Journal of Computational Mathematics, Vol.22, No.2, 2004, 309–318.

NATURAL BOUNDARY INTEGRAL METHOD AND ITS NEW DEVELOPMENT *1)

De-hao Yu
(*LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing 100080, China*)
(*E-mail: ydh@lsec.cc.ac.cn*)

Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In this paper, the natural boundary integral method, and some related methods, including coupling method of the natural boundary elements and finite elements, which is also called DtN method or the method with exact artificial boundary conditions, domain decomposition methods based on the natural boundary reduction, and the adaptive boundary element method with hyper-singular a posteriori error estimates, are discussed.

Mathematics subject classification: 65N38, 65N55, 65R20
Key words: Natural boundary integral, Artificial boundary, Domain decomposition, Hyper-singular a posteriori estimates.

1. Introduction

In many fields of scientific and engineering computing it is necessary to solve boundary value problems of partial differential equations over unbounded domains. The standard techniques such as the finite element method will meet some difficulties, even if they are very effective for bounded domains[1].

In recent twenty years many computational methods for solving problems over unbounded domains have been developed, such as: the infinite element method[19], the adaptive finite element method[1], the finite element method with approximate condition on an artificial boundary[7,9,22], the boundary element method[10], the coupling method of finite and boundary elements[21], the overlapping and non-overlapping domain decomposition methods, especially, the coupling and domain decomposition methods based on the natural boundary reduction[26−31], and so on. Each method has its advantages and disadvantages.

The natural boundary integral method and its coupling with the finite element method are suggested and developed by K. Feng, D. Yu and H. Han in early 1980 (see [4-6,9,20-22]). And then a very similar method, so-called DtN method or exact artificial boundary condition method, has also been devised by J.B. Keller and D. Givoli in later 1980 [12]. These methods are very important for solving many problems over unbounded domains. Up to now there have already been a lot of papers in this direction[11,14,15].

In this paper some new development of the natural boundary integral method is also presented. The method is applied to 3D problems, parabolic and hyperbolic equations, and anisotropic elliptic problems. Based on the natural boundary integral operators the overlapping and non-overlapping domain decomposition methods are developed for problems over

* Received January 31, 2004.
1) Supported by the Special Funds of State Major Basic Research Projects G19990328 and the Knowledge Innovation Program of the Chinese Academy of Sciences.

unbounded domain, which have very wide application background[3,18]. Besides, using hyper-singular residuals as a posteriori error estimates, the adaptive boundary element method is developed[13,32].

The natural boundary integral equations and their related computational methods can also be applied to some semi-linear and nonlinear problems. We will discuss it in our forthcoming papers.

2. Natural Boundary Reduction and DtN Operators

Finite element methods are effective for bounded domains. For unbounded domains we need use boundary integral or boundary element methods. There are different ways for reducing problems to boundary integral equations, based on which some boundary integral methods are developed. For the same physical problem, there are different mathematical formulations, which are equivalent in the theory, but often have different effects in the computing practice.

Based on natural boundary reduction, natural boundary integral method is developed by Feng and Yu. The natural boundary integral method has distinctive advantages. This method preserves basic properties of the original problem, has the same variational principle as finite element method, and can be coupled with finite elements directly and naturally . Furthermore, natural integral equation on artificial boundary is an exact artificial boundary condition. Natural integral operator is just the Dirichlet to Neumann (DtN) operator. It plays a key role in domain decomposition methods, where it has an another name: the Steklov–Poincaré operator.

There is a relation between Dirichlet data u_0 and Neumann data u_n, it is the DtN map, or natural integral equation:

$$u_n = \mathcal{K}u_0, \qquad \text{on } \partial\Omega, \tag{1}$$

where $\mathcal{K}$ is DtN operator, i.e. the natural integral operator. It is a hyper-singular integral operator, a pseudo-differential operator with positive order. $\partial\Omega$ is the boundary of domain Ω.

The solution u is given by Poisson integral formula:

$$u = Pu_0, \qquad \text{in } \Omega. \tag{2}$$

For some typical equations, when Ω is a half-plane or half-space, an interior or exterior circular or spherical domain, $\mathcal{K}$ and P can be obtained explicitly.

3. Artificial Boundary Conditions on Circle and Ellipse

Circular artificial boundary is a good selection for most 2-d exterior problems [20,22]. With natural boundary reduction, we get some DtN operators on the circle $\Gamma = \{(r,\theta)|r = R, 0 < \theta \le 2\pi\}$ as follows , where (r,θ) are polar coordinates and R is a constant (see Yu's books [26,29]).

For harmonic equation:

$$\mathcal{K} = -\frac{1}{4\pi R \sin^2\frac{\theta}{2}}*, \tag{3}$$

which satisfies $\mathcal{K}^2 = -\dfrac{\partial^2}{\partial s^2}$, $s = R\theta$.

Let

$$K(\theta) = -\frac{1}{4\pi \sin^2\frac{\theta}{2}}.$$

For biharmonic equation:

$$\mathcal{K} = \begin{bmatrix} \frac{1+\nu}{R^3}\delta''(\theta) - \frac{2}{R^3}K''(\theta) & \frac{1+\nu}{R^2}\delta''(\theta) + \frac{2}{R^2}K(\theta) \\ \frac{1+\nu}{R^2}\delta''(\theta) + \frac{2}{R^2}K(\theta) & -\frac{1+\nu}{R}\delta(\theta) + \frac{2}{R}K(\theta) \end{bmatrix}*, \tag{4}$$

where ν is Poisson ratio.

For plane elasticity equation:

$$\mathcal{K} = \frac{2}{R}\left[\begin{array}{cc} \frac{ab}{a+b}K(\theta) & \frac{b^2}{a+b}\delta'(\theta) \\ -\frac{b^2}{a+b}\delta'(\theta) & \frac{ab}{a+b}K(\theta) \end{array}\right]*, \tag{5}$$

where $a = \lambda + 2\mu$, $b = \mu$, λ and μ are Lame coefficients.

For Stokes equation:

$$\mathcal{K} = \frac{2\eta}{R}\left[\begin{array}{cc} K(\theta) & 0 \\ 0 & K(\theta) \end{array}\right]*, \tag{6}$$

where η is the viscosity.

For Helmholtz equation (see [5-7],[12]):

$$\mathcal{K} = -\frac{k}{\pi}\sum_{n=0}^{\infty}\left[\frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}\right]\cos n\theta * . \tag{7}$$

The following is DtN operator on sphere (3-D problems) for harmonic equation:

$$u_n(\theta,\varphi) = -\frac{1}{16\pi R}\int_0^{2\pi}\int_0^{2\pi}\frac{\sin\theta'}{\sin^3\frac{\gamma}{2}}u_0(\theta',\varphi')d\theta' d\varphi', \tag{8}$$

where (r,θ,φ) are spherical coordinates, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$. The operator for 3-D Helmholtz equation can be found in [7] and [12] .

The exact artificial boundary condition is nonlocal, and can be formulated by many different integral equations. The natural boundary integral equation, i.e. the DtN map, is the best one: the simplest, the most essential, convenient to applications with the unique formulation, coupled with finite element method naturally and directly.

In computational practice, we use series expansion of the hyper-singular kernel to deal with these hyper-singular integrals. This fast algorithm was first suggested in [20]. It is very simple and effective indeed (also see [26,29]).

There is an additional error as the series are truncated at the N-th term. Suppose $u \in H^1(\Omega_i) \cap H^{k-\frac{1}{2}}(\Gamma_0), k \geq 1$, R_0 be the radius of the minimum circle including Γ_0, R_1 be the radius of the circle which is the artificial boundary, $R_1 \geq R_0$, where $\Gamma_0$is the boundary of the domain. A good estimate was first given by Yu in [22]

$$\|u - u^N\|_{H^1(\Omega_i)/P_0} \leq C\frac{1}{N^{k-1}}\left(\frac{R_0}{R_1}\right)^N \|u\|_{k-\frac{1}{2},\Gamma_0}, \tag{9}$$

where C is independent of N, R_1 and u, P_0 is the set of all constants.

There are already many numerical experiments for the natural boundary integral method, which can be referred to the books [26,29], and many papers.

For problems over exterior domain of some narrow region, elliptic artificial boundary is necessary or much better.

Use elliptic coordinates (μ,φ),

$$\begin{cases} x = f_0\cosh\mu\cos\varphi, \\ y = f_0\sinh\mu\sin\varphi, \end{cases}$$

(x,y) are Cartesian coordinates, f_0 is a positive constant, $(f_0,0)$ and $(-f_0,0)$ are common focuses of ellipses Γ_μ.

The natural boundary reduction reduces the harmonic equation over an exterior elliptic domain

$$\Omega = \{(\mu,\varphi)|\mu > \mu_0\} \tag{10}$$

with elliptic boundary Γ_{μ_0} to an integral equation on the boundary.

Let $\sqrt{J}=f_0\sqrt{\cosh^2\mu-\cos^2\varphi}$, Poisson integral formula is ($\mu>\mu_0$)

$$u(\mu,\varphi)=\frac{1}{\pi}\sum_{n=1}^{\infty}e^{n(\mu_0-\mu)}\int_0^{2\pi}\cos n(\varphi-\varphi')u_0(\varphi')d\varphi'+\frac{1}{2\pi}\int_0^{2\pi}u_0(\varphi')d\varphi', \tag{11}$$

or equivalently

$$u(\mu,\varphi)=\frac{e^{2\mu}-e^{2\mu_0}}{2\pi}\int_0^{2\pi}\frac{u_0(\varphi')}{e^{2\mu}+e^{2\mu_0}-2e^{\mu+\mu_0}\cos(\varphi-\varphi')}d\varphi', \tag{12}$$

The DtN map, i.e. the natural integral equation is:

$$\mathcal{K}u_0(\varphi)=\frac{1}{\sqrt{J}}\left[-\frac{1}{4\pi\sin^2\frac{\varphi}{2}}*u_0(\varphi)\right], \tag{13}$$

or equivalently

$$\mathcal{K}u_0(\varphi)=\frac{1}{\pi\sqrt{J}}\sum_{n=1}^{\infty}n\int_0^{2\pi}\cos n(\varphi-\varphi')u_0(\mu_0,\varphi')d\varphi. \tag{14}$$

The difference between this DtN operator and that in circular case is only a factor $1/\sqrt{J}$.

Let $L_j(\varphi)$, $j=1,\cdots,N$, be piecewise linear basis functions on Γ_{μ_0}. The linear system is $QU=b$, where $Q=[q_{ij}]_{N\times N}$, $U=[U_1,\cdots,U_N]^T$, $b=[b_1,\cdots,b_N]^T$,

$$q_{ij}=\int_{\Gamma_{\mu_0}}L_j\mathcal{K}L_i ds,\qquad b_j=\int_{\Gamma_{\mu_0}}u_\nu L_j ds, \tag{15}$$

Coefficients of the stiffness matrix are

$$q_{ij}=-\int_0^{2\pi}\int_0^{2\pi}\frac{1}{4\pi\sin^2\frac{\varphi-\varphi'}{2}}L_i(\varphi')L_j(\varphi)d\varphi' d\varphi. \tag{16}$$

It is completely the same as that in the case of circular boundary.

$$q_{ij}=a_{|i-j|},$$

$$a_j=\frac{4N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin^4\frac{n\pi}{N}\cos\frac{2nj\pi}{N},\quad j=0,1,\cdots,N-1.$$

Q is symmetrical and circulant. We only need to calculate a_j, $j=0,1,\cdots,[\frac{N}{2}]$, to get it, and the infinite sum can be replaced by a finite sum approximately.

4. Application to an Anisotropic Problem

Now we consider an elliptic equation associated with anisotropic problem ($b>a>0$):

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial y^2}=0. \tag{17}$$

When $a=b$, it is just a harmonic equation.

In the case when the boundary Γ is a circle with radius R, we have Dirichlet problem:

$$\begin{cases} a\dfrac{\partial^2 u}{\partial x^2}+b\dfrac{\partial^2 u}{\partial y^2}=0, & \text{in } \Omega,\\ u=u_0, & \text{on } \Gamma, \end{cases} \tag{18}$$

and Neumann problem:

$$\begin{cases} a\dfrac{\partial^2 u}{\partial x^2}+b\dfrac{\partial^2 u}{\partial y^2}=0, & \text{in } \Omega,\\ an_x\dfrac{\partial u}{\partial x}+bn_y\dfrac{\partial u}{\partial y}=g, & \text{on } \Gamma, \end{cases} \tag{19}$$

where $\vec{n} = (n_x, n_y) = -(x/R, y/R)$ is outward normal direction on Γ with respect to Ω.

By the variable transform, $x = \sqrt{a}\xi$, $y = \sqrt{b}\eta$, the equation is reduced to a harmonic equation, while the circular boundary is replaced by an elliptic boundary, and the exterior circular domain is replaced by an exterior elliptic domain.

Introduce elliptic coordinates to above results, return to the original domain Ω, use polar coordinates (r, θ), then obtain the DtN operator on the boundary Γ as:

$$g(\theta) = -\frac{\sqrt{ab}}{4\pi R \sin^2 \frac{\theta}{2}} * u_0(\theta). \tag{20}$$

The difference between its integral operator and the operator for the harmonic equation ($a = b = 1$) is only a factor $\sqrt{ab}$.

In natural boundary element method the stiffness matrix is:

$$\sqrt{ab}Q = [\sqrt{ab}q_{ij}]_{N\times N}, \tag{21}$$

where the formula for q_{ij} is completely same as before.

In the case when the boundary Γ is an ellipse: $\Gamma = \{(x, y)|\alpha x^2 + \beta y^2 = R^2\}$, the Neumann problem can be reduced to Neumann problem of harmonic equation. The DtN map, i.e. the natural integral equation is:

$$g(\theta) = -\sqrt{\frac{a\alpha b\beta}{\alpha\cos^2\theta + \beta\sin^2\theta}} \left[\frac{1}{4\pi R\sin^2\frac{\theta}{2}} * u_0(\theta)\right]. \tag{22}$$

The difference between the matrix on elliptic boundary Γ and the matrix for harmonic equation is only a factor $\sqrt{ab}$:

$$\sqrt{ab}\int_0^{2\pi}\int_0^{2\pi}\left(-\frac{1}{4\pi\sin^2\frac{\theta-\theta'}{2}}\right)L_i(\theta')L_j(\theta)d\theta' d\theta = \sqrt{ab}q_{ij}, \tag{23}$$

where the formula for q_{ij} is completely the same as before.

We have following results.

If the stiffness matrix of natural boundary element method on circular boundary for the harmonic equation is $Q = [q_{ij}]_{N\times N}$, then:

The stiffness matrix of natural boundary element method on elliptic boundary for the harmonic equation still is $Q = [q_{ij}]_{N\times N}$.

The stiffness matrix of natural boundary element method on circular boundary for the anisotropic equation $a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial^2 u}{\partial y^2} = 0$ is $\sqrt{ab}Q = [\sqrt{ab}q_{ij}]_{N\times N}$.

The stiffness matrix of natural boundary element method on elliptic boundary for anisotropic equation $a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial^2 u}{\partial y^2} = 0$ is also $\sqrt{ab}Q = [\sqrt{ab}q_{ij}]_{N\times N}$.

5. Coupling and Domain Decomposition Methods

For problems over the exterior domain of a bounded region, by using a circular or elliptic artificial boundary, the domain is divided into a small bounded subdomain, where the finite element method is used, and an exterior elliptic subdomain, where the natural boundary reduction is applied.

We have the following error estimates for coupling method:

Theorem 5.1. (Estimate in energy norm) If $u \in H^{k+1}(\Omega_1), k \geq 1$, the interpolation operator Π satisfies

$$\|v - \Pi v\|_{1,\Omega_1} \leq Ch^j \|v\|_{j+1,\Omega_1}, \forall v \in H^{j+1}(\Omega_1), j = 1, \cdots, k, \tag{24}$$

then

$$(\|u-u_h\|_{D_1}^2+\|\gamma' u-\gamma' u_h\|_{\hat{D}_2}^2)^{\frac{1}{2}} \leq CMh^k \|u\|_{k+1,\Omega_1}. \tag{25}$$

Theorem 5.2. (Estimate in L^2 norm) If $u \in H^{k+1}(\Omega_1), k \geq 1$, the interpolation operator Π satisfies the conditions of Theorem for energy norm, then

$$\|u-u_h\|_{L^2(\Omega_1)} \leq CMh^{k+1} \|u\|_{k+1,\Omega_1}. \tag{26}$$

Here C is a positive constant, which is independent of h and u, $M=1$ for circular artificial boundary, and $M=\max\{|b/a|,|a/b|\}$ when an elliptic artificial boundary is used[30,31].

Above results can also be applied to domain decomposition methods with circular or elliptic artificial boundary (see [29]).

Overlapping domain decomposition method is Schwarz method and has geometric convergence (see [18,27]).

Non-overlapping domain decomposition method is D–N method and the convergence is correlative with the relaxation factor σ (see [3,28]).

Comparing with coupling method, advantages of domain decomposition methods are that problems in each sub-domains can be solved separately, available finite element program can be used directly, and these methods are suitable for developing parallel computation.

6. Hyper-singular A-posteriori Error Estimates for BEM

In recent years the adaptive technique with *a posteriori* error estimates is applied in many fields of scientific and engineering computing. It is at first applied in the finite element methods by I. Babuska and his cooperators, for example, we can see references [1,2,24,25] and other many papers. Then the adaptive boundary element method with a posteriori error estimates is also developed, we can find its mathematical theory in references [16,17,23] and other related papers. In both adaptive finite and boundary element methods, some reliable and efficient *a posteriori* error estimates plays a key role.

There are many ways to obtain *a posteriori* error estimates. Usually, residuals are used as a posteriori error estimates. The residual is defined as the difference between two sides of the boundary integral equations when the exact solution is replaced by the approximate solution. Since for same boundary value problem there are many kinds of boundary integral equations, we have some different classical or hyper-singular residuals, which are based on classical or hyper-singular integral equations.

In this section the boundary reduction for potential problem is introduced, the residuals for Dirichlet and Neumann problems are discussed. Three kinds of residuals for boundary element methods are compared.

Many numerical experiences show some advantages of the hyper-singular residuals, which is good *a posteriori* error indicator in many adaptive boundary element computations[13,32].

Now we discuss the potential problem as an example. There is a classical boundary integral equation:

$$\phi_0 = K\phi_0 + V\phi_n, \tag{27}$$

where the linear operators are defined as:

$$(Vu)(p) := 2\int_\Gamma G(p,Q)u(Q)ds_Q,$$

$$(Ku)(p) := -2\int_\Gamma \frac{\partial}{\partial n_Q} G(p,Q)u(Q)ds_Q.$$

And we also have a boundary integral equation with a hyper-singular operator:

$$\phi_n = L\phi_0 + M\phi_n, \tag{28}$$

where

$$(Mu)(p) := 2\int_\Gamma \frac{\partial}{\partial n_p} G(p,Q)u(Q)ds_Q,$$

$$(Lu)(p) := -2\int_\Gamma \frac{\partial}{\partial n_p}\frac{\partial}{\partial n_Q} G(p,Q)u(Q)ds_Q.$$

Here V, K, M and L are all pseudo-differential operators with integer orders. The weakly-singular operator V are of order -1. It is a smoothing operator. K and M are compact, then $I-K, I-M$ are operators of order 0, where I is the identical operator. The hyper-singular operator L is a operator of order +1.

By the theory of natural boundary reduction[26,29], we also have the natural boundary integral equation:

$$\phi_n = S\phi_0. \tag{29}$$

It is the relation between Dirichlet and Neumann data. The hyper-singular integral operator S is just the DtN operator, and it is a pseudo-differential operator of order +1.

For the Dirichlet boundary value problems, ϕ_0 is given, and ϕ_n is unknown and should be found. In this case, (27) is a Fredholm integral equation of the first kind, (28) is a singular integral equation of Cauchy type, i.e. a Fredholm integral equation of the second kind, and (29) is just a hyper-singular integral, it is unnecessary to solve any equation.

In the Neumann boundary value problems, we have given ϕ_n, and should find ϕ_0. Then (27) is a Fredholm integral equation of the second kind, (28) and (29) are hyper-singular integral equations.

Even through (28) and (29) are all hyper-singular integral equations, however, (29) is much simpler than (28), and is facile for theoretical analysis and numerical computation.

At first we discuss the Residuals for Dirichlet problem.

For same potential problem, there are several equivalent integral equations. Then we can obtain some different residuals from the computed boundary element solution.

If the boundary integral equation (27) has been used for the initial solution, the error for the approximation ϕ_n^h is:

$$e_h = \phi_n^h - \phi_n. \tag{30}$$

We can calculate the residuals by using (27), (28) or (29).

Using (27), the classical residual is defined as:

$$r_h^{(1)} = \phi_0 - (K\phi_0 + V\phi_n^h) = (\phi_0 - K\phi_0 - V\phi_n) - Ve_h = -Ve_h. \tag{31}$$

Since V is a weakly singular integral operator, it smoothes e_h, so the properties of the residual $r_h^{(1)}$ can not reflect the properties of the error exactly, it is not a proper measure of the error e_h.

Using (28), the iterate is defined by

$$\phi_n^{h\prime} = L\phi_0 + M\phi_n^h, \tag{32}$$

one obtains:

$$r_h^{(2)} = \phi_n^h - \phi_n^{h\prime} = \phi_n^h - (L\phi_0 + M\phi_n^h), = (\phi_n - L\phi_0 - M\phi_n) + e_h - Me_h, \tag{33}$$

i.e.

$$r_h^{(2)} = (I-M)e_h. \tag{34}$$

This hyper-singular residual is computable. We have following result.

Theorem 6.1. There are two positive constants $0 < C_1 < C_2$, such that

$$C_1\|r_h^{(2)}\|_0 \le \|e_h\|_0 \le C_2\|r_h^{(2)}\|_0. \tag{35}$$

Using (29), one obtains:

$$r_h^{(3)} = \phi_n^h - S\phi_0 = (\phi_n - S\phi_0) + e_h = e_h \tag{36}$$

The hyper-singular residual is just the error in the primary approximation. From (36) we know that, $\|e_h\| = \|r_h^{(3)}\|$ for both global and local norm. Therefore, when the integral of $S\phi_0$ reaches a high degree of accuracy, $r_h^{(3)}$ is the best *a posteriori* error estimator.

The difficulty of computing hyper-singular integral exists in both (28) and (29). When the expression of S is known, it is much easier and more effective to apply (29) than (28). In fact, it is convenient to use the natural boundary integral equation (29) to get the initial solution.

Then we discuss the residuals for Neumann problem.

For Neumann problems, $\phi_n = g$ is known and ϕ_0 is unknown. Assume that the solution is smooth enough. Let the approximation solution be ϕ_0^h. The error is denoted by:

$$e_h = \phi_0^h - \phi_0. \tag{37}$$

We also have three kinds of residuals.

Using boundary integral equation (27), one obtains:

$$\phi_0^{h\prime} = K\phi_0^h + V\phi_n. \tag{38}$$

The residual is defined as above and thus

$$\begin{aligned} r_h^{(1)} &= \phi_0^h - (K\phi_0^h + V\phi_n) \\ &= (\phi_0^h + e_h) - K(\phi_0^h + e_h) - Vg \\ &= (I - K)e_h. \end{aligned} \tag{39}$$

The operator K is compact (for C^1 boundary). We get the global error estimate:

$$C_1\|r_h^{(1)}\|_0 \le \|e_h\|_0 \le C_2\|r_h^{(1)}\|_0, \tag{40}$$

where $0 < C_1 \le C_2, C_1, C_2$ are constants. The error is:

$$e'_h = \phi_0^{h\prime} - \phi_0^h = K\phi_0^h + V\phi_n - \phi_0 = K(\phi_0 + e_h) - Vg - \phi_0 = Ke_h. \tag{41}$$

The compact operator K filters out all the oscillatory components terms. In this sense, the iterate result is super-convergent.

Using (28), the hyper-singular residual is defined as:

$$r_h^{(2)} = \phi_n - (L\phi_0^h + M\phi_n) = g - (L(\phi_0 + e_h) + Mg) = -Le_h. \tag{42}$$

The pseudo-differential operator L is of order $+1$. L has the effect of amplifying the high frequencies of e_h. Thus, we anticipate that $r_h^{(2)}$ will indeed be a good *a posteriori* estimator.

There exist two constants $0 < C_1 \le C_2$, such that :

$$C_1\|r_h^{(2)}\|_{-\frac{1}{2}} \le \|e_h\|_{\frac{1}{2}} \le C_2\|r_h^{(2)}\|_{-\frac{1}{2}}. \tag{43}$$

Substituting ϕ_n^h into (29), the hyper-singular residual is defined as:

$$r_h^{(3)} = \phi_n - S\phi_0^h = (\phi_n - S\phi_0) - Se_h = -Se_h. \tag{44}$$

The hyper-singular operator S is of order $+1$.

$$C_1\|r_h^{(3)}\|_{-\frac{1}{2}} \le \|e_h\|_{\frac{1}{2}} \le C_2\|r_h^{(3)}\|_{-\frac{1}{2}}, \tag{45}$$

where $0 < C_1 \le C_2, C_1, C_2$ are constants.

When computing the integral $S\phi_0$ reaches required precision, $r_h^{(3)}$ is a good *a posteriori* error estimator.

Therefor, based on different boundary integral equations, there are several residuals which can be used as *a posteriori* error estimators in the boundary element methods. Numerical results illustrate the important features of these error estimates. Especially, the hyper-singular residuals obtained from the natural boundary integral equations are good a posteriori error estimates in adaptive boundary element methods. The calculation is reliable and efficient.

7. Conclusions

The boundary reduction is an powerful means for solving boundary value problems of partial differential equations over unbounded domains. There are many different ways of boundary reduction, the best one seems to be the natural boundary reduction. Based on it the natural boundary integral method and some related computational methods, including coupling method, domain decomposition methods and adaptive method, are suggested. These methods are developed rapidly and applied widely in recent years. For the mathematical foundation of the natural boundary integral method we can see books[26,29].

Very recently, a review of Yu's English book [29] has been given by D. Givoli [8]. He has used many good words highly to appraise the book, such as: 'superb','written beautifully', 'very clear, interesting', 'definitely useful','enjoyable and eye-opening', 'one can learn quite a lot from this book', and so on. In the meantime, he has also stressed the point that, even though the natural boundary integral method and the DtN method are very similar, "so-called natural integral operator is also known as the Dirichlet-to-Neumann map", but DtN method is "devised independently in the west", since "most of Feng's and Yu's publications have appeared in Chinese and have not been generally accessible to western readers", "a very interesting piece of work that has been hidden from the western readers so far", then "this monograph is especially welcome" and "highly recommended". It just show that the natural boundary integral method or so-called DtN method is very important for scientific and engineering computing.

References

[1] Babuška, I. Strouboulis, T., The Finite Element Method and Its Reliability, Oxford Univ., Press, 2001.

[2] Babuska, I., Yu, D.H., Asymptotically exact a posteriori error estimator for biquadratic elements, *Finite Elements in Analysis and Design*, **3** (1987), 341–354.

[3] Du, Q.K., Yu, D.H., Dirichlet-Neumann alternating algorithm based on the natural boundary reduction for time-dependent problems over an unbounded domain, *Applied Numerical Mathematics*, **44** (2003), 471-486.

[4] Feng, K., Differential versus integral equation and finite versus infinite Element , *Mathematicae Numerica Sinica* , **2** :1 (1980), 100-105.

[5] Feng, K., Yu, D.H., Canonical integral equations of elliptic boundary value problems and their numerical solutions, Proc. of China–France Symp. on FEM(1982, Beijing), Scinece Press, Beijing, 1983, 211-252.

[6] Feng, K., Finite element method and natural boundary reduction, Proc. of the International Congress of Mathematicians, Warszawa, 1983, 1439-1453.

[7] Feng, K., Asymptotic radiation conditions for reduced wave equation, *J. Comp. Math.*, **2**:2 (1984), 130-138.

[8] Givoli D., Book review, *Appl. Mech. Rev.*, vol 56, no 5, Sep. 2003, B65.

[9] Han, H.D., Wu, X.N., Approximation of infinite boundary condition and its application to finite element methods . *J. Comp. Math.*, **3**:3 (1985), 179-192.

[10] Hsiao, G.C., Wendland, W.L., A finite element method for some integral equations of the first kind, *J. Math. Anal Appl.*, **58**:3 (1997), 449-481.

[11] Hu, Q.Y., Yu, D.H., Solving singularity problems in unbounded domains by coupling of natural BEM and composite grid FEM, *Applied Numerical Mathematics*, **37** (2001), 127-143.

[12] Keller, J.B., Givoli, D., Exact non-reflecting boundary conditions, *J. Comp. Phys.*, **82** (1989), 172-192.

[13] Menon, G., Paulino G.H., Mrkherjee, S., Analysis of hypersingular residual error estimates in boundary element methods for potential problems, *Comput. Methods Appl. Mech. Engrg.*, **173** (1999), 449-473.

[14] Shigeta, T., Numerical methods for magnetostatic unbounded domain/inverse problems using the mixed finite element method, Doctoral Thesis, Ibaraki University, 2000.

[15] Shi, Z.C., Ushijima, T., Proceeding of the first China-Japan Joint Seminar on Numerical Mathematics, Beijing,China, August24-29, 1992, World Scientific, 1993.

[16] Wendland, W.L., Yu, D.H., Adaptive boundary element methods for strongly elliptic integral equations, *Numer.Math.*, **53** (1988), 539-558.

[17] Wendland, W.L., Yu, D.H., A posteriori local error estimates of boundary element methods with some dopseudo-differential equations on closed curves, *J. Comput.Math.*, **10**:3 (1992), 273–289.

[18] Wu, J.M., Yu, D.H., The Overlapping DDM for biharmonic equation over exterior 3-d domain, *J. Comp. Math.*, **18**:1 (2000), 83-94.

[19] Ying, L.A., Infinite Element Methods (in Chinese), Peking University Press, Beijing, 1989.

[20] Yu, D.H., Numerical solutions of harmonic and biharmonic canonical integral equations in interior or exterior circular domains. *J.Comput.Math.*, **1**:1 (1983), 52-62.

[21] Yu, D.H., Coupling canonical element method with FEM to solve harmonic problem over cracked domain, *J. Comp. Math.*, **1**:3 (1983), 195-202.

[22] Yu, D.H., Approximation of boundary conditions at infinity for harmonic problem over cracked domain, *J. Comp. Math.*, **3**:3 (1985), 219-227.

[23] Yu, D.H., A-posteriori error estimates and adaptive approaches for some boundary element methods, Boundary Elements IX, C.A. Brebbia, W.L. Wendland and G. Kuhn eds., Springer-Verlag, 241–256, 1987.

[24] Yu, D.H., Asymptotically exact a-posteriori error estimator for elements of bi-even degree, *Math. Numer. Sinica*, **13**:1 (1991), 89-101; *Chinese J. Numer. Math. and Appl.*, **13**:2 (1991), 64-78.

[25] Yu, D.H., Asymptotically exact a-posteriori error estimator for elements of bi-odd degree, *Math. Numer. Sinica*, **13**:3 (1991), 307-314; *Chinese J. Numer. Math. and Appl.*, **13**:4 (1991), 82-90.

[26] Yu, D.H., Mathematical Theory of Natural Boundary Element Method (in Chinese), Science Press, Beijing, 1993.

[27] Yu, D.H., A domain decomposition method based on the natural boundary reduction over unbounded domain, *Math. Numer. Sinica*, **16**:4 (1994), 448-459; *Chinese J. Numer. Math. Appl.*, **17**:1 (1995), 95-105.

[28] Yu, D.H., Discretization of non-overlapping DDM for unbounded domains and its convergence, *Math. Numer. Sinica*, **18**:3 (1996), 328-436; *Chinese J. Numer. Math. Appl.*, **18**:4 (1996), 93-102.

[29] Yu, D.H., Natural Boundary Integral Method and Its Applications, Kluwer Academic Publishers, 2002.

[30] Yu, D.H., Jia, Z.P., Natural integral operator on elliptic boundary and the coupling method for an anisotropic problem, *Math. Numer. Sinica*, **24**:3 (2002), 375-384; *Chinesr J. Numer. Math. & Appl.*, **24**:4 (2002), 95–107.

[31] Yu, D.H., Du, Q.K., The coupling of natural boundary element and finite element method for 2D hyperbolic equations, *J. Comput. Math.*, **21**:5 (2003), 585-594.

[32] Yu, D.H., Zhao, L.H., Analysis of hyper-singular residuals and a posteriori error estimates in boundary element methods, Boundary Element Techniques, Tsinghua University Press/Springer, 2002, 261-266.