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SUPERCONVERGENCE OF LEAST-SQUARES MIXED FINITE ELEMENT FOR SECOND-ORDER ELLIPTIC PROBLEMS [*1)]

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Abstract

In this paper the least-squares mixed finite element is considered for solving second-order elliptic problems in two dimensional domains. The primary solution u and the flux $\boldsymbol{\sigma}$ are approximated using finite element spaces consisting of piecewise polynomials of degree k and r respectively. Based on interpolation operators and an auxiliary projection, superconvergent H^1-error estimates of both the primary solution approximation u_h and the flux approximation $\boldsymbol{\sigma}_h$ are obtained under the standard quasi-uniform assumption on finite element partition. The superconvergence indicates an accuracy of $O(h^{r+2})$ for the least-squares mixed finite element approximation if Raviart-Thomas or Brezzi-Douglas-Fortin-Marini elements of order r are employed with optimal error estimate of $O(h^{r+1})$.

Key words: Elliptic problem, Superconvergence, Interpolation projection, Least-squares mixed finite element.

1. Introduction

We are concerned with approximate solutions for the representative second-order elliptic boundary-value problem:

$$-\mathrm{div}(A\mathrm{grad}u) + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a open bounded domain with boundary Γ and A is a positive definite matrix of coefficients. Introducing the flux $\boldsymbol{\sigma} = -A\mathrm{grad}u$, the problem may be recast as the first order system

$$\boldsymbol{\sigma} + A\mathrm{grad}u = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\mathrm{div}\boldsymbol{\sigma} + cu = f \quad \text{in } \Omega, \tag{1.4}$$

$$u = 0 \quad \text{on } \Gamma. \tag{1.5}$$

In many applications such as reservoir simulation, second-order elliptic equations are coupled with other partial differential equations through the velocity terms. So, The mixed finite element methods are usually used. The classical mixed method for (1.3)-(1.5) is based on the stationary principle for a saddle-point problem and is subject to the inf-sup condition on the spaces for u and $\boldsymbol{\sigma}$ (see Brezzi [1]), This implies certain restrictions on the polynomial degree k and r for the element bases defining approximations u_h and $\boldsymbol{\sigma}_h$ respectively. In the least-squares mixed (LSM) approach a least-squares residual minimization is introduced for the mixed system

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(1.3)-(1.5) of u and $\boldsymbol{\sigma}$. The finite element approximation yields a symmetric discrete system for the solution $u_h \in V_h$ and $\boldsymbol{\sigma} \in \boldsymbol{W}_h$, where V_h and $\boldsymbol{W}_h$ are the respective approximation subspaces which needn't to be subject to the consistency requirement. In [16-18], Pehlivanov et al. presented a least-squares mixed (LSM) finite elements method for second-order elliptic problems. It have been proved that the LSM method is not subject to the LBB condition and error estimates for various choices of approximation spaces have been obtained.

The objective of this paper is to investigate superconvergence phenomena for second-order elliptic problems by using the LSM method. Such a study is important in applications to mathematical modeling of fluid flow in porous media since the modeling process requires the determination of a very accurate fluid velocity. Various superconvergence results have been established for the mixed finite element for elliptic problems [11-12, 14] and, for miscible displacement problems [2-6, 9, 13]. In the 1990s, Lin et al.[14-15] introduced a so-called interpolation postprocessing technique into the finite element mthods and obtained the globally high-accuracy approximation for solution problems. C.M.Chen and Y.Q.Huang [6] presented an element analysis methods for the high-accuracy theory of the finite element methods.

The paper is organized as follows: In Section 2 we formulate the problem and its LSM finite element approximation and the coerciveness of the bilinear form in appropriate spaces are stated. In Section 3 the interpolation operators and an auxiliary projection are defined and some identity technique results are presented. The superconvergent approximation properties are derived for the LSM method.

2. Problem Formulation and the LSM Approach

We assume that the matrix of coefficients $A = (a_{ij}(x))_{i,j=1}^2$, $x \in \overline{\Omega}$, is positive definite and the coefficients $a_{ij}(x)$ are bounded; i.e. there exist constants α_1 and α_2 such that

$$\alpha_1 \zeta^T \zeta \le \zeta^T A \zeta \le \alpha_2 \zeta^T \zeta, \tag{2.1}$$

for all vectors $\zeta \in \mathbb{R}^2$ and all $x \in \overline{\Omega}$.

The standard notations for Sobolev spaces $H^m(\Omega)$ with norm $||\cdot||_{m,\Omega}$ and seminorms $|\cdot|_{i,\Omega}$, $0 \le i \le m$, are employed throughout. as usual, $L^2(\Omega) = H^0(\Omega)$ and let $(H^m(\Omega))^2$ be the corresponding product space. Also, we shall use the spaces $H^s(\Gamma)$. Let

$$V = \{v \in H^1(\Omega): \quad v = 0 \ \text{ on } \ \Gamma\}.$$

By the Poincaré-Friedrichs inequality

$$||v||_{0,\Omega} \le C_F |v|_{1,\Omega} \quad \text{for all} \quad v \in V. \tag{2.2}$$

Let

$$c_0 = \min\left\{\inf_{x\in\Omega} c(x),\ 0\right\}. \tag{2.3}$$

We make the following assumptions with respect to the coefficients of our equation: there exist constants α_0 and c_1 such that

$$|c(x)| \le c_1 \quad \text{for all} \quad x \in \overline{\Omega}, \tag{2.4}$$

$$0 < \alpha_0 \le \alpha_1 + c_0 C_F^2, \tag{2.5}$$

where C_F is the constant from the Poincaré-Friedrichs inequality above. Hence, the coefficient c(x) may be negative provided that α_1 is sufficiently large.

Let $\boldsymbol{\tau} = (\tau_1, \tau_2)$ be a smooth vector function and $v \in H^1(\Omega)$, we denote that

$$\text{div}\boldsymbol{\tau} = \partial_1 \tau_1 + \partial_2 \tau_2, \qquad \text{grad}v = (\partial_1 v, \partial_2 v).$$

Introducing the following spaces:

$$\boldsymbol{W} = \{\boldsymbol{\tau} \in (L^2(\Omega))^2, \ \ \text{div}\boldsymbol{\tau} \in L^2(\Omega)\}, \tag{2.6}$$

with norm

$$||\boldsymbol{\tau}||^2_{H(\mathrm{div})} = ||\boldsymbol{\tau}||^2_{0,\Omega} + ||\mathrm{div}\boldsymbol{\tau}||^2_{0,\Omega}. \tag{2.7}$$

The least-squares functional $J(v, \boldsymbol{\tau})$ for the mixed system (1.3)-(1.5) is defined as in [17]

$$\begin{aligned} J(v, \boldsymbol{\tau}) &= (\mathrm{div}\boldsymbol{\tau} + cv - f, \mathrm{div}\boldsymbol{\tau} + cv - f)_{0,\Omega} \\ &\quad + (\boldsymbol{\tau} + A\mathrm{grad}v, \boldsymbol{\tau} + A\mathrm{grad}v)_{0,\Omega}, \end{aligned} \tag{2.8}$$

where $(\cdot,\cdot)_{0,\Omega}$ is the standard inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$. Then the least-squares minimization problem is: find $u \in V$, $\boldsymbol{\sigma} \in \boldsymbol{W}$ such that

$$J(u, \boldsymbol{\sigma}) = \inf_{v \in V, \boldsymbol{\tau} \in \boldsymbol{W}} J(v, \boldsymbol{\tau}).$$

The corresponding variational statement is: find $u \in V$, $\boldsymbol{\sigma} \in \boldsymbol{W}$ such that

$$a(u, \boldsymbol{\sigma}; v, \boldsymbol{\tau}) = (f, \mathrm{div}\boldsymbol{\tau} + cv) \qquad \text{for all } v \in V, \quad \boldsymbol{\tau} \in \boldsymbol{W}, \tag{2.9}$$

where

$$\begin{aligned} a(u, \boldsymbol{\sigma}; v, \boldsymbol{\tau}) &= (\mathrm{div}\boldsymbol{\sigma} + cu, \mathrm{div}\boldsymbol{\tau} + cv)_{0,\Omega} \\ &\quad + (\boldsymbol{\sigma} + A\mathrm{grad}u, \boldsymbol{\tau} + A\mathrm{grad}v)_{0,\Omega}. \end{aligned} \tag{2.10}$$

The coerciveness of the bilinear form $a(\cdot;\cdot)$ and the existence and uniqueness of problem (2.9) were proved in [16].

Theorem 2.1. *There exists a constant $C > 0$ such that for all $v \in V$, $\boldsymbol{\tau} \in \boldsymbol{W}$,*

$$C\left(||v||^2_{1,\Omega} + ||\boldsymbol{\tau}||^2_{H(div)}\right) \le a(v, \boldsymbol{\tau}; v, \boldsymbol{\tau}). \tag{2.11}$$

Theorem 2.2. *Let $f \in L^2(\Omega)$. Then the problem (2.11) has a unique solution $u \in V$, $\boldsymbol{\sigma} \in \boldsymbol{W}$.*

For simplicity, we assume that Ω is a convex polygonal domain which is partitioned into rectangular elements. Let $\mathcal{T}_h$ be the quasi-uniform rectangular partition of the domain Ω, where h is the mesh parameter, generally denoting the biggest one of diameters of elements in partitions $\mathcal{T}_h$.

For two integers $k > 0, r > 0$, we construct the finite element subspaces: the finite-dimensional subspaces of V is the span of piecewise polynomial basis of degree k:

$$V_h = \left\{v_h \in C^0(\Omega): \quad v_h|_e \in Q_k(e), \quad \forall e \in \mathcal{T}_h, \ v_h = 0 \ \text{ on } \ \Gamma\right\}. \tag{2.12}$$

A choice for the finite-dimensional subspaces of $\boldsymbol{W}_h \subset \boldsymbol{W}$ is the Raviart-Thomas spaces (see [10], [19]) where the continuity requirement is weaker:

$$\boldsymbol{W}_h = \{\boldsymbol{\tau}_h \in \boldsymbol{W}: \quad \boldsymbol{\tau}_h|_e \in Q_{r+1,r}(e) \times Q_{r,r+1}(e), \quad \forall e \in \mathcal{T}_h\}, \tag{2.13}$$

and another for $\boldsymbol{W}_h$ is the Brezzi-Douglas-Fortin-Marini (BDFM) mixed elements [1]:

$$\boldsymbol{W}_h = \left\{\boldsymbol{\tau}_h \in \boldsymbol{W}: \quad \boldsymbol{\tau}_h|_e \in P_{r+1}(e)\backslash\{y^{r+1}\} \times P_{r+1}(e)\backslash\{x^{r+1}\}, \quad \forall e \in \mathcal{T}_h\right\}, \tag{2.14}$$

where $Q_{m,n}(e)$ indicates the space of polynomials of degree no more than m and n in x and y variables, respectively,

$$Q_{m,n}(e) = \mathrm{span}\{x^i y^j, \quad 0 \le i \le m, \quad 0 \le j \le n, \quad (x,y) \in e\}, \qquad Q_{m,m} = Q_m,$$

P_m indicates the space of complete polynomials of degree m,

$$P_m(e) = \mathrm{span}\{x^i y^j, \quad 0 \le i + j \le m, \quad (x,y) \in e\}.$$

These spaces possess the following approximation properties:

$$\inf_{v_h \in V_h} \{||v - v_h||_{0,\Omega} + h||\mathrm{grad}(v - v_h)||_{0,\Omega}\} \le Ch^{k+1}||v||_{k+1,\Omega}; \tag{2.15}$$

$$\inf_{\boldsymbol{\tau}_h \in \boldsymbol{W}_h} ||\boldsymbol{\tau} - \boldsymbol{\tau}_h||_{0,\Omega} \le Ch^{r+1}||\boldsymbol{\tau}||_{r+1,\Omega}; \tag{2.16}$$

$$\inf_{\boldsymbol{\tau}_h \in \boldsymbol{W}_h} ||\mathrm{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h)||_{0,\Omega} \le Ch^{r+1}||\boldsymbol{\tau}||_{r+2,\Omega}; \tag{2.17}$$

for any $v \in H^{k+1}(\Omega) \cap V$ and $\boldsymbol{\tau} \in (H^{r_1+1}(\Omega))^2 \cap \boldsymbol{W}$ in the case of R-T mixed elements and in the case of BDFM mixed elements.

The finite element approximation to (2.9) is: find $u_h \in V_h$, $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$ such that

$$a(u_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{\tau}_h) = (f, \mathrm{div}\boldsymbol{\tau}_h + cv_h) \qquad \text{for all} \ \ v_h \in V_h, \ \ \boldsymbol{\tau}_h \in \boldsymbol{W}_h. \tag{2.18}$$

From Theorem 2.1 we conclude that problem (2.18) has a unique solution. Moreover, the error has the orthogonality property:

$$a(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; v_h, \boldsymbol{\tau}_h) = 0 \qquad \text{for all} \ \ v_h \in V_h, \ \ \boldsymbol{\tau}_h \in \boldsymbol{W}_h. \tag{2.19}$$

3. Interpolation Projections

There are two major steps in our superconvergence analysis. First, we compare the finite element approximation with an appropriately chosen interpolation of the exact solution in the finite element space. This difference is often far smaller than the global optimal error estimate. Second, we investigate the relation between the exact solution and its interpolation. The objective here is to find special points where the interpolant superapproximates the exact solution. This means that the finite element approximation is super-close to the exact solution at some special points or lines. The interpolant is usually defined locally so that the second step is easily carried out.

Now, we define locally the standard L^2 projection function $P_h u \in V_h$ of u:

$$\int_e (u - P_h u) v dxdy = 0, \qquad \forall v \in V_h, \ \ e \in \mathcal{T}_h. \tag{3.1}$$

Next, in the case of R-T mixed elements, let us define the interpolation function $\Pi\boldsymbol{\sigma} \in \boldsymbol{W}_h$ of $\boldsymbol{\sigma}$:

$$\begin{aligned} \int_{l_i} (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}) \cdot \boldsymbol{\nu} v dl &= 0, \ \ \forall\, v \in P_r(l_i), \ i = 1,2,3,4, \\ \int_e (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} &= 0, \ \ \forall\, \boldsymbol{\tau} \in Q_{r-1,r}(e) \times Q_{r,r-1}(e), \ \ e \in \mathcal{T}_h, \end{aligned}$$

and in the case of BDFM mixed elements, we also define

$$\begin{aligned} \int_{l_i} (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}) \cdot \boldsymbol{\nu} v ds &= 0, \ \ \forall\, v \in P_r(l_i), \ i = 1,2,3,4, \\ \int_e (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} &= 0, \ \ \forall\, \boldsymbol{\tau} \in [P_{r-1}(e)]^2, \ \ e \in \mathcal{T}_h. \end{aligned}$$

Here $\boldsymbol{\nu} = (\nu_1, \nu_2)$ be the outward normal to the boundary Γ, $\{l_i\}$ are edges of rectangular element e and $P_k(l_i)$ denotes the set of polynomials of total degree no more than k. Then, from approximation theory (see [8]),

$$\begin{aligned} ||u - P_h u||_{0,\Omega} + h||\mathrm{grad}(u - P_h u)||_{0,\Omega} &\le Ch^{k+1}||u||_{k+1,\Omega}, & (3.2) \\ ||\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}||_{0,\Omega} &\le Ch^{r+1}||\boldsymbol{\sigma}||_{r+1,\Omega}, & (3.3) \\ ||\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})||_{0,\Omega} &\le Ch^{r+1}||\boldsymbol{\sigma}||_{r+2,\Omega}, & (3.4) \end{aligned}$$

From Lemma 1.25-1.29 and Theorem 1.4 in [14], using the identity technique, it can be proved that:

Lemma 3.1. *If the finite element partition $\mathcal{T}_h$ is quasi-uniform and $\Pi\boldsymbol{\sigma}$ is the interpolation of $\boldsymbol{\sigma}$ defined above, then there exists a constant C such that*

$$(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}, \boldsymbol{\tau}_h)_{0,e} \le Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,e}||\boldsymbol{\tau}_h||_{0,e}, \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \quad e \in \mathcal{T}_h. \tag{3.5}$$

Now, we assume that $k = r + 1$.

Lemma 3.2. *If the finite element partition $\mathcal{T}_h$ is quasi-uniform and $\Pi\boldsymbol{\sigma}$ is the interpolation of $\boldsymbol{\sigma}$ defined above, then*

$$(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}, gradv_h)_{0,e} \leq Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,e}||v||_{1,e}, \tag{3.6}$$

$$(div(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}), v_h)_{0,e} \leq Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,e}||v||_{1,e}, \tag{3.7}$$

for $\forall v_h \in V_h, \quad e \in \mathcal{T}_h$.

Proof. We only give the proof in the case of of R-T mixed elements.

For any $e=[x_e-h_e, x_e+h_e]\times[y_e-h_e, y_e+h_e]\in\mathcal{T}_h$, as in [14], we introducing the error function:

$$E(x)=\frac{1}{2}((x-x_e)^2-h_e^2).$$

For any $v_h\in V_h$, from the Taylor expansion and the fact that grad$v_h \in Q_{r,r+1}(e)\times Q_{r+1,r}(e)$, we see that

$$\partial_1 v_h(x,y)=\sum_{i=0}^{r}\frac{1}{i!}(x-x_e)^i\partial_1^{i+1}v_h(x_e,y).$$

By the definition of the interpolation function $\Pi\boldsymbol{\sigma}$,

$$\int_e(\sigma-\Pi\sigma)_1\sum_{i=0}^{r-1}\frac{1}{i!}(x-x_e)^i\partial_1^{i+1}v_h(x_e,y)=0.$$

Using (1.4.12) in [14], there exists $H(x)\in P_{r-1}(x)$ such that

$$\frac{1}{r!}(x-x_e)^r=\frac{2^{r+1}}{(2r+2)!}\Big((E^{r+1}(x))^{(r+2)}+H(x)\Big).$$

We also use the definition of the interpolation function $\Pi\boldsymbol{\sigma}$:

$$\int_{l_i}(\sigma-\Pi\sigma)_1 v dy=0, \quad \forall\ v\in P_r(l_i),\ i=2,4.$$

With the integration by parts, we can get

$$\begin{aligned}
&\int_e(\sigma-\Pi\sigma)_1\frac{1}{r!}(x-x_e)^r\partial_1^{r+1}v_h(x_e,y)\\
&=\frac{2^{r+1}}{(2r+2)!}\int_e\Big((E^{r+1}(x))^{(r+2)}+H(x)\Big)(\sigma-\Pi\sigma)_1\partial_1^{r+1}v_h(x_e,y)\\
&=-\frac{(-2)^{r+1}}{(2r+2)!}\int_e E^{r+1}(x)\partial_1^{r+2}\sigma_1\partial_1^{r+1}v_h(x_e,y)+0\\
&\leq Ch^{r+2}||\sigma_1||_{r+2,e}||v_h||_{1,e}.
\end{aligned}$$

Thus,

$$((\sigma-\Pi\sigma)_1,\partial_1 v_h)_{0,e}\leq Ch^{r+2}||\sigma_1||_{r+2,e}||v_h||_{1,e}, \qquad \forall v_h\in V_h, \quad e\in\mathcal{T}_h.$$

Similarly,

$$((\sigma-\Pi\sigma)_2,\partial_2 v_h)_{0,e}\leq Ch^{r+2}||\sigma_2||_{r+2,e}||v_h||_{1,e}, \qquad \forall v_h\in V_h, \quad e\in\mathcal{T}_h.$$

Hence, (3.6) has been derived. Using integration by parts we have

$$(\mathrm{div}(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}),v_h)_{0,e}=\int_{\partial e}(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma})\cdot\boldsymbol{\nu}v_h ds-\int_e(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma})\cdot\mathrm{grad}v_h\Big].$$

By the similar argument above, we also can obtain (3.7). Thus, the proof of Lemma 3.2 is completed.

Set,

$$\widetilde{V}_h=\left\{v_h\in L^2(\Omega): \quad v_h|_e\in Q_r(e), \quad \forall\ e\in\mathcal{T}_h,\right\}. \tag{3.8}$$

It is obvious that

$$\mathrm{div}\boldsymbol{\sigma} \in \widetilde{V}_h, \qquad \forall\ \boldsymbol{\sigma} \in \boldsymbol{W}_h, \tag{3.9}$$

in two cases of R-T elements and BDFM elements.

From the definitions of the interpolation functions $P_h u \in V_h$ and $\Pi\boldsymbol{\sigma} \in \boldsymbol{W}_h$, it is clearly to see that

Lemma 3.3. *If the finite element partition $\mathcal{T}_h$ is quasi-uniform. $P_h u$ and $\Pi\boldsymbol{\sigma}$ are the interpolant projections of u and $\boldsymbol{\sigma}$ defined above, then*

$$(u - P_h u, div\boldsymbol{\tau}_h)_{0,e} = 0, \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{W}_h \quad e \in \mathcal{T}_h, \tag{3.10}$$

$$(div(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}), v_h)_{0,e} = 0, \qquad \forall v_h \in \widetilde{V}_h \quad e \in \mathcal{T}_h. \tag{3.11}$$

Since A possesses the positive above-below boundness, so we can define an auxiliary projection $S_h v \in V_h$ of $v \in V$ by

$$(A\mathrm{grad}(v - S_h v), A\mathrm{grad}v_h)_{0,\Omega} + (c(v - S_h v), cv_h)_{0,\Omega} = 0 \quad \text{for all} \ \ v_h \in V_h. \tag{3.12}$$

From the standard convergence theory of the finite element methods [8], if $v \in H^{k+1}(\Omega)$ then we have the estimate

$$||v - S_h v||_{0,\Omega} + h||v - S_h v||_{1,\Omega} \le Ch^{k+1}||v||_{k+1,\Omega}. \tag{3.13}$$

By the high-accuracy theory of the finite element methods[7, 14-15], we also have

$$||S_h u - P_h u||_{1,\Omega} \le Ch^{k+1}||u||_{k+2,\Omega}. \tag{3.14}$$

4. Main Results of Superconvergence

Now, let us present the main super-approximation result.

Theorem 4.1. *Assume that the finite element partition $\mathcal{T}_h$ are quasi-uniform and $(u_h, \boldsymbol{\sigma}_h)$ is the solution of (2.18) by using rectangular elements of Raviart-Thomas or BDFM of order r. If the exact solution u and $\boldsymbol{\sigma}$ satisfies*

$$u \in H^{k+2}(\Omega), \qquad \boldsymbol{\sigma} \in [H^{r+2}(\Omega)]^2,$$

let $k = r + 1$, then

$$||u_h - P_h u||_{1,\Omega} + ||\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}||_{H(div)} \le Ch^{r+2}\left(||u||_{r+3,\Omega} + ||\boldsymbol{\sigma}||_{r+2,\Omega}\right). \tag{4.1}$$

Proof. From the coercivity of the bilinear form in Theorem 2.1 and (2.19),

$$\begin{aligned}
&C\left(||u_h - P_h u||^2_{1,\Omega} + ||\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}||^2_{H(\mathrm{div})}\right)\\
&\quad \le a(u_h - P_h u, \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}; u_h - P_h u, \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})\\
&\quad = a(u - P_h u, \boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}; u_h - P_h u, \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})\\
&\quad = (\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}), \mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}))_{0,\Omega} + (\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}), c(u_h - P_h u))_{0,\Omega}\\
&\qquad + (c(u - P_h u), \mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}))_{0,\Omega} + (c(u - P_h u), c(u_h - P_h u))_{0,\Omega}\\
&\qquad + (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}, A\mathrm{grad}(u_h - P_h u))_{0,\Omega} + (A\mathrm{grad}(u - P_h u), \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})_{0,\Omega}\\
&\qquad + (\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}, \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})_{0,\Omega} + (A\mathrm{grad}(u - P_h u), A\mathrm{grad}(u_h - P_h u))_{0,\Omega}\\
&\quad = \sum_{i=1}^{8} I_i
\end{aligned} \tag{4.2}$$

From(3.9), it is obvious that

$$\mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}) \in \widetilde{V}_h.$$

By (3.11) in Lemma 3.3, we have

$$I_1 = (\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}), \mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}))_{0,\Omega} = 0. \tag{4.3}$$

The second term and the fifth term can be handled by splitting the integral and we use Lemma 3.2 and (3.3)-(3.4),

$$\begin{aligned}
|I_2| &= \left| \sum_{e\in\mathcal{T}_h} \int_e c\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})(u_h - P_h u)dxdy \right| \\
&= \left| \sum_{e\in\mathcal{T}_h} \left[\int_e (c-\bar{c})\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})(u_h - P_h u)dxdy \right.\right. \\
&\quad \left.\left. + \int_e \bar{c}\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})(u_h - P_h u)dxdy \right] \right| \\
&\le Ch||c||_{1,\infty} \sum_{e\in\mathcal{T}_h} \int_e |\mathrm{div}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})|\,|u_h - P_h u|\,dxdy \\
&\quad + Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,\Omega}||u_h - P_h u||_{1,\Omega} \\
&\le Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,\Omega}||u_h - P_h u||_{1,\Omega}, \qquad (4.4) \\
|I_5| &= \left| \sum_{e\in\mathcal{T}_h} \left[\int_e (A - \overline{A})(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})\mathrm{grad}(u_h - P_h u)dxdy \right.\right. \\
&\quad \left.\left. + \int_e \overline{A}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma})\mathrm{grad}(u_h - P_h u)dxdy \right] \right| \\
&\le Ch^{r+2}||A||_{1,\infty}||\boldsymbol{\sigma}||_{r+1,\Omega}||u_h - P_h u||_{1,\Omega} \\
&\quad + Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,\Omega}||u_h - P_h u||_{1,\Omega} \\
&\le Ch^{r+2}||\boldsymbol{\sigma}||_{r+2,\Omega}||u_h - P_h u||_{1,\Omega}, \qquad (4.5)
\end{aligned}$$

where $\bar{c}$ and $\overline{A}$ denote the averages of c and A on the element e, respectively. Similarly, using Lemma 3.3 and (3.1)-(3.2), we also have

$$\begin{aligned}
|I_3| &\le Ch^{k+1}||c||_{1,\infty}||u||_{k+1,\Omega}||\mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})||_{0,\Omega}, \qquad (4.6) \\
|I_4| &\le Ch^{k+1}||c||_{1,\infty}||u||_{k+1,\Omega}||u_h - P_h u||_{0,\Omega}, \qquad (4.7) \\
|I_6| &\le Ch^{k+1}||A^T||_{1,\infty}||u||_{k+1,\Omega}||\mathrm{div}(\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma})||_{0,\Omega}. \qquad (4.8)
\end{aligned}$$

Now, it follows from Lemma 3.1 that

$$I_7 \le Ch^{r+2}||A||_{1,\infty}||\boldsymbol{\sigma}||_{r+1,\Omega}||\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}||_{0,\Omega}. \tag{4.9}$$

Next, from (3.12) we can see that

$$\begin{aligned}
I_8 &= (A\mathrm{grad}(S_h u - P_h u), A\mathrm{grad}(u_h - P_h u))_{0,\Omega} \\
&\quad -(c(u - S_h u), c(u_h - P_h u))_{0,\Omega} = T_1 + T_2. \qquad (4.10)
\end{aligned}$$

By (3.13)-(3.14),

$$\begin{aligned}
T_1 &\le ||A^T A||_{0,\infty}||S_h u - P_h u||_{1,\Omega}||u_h - P_h u||_{1,\Omega} \\
&\le Ch^{k+1}||A^T A||_{0,\infty}||u||_{k+2,\Omega}||u_h - P_h u||_{1,\Omega}, \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
T_2 &\le ||c^2||_{0,\infty}||u - S_h u||_{0,\Omega}||u_h - P_h u||_{0,\Omega} \\
&\le Ch^{k+1}||c^2||_{0,\infty}||u||_{r+2,\Omega}||u_h - P_h u||_{0,\Omega}. \qquad (4.12)
\end{aligned}$$

Since $k = r + 1$, combine the above estimates of $\{I_i\}$ to get

$$\begin{aligned} &a(u - P_h u, \boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}; u_h - P_h u, \boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}) \\ &\le Ch^{r+2}\Big(||\boldsymbol{\sigma}||_{r+2,\Omega} + ||u||_{r+3,\Omega}\Big) \\ &\quad \cdot \Big(||u_h - P_h u||_{1,\Omega}^2 + ||\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}||_{H(\mathrm{div})}^2\Big), \end{aligned} \tag{4.13}$$

where C depends on the C^1 norms of the coefficient functions A and c. Hence, from (4.2)

$$||u_h - P_h u||_{1,\Omega} + ||\boldsymbol{\sigma}_h - \Pi\boldsymbol{\sigma}||_{H(\mathrm{div})} \le Ch^{r+2}\Big(||\boldsymbol{\sigma}||_{r+2,\Omega} + ||u||_{r+3,\Omega}\Big). \tag{4.14}$$

The proof of Theorem 4.1 is completed.

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