbf{p}, \mathbf{q}] = (\mathbf{p}, \mathbf{q}) + (div\mathbf{p}, div\mathbf{q}). \tag{2.33}$$

Then the problem (2.27) can be rewritten as follows:

$$\begin{aligned} &\inf_{v\in\tilde{C}}\sup_{\mathbf{q}\in\tilde{K}}\{-\tfrac{1}{2}[\mathbf{q},\mathbf{q}]-(f,div\mathbf{q})+<g,q_\nu>_\Gamma\}-\tfrac{1}{2}(f,f)\\ &=-\inf_{\mathbf{q}\in\tilde{K}}\{\tfrac{1}{2}[\mathbf{q},\mathbf{q}]+(f,div\mathbf{q})-<g,q_\nu>_\Gamma\}-\tfrac{1}{2}(f,f),\end{aligned} \tag{2.34}$$

which is equivalent to the problem (see [2]): Find $\mathbf{p}\in\tilde{K}$ such that

$$[\mathbf{p},\mathbf{q}-\mathbf{p}]\geq<g,q_\nu-p_\nu>_\Gamma-(f,div(\mathbf{q}-\mathbf{p}))\quad\forall\mathbf{q}\in\tilde{K}. \tag{2.35}$$

Due to the inequality (2.30), the constraint: $div\mathbf{p}-u+f=0$ in Ω is absent from $\tilde{K}$ in the problem (2.35), while the constraint: $div\mathbf{p}+f=0$ in Ω appears in K in the problem (2.6) without the lower order term.

3. Existence and Uniqueness

In this section, we consider the existence and uniqueness of the solution of the dual mixed problem (2.24). It can be seen that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on $H_{div}(\Omega)$, and then by Theorem 2.1 in [2], in order to ensure the existence and uniqueness of the solution of the problem (2.24), it is sufficient to prove the following lemma.

Lemma 3.1. *There exists a constant $\beta\geq 0$, such that*

$$\sup_{\mathbf{q}\in Q_0^*}\frac{b(\mathbf{q};[\mu,v])}{\|\mathbf{q}\|_{div}}\geq\beta(\|\mu\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_C)}+\|v\|^2_{0,\Omega})^{\frac{1}{2}}\quad\forall\quad[\mu,v]\in\Lambda\times L^2(\Omega). \tag{3.1}$$

Proof. For any given $\mu\in\Lambda, v\in L^2(\Omega)$, there exists a solution $w\in H^2(\Omega)$, such that

$$\begin{cases}\Delta w=v\quad in\quad\Omega,\\ w=0\quad on\quad\Gamma_D,\partial_\nu w=0\quad on\quad\Gamma_F\quad and\quad w=\mu\quad on\quad\Gamma_C.\end{cases} \tag{3.2}$$

Let $\mathbf{q}=\nabla w$, then

$$b(\mathbf{q};[\mu,v])=(div\mathbf{q},v)+<q_\nu,\mu>_{\Gamma_C}=\|v\|^2_{0,\Omega}+<q_\nu,\mu>_{\Gamma_C}. \tag{3.3}$$

We now want to prove, on the one hand, that there exists $\beta=$ const. >0, such that

$$\|v\|^2_{0,\Omega}+<q_\nu,\mu>_{\Gamma_C}\geq\beta\{\|v\|^2_{0,\Omega}+\|\mu\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_C)}\}. \tag{3.4}$$

To do this, by the definition of $H_{00}^{\frac{1}{2}}(\Gamma_C)$-norm and Poincare inequality (see the Appendix),

$$\|\mu\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_C)}\leq C\inf_{\zeta\in H^1(\Omega),\zeta|_{\Gamma_C}=\mu}\|\zeta\|^2_{1,\Omega}\leq C\|w\|^2_{1,\Omega}\leq C|w|^2_{1,\Omega}. \tag{3.5}$$

By Green's formula,

$$|w|^2_{1,\Omega}=<q_\nu,\mu>_{\Gamma_C}-(v,w)\leq<q_\nu,\mu>_{\Gamma_C}+C\|v\|_{0,\Omega}|w|_{1,\Omega},$$

from which it can be seen that

$$|w|^2_{1,\Omega}\leq C\{\|v\|^2_{0,\Omega}+<q_\nu,\mu>_{\Gamma_C}\}, \tag{3.6}$$

thus the inequality (3.4)is proved. On the other hand, we want to prove that

$$\|\mathbf{q}\|^2_{div}\leq C\{\|v\|^2_{0,\Omega}+\|\mu\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_C)}\}. \tag{3.7}$$

In fact, we have

$$<q_\nu,\mu>_{\Gamma_C}\leq\|\partial_\nu w\|_{-\frac{1}{2},\Gamma_C}\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)}\leq(|w|^2_{1,\Omega}+\|v\|^2_{0,\Omega})^{\frac{1}{2}}\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)},$$

from which and (3.6), it can be seen that

$$|w|^2_{1,\Omega}\leq C\{\|v\|^2_{0,\Omega}+\|\mu\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_C)}\}. \tag{3.8}$$

Thus

$$\|\mathbf{q}\|_{div}^2 = \|\mathbf{q}\|_{0,\Omega}^2 + \|div\mathbf{q}\|_{0,\Omega}^2 = |w|_{1,\Omega}^2 + \|v\|_{0,\Omega}^2,$$

from which and (3.8), the inequality (3.7) is proved. Thus from (3.3), (3.4) and (3.7) the proof of the lemma is completed.

From the Lemma 3.1, we have

Theorem 3.1. *Assume that* $f \in L^2(\Omega), t \in L^2(\Gamma_F)$ *and* $g \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, *then there exists unique solution* $(\mathbf{p}, [\lambda, u])$ *of the problem (2.24) (or (2.22), or (2.19)).*

And we also have

Theorem 3.2. *Assume that* $f \in L^2(\Omega), t \in L^2(\Gamma_F), g \in H_{00}^{\frac{1}{2}}(\Gamma_C)$ *and* u *is the solution of the problem (1.1), then* $(\mathbf{p}, [\lambda, u]) = (\nabla u, [g-u, u])$ *is the solution of the problem (2.19).*

Proof. (i) Since u is the solution of the problem (1.1), and taking account of (1.6), then $\forall\ \mathbf{q} \in Q_0^*$, by Green's formula, we have

$$\begin{aligned}&(\nabla u, \mathbf{q}) + (div\mathbf{q}, u) + < q_\nu, -u >_{\Gamma_C}\\ &= \int_{\Gamma_C} q_\nu u ds - < q_\nu, u >_{\Gamma_C} = 0.\end{aligned}$$

(ii) In order to prove the second inequality of (2.19), it is sufficient to prove the relations (2.20) and (2.21). In fact, by the first equality of (1.6), it can be seen that the equality (2.20) holds. And by the relation on Γ_C in (1.6), it can be seen that $\forall \mu \in \Lambda$

$$< \partial_\nu u, \mu + (u-g) >_{\Gamma_C} = < \partial_\nu u, \mu >_{\Gamma_C} \le 0.$$

Thus the proof is completed.

Theorem 3.3. *Let* $(\mathbf{p}; [\lambda, u])$ *be the solution of the problem (2.23)–(2.24), then* $\mathbf{p}$ *is the solution of the dual problem (2.6).*

Proof. Firstly, by the lerations (2.20), (2.21), it can be seen that $\mathbf{p} \in K$. Next $\forall \mathbf{q} \in K$, then $\mathbf{q} - \mathbf{p} \in Q_0^*$, and noting (2.21), we have

$$\begin{aligned}&(\mathbf{p}, \mathbf{q}-\mathbf{p}) - < g, q_\nu - p_\nu >_{\Gamma_C} = a(\mathbf{p}, \mathbf{q}-\mathbf{p}) - < g, q_\nu - p_\nu >_{\Gamma_C}\\ &= b(\mathbf{p}-\mathbf{q}; [\lambda, u]) = (div(\mathbf{p}-\mathbf{q}), u) + < p_\nu - q_\nu, \lambda >_{\Gamma_C}\\ &= < p_\nu - q_\nu, \lambda >_{\Gamma_C} \ge 0.\end{aligned} \tag{3.9}$$

4. Finite Element Approximation

We now consider the mixed finite element approximation of the problem (2.23)–(2.24). Let $\mathcal{T}_h$ be the quasi uniformly triangulation of the convex polygonal domain Ω. Let Q_h^* be Raviart–Thomas space ($k \ge 0$) (see [3]) associated with $\mathcal{T}_h$:

$$Q_h^* = \{\mathbf{q}_h \in H_{div}(\Omega) : \mathbf{q}_h|_\tau \in RT_k(\tau)\ \ \forall\ \ \tau \in \mathcal{T}_h\}, \tag{4.1}$$

and

$$Q_{th}^* = \{\mathbf{q}_h \in Q_h^* : \int_e q_{h\nu}\psi_h ds = \int_e t\psi_h ds\ \ \forall\ \ \psi_h \in P_k(e),\ \ e \subset \Gamma_F\}, \tag{4.2}$$

and

$$Q_{0h}^* = \{\mathbf{q}_h \in Q_h^* : \int_e q_{h\nu}\psi_h ds = 0\ \ \forall\ \ \psi_h \in P_k(e),\ \ e \subset \Gamma_F\}. \tag{4.3}$$

Let $M_h \subset L^2(\Omega)$ be the space of piecewise polynomials of degree k on Ω, and let Λ_h be the nonpositive piecewise polynomials of degree k on Γ_C (not necessarily continuous):

$$\Lambda_h = \{\mu_h \in L^2(\Gamma_C) : \mu_h|_e \in P_k(e)\ \forall\ \ e \subset \Gamma_C,\ \textit{and}\ \mu_h \le 0\ \textit{on}\ \Gamma_C\}. \tag{4.4}$$

Then the finite element approximation of the problem (2.23)–(2.24) is as follows:

$$\begin{cases} find\ \mathbf{p}_h \in Q^*_{th}, [\lambda_h, u_h] \in \Lambda_h \times M_h, \quad such\ that \\ a(\mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h; [\lambda_h, u_h]) = -(f, div\mathbf{q}_h) + < q_{h\nu}, g >_{\Gamma_C} \\ \qquad \forall\ \ \mathbf{q}_h \in Q^*_{0h}, \\ b(\mathbf{p}_h; [\mu_h, v_h] - [\lambda_h, u_h]) \leq -(f, v_h - u_h) \ \ \forall\ \ [\mu_h, v_h] \in \Lambda_h \times M_h, \end{cases} \tag{4.5}$$

where the bilinear $a(\cdot,\cdot)$ and $b(\cdot;[\cdot,\cdot])$ have been defined as (2.23).

We now introduce the finite element approximation of the problem (2.6) as follows:

$$\begin{cases} find\ \ \mathbf{p}_h \in K_h, \quad such\ that \\ a(\mathbf{p}_h, \mathbf{q}_h - \mathbf{p}_h) \geq < g, q_{h\nu} - p_{h\nu} >_{\Gamma_C} \quad \forall\ \ \mathbf{q}_h \in K_h, \end{cases} \tag{4.6}$$

where

$$\begin{aligned} K_h = \{\mathbf{q}_h \in Q^*_{th} : (div\mathbf{q}_h + f, v_h) = 0 \ \ \forall\ \ v_h \in M_h, \\ and \quad < q_{h\nu}, \mu_h >_{\Gamma_C} \leq 0 \ \ \forall\ \ \mu_h \in \Lambda_h\}. \end{aligned} \tag{4.7}$$

We have

Theorem 4.1. *Let $(\mathbf{p}_h; [\lambda_h, u_h])$ be the solution of the problem(4.5), then $\mathbf{p}_h$ satisfies the problem (4.6).*

Proof. Firstly, from the second inequality in (4.5), it can be seen that

$$\begin{cases} (div\mathbf{p}_h + f, v_h) = 0 \ \ \forall\ \ v_h \in M_h, \\ < p_{h\nu}, \mu_h >_{\Gamma_C} \leq 0 \ \ \forall\ \ \mu_h \in \Lambda_h, \\ < p_{h\nu}, \lambda_h >_{\Gamma_C} = 0, \end{cases}$$

which means that $\mathbf{p}_h \in K_h$. Next, for any given $\mathbf{q}_h \in K_h$, then $\mathbf{q}_h - \mathbf{p}_h \in Q^*_{0h}$, and from the first equation of (4.5) and (2.23), we have

$$\begin{aligned} & a(\mathbf{p}_h, \mathbf{q}_h - \mathbf{p}_h) - < g, q_{h\nu} - p_{h\nu} >_{\Gamma_C} \\ & = b(\mathbf{p}_h - \mathbf{q}_h; [\lambda_h, u_h]) + (f, div(\mathbf{p}_h - \mathbf{q}_h)) \\ & = (div(\mathbf{p}_h - \mathbf{q}_h), u_h) + < p_{h\nu} - q_{h\nu}, \lambda_h >_{\Gamma_C} + (f, div(\mathbf{p}_h - \mathbf{q}_h)) \\ & = < p_{h\nu} - q_{h\nu}, \lambda_h >_{\Gamma_C} = - < q_{h\nu}, \lambda_h > \geq 0, \end{aligned}$$

here we have used that $(div(\mathbf{p}_h - \mathbf{q}_h), u_h) = 0$ since $\mathbf{p}_h, \mathbf{q}_h \in K_h$, and also used that $(f, div(\mathbf{p}_h - \mathbf{q}_h)) = (div\mathbf{p}_h + f, div(\mathbf{p}_h - \mathbf{q}_h)) - (div\mathbf{p}_h, div(\mathbf{p}_h - \mathbf{q}_h)) = 0$ since $\mathbf{p}_h, \mathbf{q}_h \in K_h$ and $div\mathbf{p}_h, div\mathbf{q}_h \in M_h$, and $< p_{h\nu}, \lambda_h >_{\Gamma_C} = 0$. Thus the proof is completed.

In order to insure the existence of the unique solution of the finite element approximation (4.5) and establish the error estimate, we need the following lemma and the discrete inf-sup condition.

Lemma 4.1 (see [3]). *Let the interpolation operator $\Pi_h : H_{div}(\Omega) \to Q^*_h$ be defined as follows: For any given $\mathbf{q} \in H_{div}(\Omega), \Pi_h\mathbf{q} \in Q^*_h, \Pi_\tau\mathbf{q} = \Pi_h\mathbf{q}|_\tau$, such that*

$$\begin{cases} \int_\tau (\mathbf{q} - \Pi_\tau\mathbf{q})\mathbf{p}_{k-1} dx = 0 \ \ \forall\ \ \mathbf{p}_{k-1} \in (P_{k-1}(\tau))^2, \\ \int_e (q_\nu - (\Pi_\tau\mathbf{q})_\nu) p_{e,k} ds = 0 \ \ \forall\ \ p_{e,k} \in P_k(e), e \subset \partial\tau, \end{cases} \tag{4.8}$$

with $k \geq 0 (\mathbf{p}_{k-1} = 0$ for $k = 0)$, then there exists a constant C depending only on k and on the shape of τ, such that

$$\begin{cases} \|\mathbf{q} - \Pi_h\mathbf{q}\|_{0,\Omega} \leq Ch^m |\mathbf{q}|_{m,\Omega}, \\ \|div(\mathbf{q} - \Pi_h\mathbf{q})\|_{0,\Omega} \leq Ch^m |div\mathbf{q}|_{m,\Omega} \end{cases} \tag{4.9}$$

with $1 \leq m \leq k+1$, and

$$\|\Pi_h\mathbf{q}\|_{div} \leq C\|\mathbf{q}\|_{div}. \tag{4.10}$$

Theorem 4.2. *Let Q^*_h be Raviart-Thomas space with $k \geq 0$, then there exists a constant $\beta' \geq 0$, such that*

$$\begin{aligned} \sup_{\mathbf{q}_h \in Q^*_{0h}} \frac{b(\mathbf{q}_h; [\mu_h, v_h])}{\|\mathbf{q}_h\|_{div}} \geq \beta' (\|\mu_h\|^2_{0,\Gamma_C} + \|v_h\|^2_{0,\Omega})^{\frac{1}{2}} \\ \forall\ \ [\mu_h, v_h] \in \Lambda_h \times M_h. \end{aligned} \tag{4.11}$$

Proof. For any given $\mu_h \in \Lambda_h$, and $v_h \in M_h$, let α satisfy the equations:

$$\Delta\alpha = v_h \ in \ \Omega, \qquad \partial_\nu \alpha = \mu_h \ on \ \Gamma_C, \qquad \alpha = 0 \ on \ \partial\Omega \setminus \Gamma_C. \tag{4.12}$$

Let $\mathbf{q} = \nabla\alpha$ and $\mathbf{q}_h = \Pi_h \mathbf{q}$, then $\mathbf{q}_h \in Q_{0h}^*$ since the second equality of (4.8). Thus from (4.8), we have

$$\begin{aligned} b(\Pi_h \mathbf{q}; [\mu_h, v_h]) &= (div \Pi_h \mathbf{q}, v_h) + < \Pi_h \mathbf{q} \cdot \nu, \mu_h >_{\Gamma_C} \\ &= (div \mathbf{q}, v_h) + < \mathbf{q} \cdot \nu, \mu_h >_{\Gamma_C} \\ &= \|v_h\|_{0,\Omega}^2 + \|\mu_h\|_{0,\Gamma_C}^2. \end{aligned} \tag{4.13}$$

From (4.12) we have

$$\int_\Omega \Delta\alpha \cdot \alpha dx = \int_\Omega \alpha \cdot v_h dx,$$

from which, and by Green's formula, we have

$$\begin{aligned} &\|\nabla\alpha\|_{0,\Omega}^2 = \int_{\Gamma_C} \mu_h \cdot \alpha ds - \int_\Omega v_h \cdot \alpha dx \\ &\le \|\mu_h\|_{0,\Gamma_C} \|\alpha\|_{0,\Gamma_C} + \|\alpha\|_{0,\Omega} \|v_h\|_{0,\Omega} \le C\|\alpha\|_{1,\Omega}(\|v_h\|_{0,\Omega} + \|\mu_h\|_{0,\Gamma}), \end{aligned}$$

Poincare inequality yields

$$\|\nabla\alpha\|_{0,\Omega} \le C(\|v_h\|_{0,\Omega} + \|\mu_h\|_{0,\Gamma_C}). \tag{4.14}$$

By (4.10) and (4.14)

$$\begin{aligned} &\|\Pi_h \mathbf{q}\|_{div}^2 \le C\|\mathbf{q}\|_{div}^2 = C(\|\mathbf{q}\|_{0,\Omega}^2 + \|div \mathbf{q}\|_{0,\Omega}^2) \\ &= C(\|\nabla\alpha\|_{0,\Omega}^2 + \|\Delta\alpha\|_{0,\Omega}^2) \le C(|\alpha|_{1,\Omega}^2 + \|v_h\|_{0,\Omega}^2) \\ &\le C(\|v_h\|_{0,\Omega}^2 + \|\mu_h\|_{0,\Gamma_h}^2). \end{aligned} \tag{4.15}$$

From inequalities (4.13) and (4.15), the proof is completed.

We now turn to consider the error estimate.

Lemma 4.2 (see [2]). *Let* $(\mathbf{p}; [\lambda, u])$, *and* $(\mathbf{p}_h; [\lambda_h, u_h])$ *be the solutions of the problems (2.24) and (4.5) respectively, then* $\forall \ \ \mathbf{q}_h \in K_h$, *the following abstract error estimate holds*

$$a(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{p}_h) \le a(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{q}_h) + b(\mathbf{p}_h - \mathbf{q}_h; [\lambda, u]). \tag{4.16}$$

Proof. By the Theorem 4.1, $\forall \ \ \mathbf{q}_h \in K_h$, we have

$$\begin{aligned} &a(\mathbf{p} - \mathbf{p}_h, \mathbf{q}_h - \mathbf{p}_h) = a(\mathbf{p}, \mathbf{q}_h - \mathbf{p}_h) - a(\mathbf{p}_h, \mathbf{q}_h - \mathbf{p}_h) \\ &\le a(\mathbf{p}, \mathbf{q}_h - \mathbf{p}_h) - < g, q_{h\nu} - p_{h\nu} >_{\Gamma_C} \\ &= b(\mathbf{p}_h - \mathbf{q}_h; [\lambda, u]) + (f, div(\mathbf{p}_h - \mathbf{q}_h)) \\ &= b(\mathbf{p}_h - \mathbf{q}_h; [\lambda, u]), \end{aligned}$$

in which the last second equality holds since $\mathbf{p}_h - \mathbf{q}_h \in Q_{0h}^* \subset Q_0^*$, and the last equality holds since $\mathbf{p}_h, \mathbf{q}_h \in K_h$ and then $(f, div(\mathbf{p}_h - \mathbf{q}_h)) = 0$ (by the arguing in the proof of Theorem 4.1). Thus

$$\begin{aligned} &a(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{p}_h) = a(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{q}_h) + a(\mathbf{p} - \mathbf{p}_h, \mathbf{q}_h - \mathbf{p}_h) \\ &\le a(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{q}_h) + b(\mathbf{p}_h - \mathbf{q}_h; [\lambda, u]). \end{aligned}$$

The proof is completed.

Firstly, we present the following error estimate of the lowest order approximation (i.e., $k = 0$) as follows.

Theorem 4.3. *Let* $\mathcal{T}_h$ *be the quasi uniformly triangulation of the convex polygonal domain* Ω, Q_h^* *be Raviart-Thomas element space with* $k = 0$,*with respect to* $\mathcal{T}_h$, *and* $(\mathbf{p}; [\lambda, u])$ *and*

$(\mathbf{p}_h;[\lambda_h,u_h])$ be the solutions of the problems (2.24) and (4.5) respectively, with the regularities: $div\mathbf{p}\in H^1(\Omega)$ and $u\in H^2(\Omega)(i.e., \mathbf{p}\in H^1(\Omega),\lambda\in H^{\frac{3}{2}}(\Gamma_C))$, then the following error estimate holds:

$$\|\mathbf{p}-\mathbf{p}_h\|_{div}\le Ch^{\frac{3}{4}}(\|\mathbf{p}\|_{1,\Omega}+|div\mathbf{p}|_{1,\Omega}+|u|_{1,\Omega}+\|\lambda\|_{1,\Gamma_C}), \tag{4.17}$$

with $C=Const.>0$ independent of h.

Proof. By the definition (2.23) and let $\mathbf{q}_h=\Pi_h\mathbf{p}\in K_h$ in Lemma 4.2, then

$$b(\mathbf{p}_h-\mathbf{q}_h;[\lambda,u])=(div(\mathbf{p}_h-\Pi_h\mathbf{p}),u)+<p_{h\nu}-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}. \tag{4.18}$$

Firstly we estimate the first term on the right hand side of (4.18). Let

$$P_\tau v=\frac{1}{|\tau|}\int_\tau vdx,\quad Pv|_\tau=P_\tau v,\quad\forall\ \ v\in L^2(\Omega),$$

then $P:L^2(\Omega)\longrightarrow M_h$ is the L^2- interpolation operator. Thus

$$\begin{aligned}&(div(\mathbf{p}_h-\Pi_h\mathbf{p}),u)=(div(\mathbf{p}_h-\mathbf{p}),u)+(div(\mathbf{p}-\Pi_h\mathbf{p}),u)\\&=(div(\mathbf{p}_h-\mathbf{p}),u-Pu)+(div(\mathbf{p}_h-\mathbf{p}),Pu)\\&+(div(\mathbf{p}-\Pi_h\mathbf{p}),u-Pu)+(div(\mathbf{p}-\Pi_h\mathbf{p}),Pu)\\&=(div(\mathbf{p}_h-\mathbf{p}),u-Pu)+(div(\mathbf{p}-\Pi_h\mathbf{p}),u-Pu),\end{aligned}$$

here we have used $(div(\mathbf{p}_h-\mathbf{p}),Pu)=(div\mathbf{p}_h+f,Pu)-(div\mathbf{p}+f,Pu)=0$ since $\mathbf{p}_h\in K_h$ and $\mathbf{p}\in K$, and $(div(\mathbf{p}-\Pi_h\mathbf{p}),Pu)=0$ since the Lemma 4.1. Then by the interpolation error estimate and (4.9), we have

$$\begin{aligned}&div(\mathbf{p}_h-\Pi_h\mathbf{p},u)\\&\le\|div(\mathbf{p}-\mathbf{p}_h)\|_{0,\Omega}\|u-Pu\|_{0,\Omega}+\|div(\mathbf{p}-\Pi_h\mathbf{p})\|_{0,\Omega}\|u-Pu\|_{0,\Omega}\\&\le Ch\|div(\mathbf{p}-\mathbf{p}_h)\|_{0,\Omega}|u|_{1,\Omega}+Ch^2|div\mathbf{p}|_{1,\Omega}|u|_{1,\Omega}.\end{aligned} \tag{4.19}$$

Next we estimate the second term on the right hand side of (4.18). Let $\bar\lambda\in\Lambda_h$ be the L^2- projection of λ, then

$$<p_{h\nu}-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}=<p_{h\nu}-p_\nu,\lambda>_{\Gamma_C}+<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}. \tag{4.20}$$

And by Lemma 4.1, we have

$$\begin{aligned}&<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}\\&=<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda-\bar\lambda>_{\Gamma_C}+<p_\nu-(\Pi_h\mathbf{p})_\nu,\bar\lambda>_{\Gamma_C}\\&=<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda-\bar\lambda>_{\Gamma_C}\le\|p_\nu-(\Pi_h\mathbf{p})_\nu\|_{0,\Gamma_C}\|\lambda-\bar\lambda\|_{0,\Gamma_C}\\&\le Ch\|p_\nu-(\Pi_h\mathbf{p})_\nu\|_{0,\Gamma_C}\|\lambda\|_{1,\Gamma_C}.\end{aligned} \tag{4.21}$$

By interpolation error estimates (see [4], [9]),it can be seen that

$$\begin{aligned}&\|p_\nu-(\Pi_h\mathbf{P})_\nu\|^2_{0,\Gamma_C}=\textstyle\sum_{e\subset\Gamma_C}\|p_\nu-(\Pi_h\mathbf{p})_\nu\|^2_{0,e}\\&\le C\textstyle\sum_{e\subset\Gamma_C,e\subset\partial\tau}(h^{-1}\|\mathbf{p}-\Pi_h\mathbf{p}\|^2_{0,\tau}+h|\mathbf{p}-\Pi_h\mathbf{p}|^2_{1,\tau})\le Ch|\mathbf{p}|^2_{1,\Omega},\end{aligned} \tag{4.22}$$

from which and (4.21), we have

$$<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}\le Ch^{\frac{3}{2}}|\mathbf{p}|_{1,\Omega}\|\lambda\|_{1,\Gamma_C}. \tag{4.23}$$

As to the first term on the right hand side of (4.20), since $\mathbf{p}_h\in K_h,<p_{h\nu},\mu_h>_{\Gamma_C}\le0\,\forall\,\mu_h\in\Lambda_h$, and $p_{h\nu}\in P_0(e)\,\forall\,e\subset\Gamma_C$, then $p_{h\nu}\ge0$ on Γ_C, and from that $\lambda\le0$ on Γ_C, and $<p_\nu,\lambda>_{\Gamma_C}=0$, we have

$$<p_{h,\nu}-p_\nu,\lambda>_{\Gamma_C}=<p_{h\nu},\lambda>_{\Gamma_C}\le0. \tag{4.24}$$

From (4.18), (4.19)–(4.21), (4.23) and (4.24), the proof is completed.

Lemma 4.3 (see [2]). *Let $(\mathbf{p},[\lambda,u])$ and $(\mathbf{p}_h,[\lambda_h,u_h])$ be the solutions of the problems (2.24) and (4.5) respectively, then the following error estimate holds*

$$\begin{aligned}&\|\lambda-\lambda_h\|_{0,\Gamma_C}+\|u-u_h\|_{0,\Omega}\\&\le C\{\|\mathbf{p}-\mathbf{p}_h\|_{div}+\inf_{\mu_h\in\Lambda_h}\|\lambda-\mu_h\|_{0,\Gamma_C}+\inf_{v_h\in M_h}\|u-v_h\|_{0,\Omega}\},\end{aligned} \tag{4.25}$$

where $C = const. > 0$ independent of h.

By the Lemma 4.3 and Theorem 4.3, we have the following error estimate

Theorem 4.4. *Under the assumptions of Theorem 4.3, the following error estimate holds*

$$\begin{aligned}&\|\lambda-\lambda_h\|_{0,\Gamma_C}+\|u-u_h\|_{0,\Omega}\\&\le Ch^{\frac{3}{4}}\{|\mathbf{p}|_{1,\Omega}+|div\mathbf{p}|_{1,\Omega}+|u|_{1,\Omega}+\|\lambda\|_{1,\Gamma_C}\}.\end{aligned} \tag{4.26}$$

Remark 4.1. The optimal error bound $O(h)$ has been obtained for the lowest order approximation of the mixed finite element method of the unilateral problem by Brezzi et al., [2] under the regularity: $u \in H^1(\Omega)\cap W^{1,\infty}(\Omega)$, and the assumption for the free boundary:

$$\begin{aligned}&\textit{the number of points in the free boundary set}\\&\textit{where the constraint } u\ge g \textit{ changes from binding to}\\&\textit{nonbinding is finite.}\end{aligned} \tag{4.27}$$

Next we present the error estimate of the mixed finite element approximation (4.5) for $k=1$, under the regularity: $u\in H^3(\Omega)$, and without the assumption (4.27).

Theorem 4.5. *Let $\mathcal{T}_h$ be the quasi uniformly triangulation of the convex polygonal domain Ω, Q_h^* be Raviart-Thomas element space for $k=1$, and M_h and Λ_h be defined as before for $k=1$, with respect to $\mathcal{T}_h$. Let $(\mathbf{p};[\lambda,u])$ and $(\mathbf{p}_h;[\lambda_h,u_h])$ be solutions of problems (2.24) and (4.5) respectively, with regularity: $u\in H^3(\Omega)$, then the following error estimate holds:*

$$\begin{aligned}&\|\mathbf{p}-\mathbf{p}_h\|_{div}+\|\lambda-\lambda_h\|_{0,\Gamma_C}+\|u-u_h\|_{0,\Omega}\\&\le Ch^{\frac{3}{2}}(|\mathbf{p}|_{2,\Omega}+\|\lambda\|_{\frac{3}{2},\Gamma_C}).\end{aligned} \tag{4.28}$$

Proof. It is sufficient to modify the proof of Theorem 4.3 as follows.

Firstly, in order to estimate the first term on the right hand side of (4.18), let $v^I\in M_h$ be the piecewise linear interpolation of $v\in H^2(\Omega)$, then

$$\begin{aligned}&(div(\mathbf{p}_h-\Pi_h\mathbf{p}),u)=(div(\mathbf{p}_h-\mathbf{p}),u-u^I)+(div(\mathbf{p}-\Pi_h\mathbf{p}_h),u-u^I)\\&\le\|div(\mathbf{p}_h-\mathbf{p})\|_{0,\Omega}\|u-u^I\|_{0,\Omega}+\|div(\mathbf{p}-\Pi_h\mathbf{p})\|_{0,\Omega}\|u-u^I\|_{0,\Omega}\\&\le Ch^2\|div(\mathbf{p}-\mathbf{p}_h)\|_{0,\Omega}|u|_{2,\Omega}+Ch^3|div\mathbf{p}|_{1,\Omega}|u|_{2,\Omega},\end{aligned} \tag{4.29}$$

here we also have used $(div(\mathbf{p}_h-\mathbf{p}),u^I)=(div\mathbf{p}_h+f,u^I)-(div\mathbf{p}+f,u^I)=0$ since $\mathbf{p}_h\in K_h$ and $\mathbf{p}\in K$, and $(div(\mathbf{p}-\Pi_h\mathbf{p}),u^I)=0$ by Lemma 4.1.

Next we estimate the second term on the right hand side of (4.18). Let $\lambda^I\in\Lambda_h$ be the piecewise linear interpolation of $\lambda\in H^{\frac{3}{2}}(\Gamma_C)$, then

$$\begin{aligned}&<p_{h\nu}-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}=<p_{h\nu}-p_\nu,\lambda>_{\Gamma_C}+<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}\\&=<p_{h\nu},\lambda>_{\Gamma_C}+<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}\qquad(since\quad<p_\nu,\lambda>_{\Gamma_C}=0)\\&\le<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda>_{\Gamma_C}\qquad(since\ \ p_{h\nu}\ge0,\lambda\le0\ \ on\ \ \Gamma_C)\\&=<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda-\lambda^I>_{\Gamma_C}+<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda^I>_{\Gamma_C}\\&=<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda-\lambda^I>_{\Gamma_C}\qquad(since\quad<p_\nu-(\Pi_h\mathbf{p})_\nu,\lambda^I>_{\Gamma_C}=0,\\&\qquad\qquad by\ Lemma\ 4.1)\\&\le\|p_\nu-(\Pi_h\mathbf{p})_\nu\|_{0,\Gamma_C}\|\lambda-\lambda^I\|_{0,\Gamma_C}\le Ch^{\frac{3}{2}}\|p_\nu-(\Pi_h\mathbf{p})_\nu\|_{0,\Gamma_C}\|\lambda\|_{\frac{3}{2},\Gamma_C}\\&\le Ch^{\frac{3}{2}}\{\textstyle\sum_{e\subset\Gamma_C,e\subset\partial\tau}(h^{-1}\|\mathbf{p}-\Pi_h\mathbf{p}\|_{0,\tau}^2+h|\mathbf{p}-\Pi_h\mathbf{p}|_{1,\tau}^2)\}^{\frac{1}{2}}\|\lambda\|_{\frac{3}{2},\Gamma_C}\\&\le Ch^3|\mathbf{p}|_{2,\Omega}\|\lambda\|_{\frac{3}{2},\Gamma_C}.\end{aligned} \tag{4.30}$$

From (4.18), (4.29) and (4.30) we have

$$\|\mathbf{p}-\mathbf{p}_h\|_{div}\le Ch^{\frac{3}{2}}(|\mathbf{p}|_{2,\Omega}+\|\lambda\|_{\frac{3}{2},\Gamma_C}). \tag{4.31}$$

Then by Lemma 4.3, the proof is completed.

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Appendix

In this appendix, we will present a result, which we have used in (3.5).

We assume that the convex bounded domain Ω in R^2 satisfies the following hypotheses: the boundary $\partial\Omega$ of Ω is smooth enough, the part $\Gamma_C \subset \partial\Omega$ of boundary with the end points P_{s_1}, P_{s_2}, and the right triangle with hypotenuse $\overline{P_{s_1}P_{s_2}}$ is contained in Ω.

We have the following

Proposition. *Let Ω satisfy the above hypotheses, then for any given $\mu \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, the norm $\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)}$ is equivalent to $\inf_{v\in H^1(\Omega),\gamma v=\mu \ on \ \Gamma_C} \|v\|_{1,\Omega}$, where $\gamma : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator.*

Proof. By the definition (see [A3]),

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)} = \{\|\mu\|_{0,\Gamma_C}^2 + \int_{\Gamma_C}\int_{\Gamma_C} (\frac{\mu(P_s) - \mu(P_{s'})}{r(P_s, P_{s'})})^2 dP_s dP_{s'}$$

$$+ \int_{\Gamma_C} \frac{\mu(P_s)^2}{r(P_s, P_{s_1})} dP_s + \int_{\Gamma_C} \frac{\mu(P_s)^2}{r(P_s, P_{s_2})} dP_s\}^{\frac{1}{2}}. \tag{A.1}$$

Thus by [A1], it can be seen that the norm $\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)}$ is equivalent to

$$\inf_{v\in H^1(\Omega),\gamma v=\tilde{\mu} \ on \ \partial\Omega} \|v\|_{1,\Omega}, \tag{A.2}$$

where $\tilde{\mu}$ is the zero extension of μ on $\partial\Omega$:

$$\tilde{\mu} = \begin{cases} \mu & on \quad \Gamma_C, \\ 0 & on \quad \partial\Omega \setminus \Gamma_C. \end{cases} \tag{A.3}$$

Thus there exists a constant $C_1 > 0$ independent of μ, such that

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)} \geq C_1 \inf_{\substack{v \in H^1(\Omega) \\ \gamma v = \mu \ on \ \Gamma_C}} \|v\|_{1,\Omega} \quad \forall \quad \mu \in H_{00}^{\frac{1}{2}}(\Gamma_C). \tag{A.4}$$

We now want to prove that there exists another constant $C_2 > 0$ independent of μ, such that

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma_C)} \leq C_2 \inf_{\substack{v \in H^1(\Omega) \\ \gamma v = \mu \ on \ \Gamma_C}} \|v\|_{1,\Omega} \quad \forall \quad \mu \in H_{00}^{\frac{1}{2}}(\Gamma_C). \tag{A.5}$$

In fact, for any given $\mu \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, and $\forall \ \ v \in H^1(\Omega), \gamma v = \mu$ on Γ_C, firstly it is obviously seen that

$$\|\mu\|_{0,\Gamma_C}^2 \leq \|\gamma v\|_{0,\partial\Omega}^2. \tag{A.6}$$

Next, by Schwarz inequality, we have (see Fig.1)

$$\begin{aligned}
&\int_{\Gamma_C} \frac{\mu(P_s)^2}{r(P_s, P_{s_1})} dP_s \\
&\leq 2\{\int_{\Gamma_C} \frac{(v(P_s) - v(R_{ss_1}))^2}{r(P_s, R_{ss_1})} dP_s + \int_{\Gamma_C} \frac{(v(P_{s_1}) - v(R_{ss_1}))^2}{r(P_{s_1}, R_{ss_1})} dP_s\} \\
&= 2\{\int_{\Gamma_C} \frac{(\int_{y_{P_{s_1}}}^{y_{P_s}} \partial_y v(x_{P_{s_1}}, y) dy)^2}{|y_{P_s} - y_{P_{s_1}}|} dP_s + \int_{\Gamma_C} \frac{(\int_{x_{P_{s_1}}}^{x_{P_s}} \partial_x v(x, y_{P_{s_1}}) dx)^2}{|x_{P_s} - x_{P_{s_1}}|} dP_s\} \\
&\leq 2\{\int_{\Gamma_C} \int_{y_{P_{s_1}}}^{y_{P_s}} (\partial_y v(x_{P_{s_1}}, y))^2 dy dP_s + \int_{\Gamma_C} \int_{x_{P_{s_1}}}^{x_{P_s}} (\partial_x v(x, y_{P_{s_1}}))^2 dx dP_s\} \\
&\leq 2|v|_{1,\Omega}^2,
\end{aligned} \tag{A.7}$$

and similarly

$$\int_{\Gamma_C} \frac{\mu(P_s)^2}{r(P_s, P_{s_1})} dP_s \le 2|v|^2_{1,\Omega}. \tag{A.8}$$

Finally we estimate the second term on the right hand side of (A.1) in the way as [A.2]. To do this, we need a classical inequality, Hardy inequality (see [A.4])

$$\int_0^A \Big(\frac{\int_0^x f(\xi)d\xi}{x}\Big)^2 dx < 4\int_0^A f^2(x)dx, \tag{A.9}$$

where $0 < A \le \infty$. By the assumption of the domain Ω, we have (see Fig.2),

$$\begin{aligned}
&\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_s)-v(P_{s'})}{r(P_s,P_{s'})}\Big)^2 dP_{s'}dP_s \\
&\le 2\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_s)-v(R_{ss'})}{r(P_s,P_{s'})}\Big)^2 dP_{s'}dP_s \\
&+2\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_{s'})-v(R_{ss'})}{r(P_s,P_{s'})}\Big)^2 dP_s dP_{s'}.
\end{aligned} \tag{A.10}$$

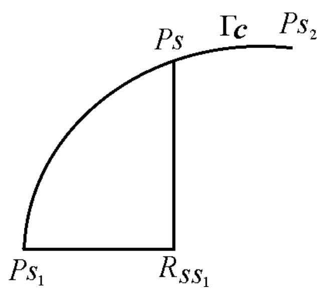

Fig.1.

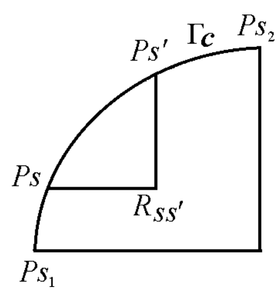

Fig.2.

We now estimate the first term on the right hand side of (A.10)

$$\begin{aligned}
&\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_s)-v(R_{ss'})}{r(P_s,P_{s'})}\Big)^2 dP_{s'}dP_s = \int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{\int_{x_{P_s}}^{x_{P_{s'}}} \partial_x v(x,y_{P_s})dx}{|x_{P_{s'}}-x_{P_s}}\Big)^2 dP_{s'}dP_s \\
&\le C\int_{\Gamma_C}\int_{x_{P_s}}^{x_{P_{s2}}} \Big(\frac{\int_{x_{P_s}}^{x_{P_{s'}}} \partial_x v(x,y_{P_s})dx}{|x_{P_{s'}}-x_{P_s}|}\Big)^2 dx_{P_{s'}}dP_s,
\end{aligned}$$

here we used $dP_{s'} \le C dx_{P_{s'}}$ on Γ_C, since $\partial\Omega$ is smooth enough. Then by Hardy inequality, we have

$$\begin{aligned}
&\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_s)-v(R_{ss'})}{r(P_s,P_{s'})}\Big)^2 dP_{s'}dP_s \\
&\le C\int_{\Gamma_C}\int_{x_{P_s}}^{x_{P_{s2}}} (\partial_x v(x,y_{P_s}))^2 dx dP_s \le C\int\int_{\Omega} (\partial_x v)^2 dxdy,
\end{aligned} \tag{A.11}$$

and similarly,

$$\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_{s'})-v(R_{ss'})}{r(P_s,P_{s'})}\Big)^2 dP_s dP_{s'} \le C\int\int_{\Omega} (\partial_y v)^2 dxdy. \tag{A.12}$$

Thus

$$\int_{\Gamma_C}\int_{\Gamma_C} \Big(\frac{v(P_s)-v(P_{s'})}{r(P_s,P_{s'})}\Big)^2 dP_s dP_{s'} \le C|v|^2_{1,\Omega}. \tag{A.13}$$

From (A.6)–(A.8) and (A.13), the inequality (A.5) is proved. Thus the proof of proposition is completed.

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