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POLYNOMIAL PRESERVING RECOVERY FOR ANISOTROPIC AND IRREGULAR GRIDS [*1)]

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

Some properties of a newly developed polynomial preserving gradient recovery technique are discussed. Both practical and theoretical issues are addressed. Bounded-ness property is considered especially under anisotropic grids. For even-order finite element space, an ultra-convergence property is established under translation invariant meshes; for linear element, a superconvergence result is proven for unstructured grids generated by the Delaunay triangulation.

Mathematics subject classification: 65N30, 65N15
Key words: Finite element, Superconvergence, Gradient recovery, *A posteriori* error estimate.

1. Introduction

The Zienkiewicz-Zhu error estimator [15] using recovered gradient by the superconvergent patch recovery (SPR) [16] has proven to be an effective way to access the error in computed data. The idea of their recovery is to fit higher-order polynomials, in the least-squares sense, with computed gradients on element patches. Recently, we proposed an alternative recovery method [14]. The idea is to fit higher-order polynomials with computed solution values (instead of gradient values) at some local sampling points, and obtain the recovered gradient at a nodal point by evaluating the gradient of the resultant polynomial at the same nodal point. One significant feature of this recovery is polynomial preserving. For this reason, we call it PPR.

In an earlier work [11], Wiberg-Li used function value fitting to improve convergence in the L_2-norm. In a more recent work [10], Wang used a semi-local L_2-projection and proved a superconvergent result under quasi-uniform mesh assumption.

Superconvergence properties of the SPR and its effectiveness in *a posteriori* error estimates have been studied by the author and his colleagues, see e.g., [5, 12, 13]. In this paper, we discuss PPR. Other than theoretical discussions, some practical aspects, including the selection of polynomial basis functions in the least-squares fitting and anisotropic grids are considered. Finally, we establish an ultra-convergence (two-order superconvergence) property for even-order finite elements under translation invariant meshes and a superconvergence result with irregular meshes by the Delaunay triangulation.

Numerical tests of PPR and its comparison with SPR can be found in [7, 14]. Our tests indicate that PPR is as good as, or better than SPR in practice.

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As for the literature regarding superconvergence and *a posteriori* error estimates, the reader is referred to [1, 2, 3, 4, 6, 8, 9].

2. Recovery Procedure

Let $S_{h,k}$ be a polynomial finite element space of degree k over a triangulation $\mathcal{T}_h$. We define a gradient recovery operator $G_h : S_{h,k} \to S_{h,k}^d$, with $d = 1, 2, 3$. Given a finite element solution u_h, we first define $G_h u_h$ at certain nodes. When $d = 2$, there are three types of nodes: vertices, edge nodes, and internal nodes. When $d = 3$, there is one more type: the surface node. For the linear element, all nodes are vertices. For the quadratic element, there are vertices and edge-center nodes. For the cubic or higher-order elements, all types of nodes are present. After defining values of $G_h u_h$ at all nodes, we obtain $G_h u_h \in S_{h,k}^d$ on the whole domain by interpolation using the original nodal shape functions of $S_{h,k}$.

Given a node $\boldsymbol{z}_i$, we need to determine $G_h u_h(\boldsymbol{z}_i)$. This is achieved by first selecting $n \geq m = \frac{1}{d!}\Pi_{j=1}^d (k+1+j)$ sampling points adjacent to $\boldsymbol{z}_i$ (including $\boldsymbol{z}_i$), and then fitting a polynomial of degree $k+1$, in the least-squares sense, with values of u_h at those sampling points. In other words, we are looking for $p_{k+1} \in \mathcal{P}_{k+1}$ such that

$$\sum_{j=1}^n (p_{k+1} - u_h)^2(\boldsymbol{z}_{ij}) = \min_{q \in \mathcal{P}_{k+1}} \sum_{j=1}^n (q - u_h)^2(\boldsymbol{z}_{ij}). \tag{2.1}$$

Using the local coordinates (x, y) with $\boldsymbol{z}_i$ as the origin, the fitting polynomial is denoted as $p_{k+1}(x, y; \boldsymbol{z}_i)$, we then define

$$G_h u_h(\boldsymbol{z}_i) = \nabla p_{k+1}(0, 0; \boldsymbol{z}_i). \tag{2.2}$$

Comparing with Zienkiewicz-Zhu's patch recovery [16], here we fit u_h instead of ∇u_h. The above procedure generates a finite difference scheme

$$G_h v(\boldsymbol{z}_i) = \sum_{j=1}^n \vec{C}_j v(\boldsymbol{z}_{ij}), \quad \sum_{j=1}^n \vec{C}_j = \vec{0}. \tag{2.3}$$

The task now is to determine the coefficients $\vec{C}_j$s.

Usually, we select sampling points as nodal points of all triangles that share a common vertex $\boldsymbol{z}_i$. These triangles naturally form an element patch as used in [16]. Figures 4-6 depict some possible interior and boundary patches when $d = 2$. Among them, only the last two interior patches (with 4 and 5 triangles, respectively) in Figure 4 and the two boundary patches in Figure 5 appear in meshes constructed by a sophisticated automatic mesh generator (based on the Delaunay triangulation). Indeed, for an interior patch that has only three triangles (the first patch in Figure 4), a mesh generator simply removes the center node and three connecting edges; for a patch that contains four triangles (the second patch in Figure 4), a mesh generator removes the center node and related edges, then adds one of the diagonals of the quadrilateral to form two new triangles. As for a boundary vertex, a mesh generator always seeks to connect it with two interior vertices, and for a corner vertex, a mesh generator always bisects the angle that is less than $\pi/2$. Therefore, situations in Figure 6 and the first two cases in Figure 4 almost never happen in practice.

Actually, the sampling points selection can be very flexible. The rule of thumb is to make an interior node $\boldsymbol{z}_i$ as close as possible to the geometric center of all $\boldsymbol{z}_{ij}$s. The perfect situation is when $\boldsymbol{z}_{ij}$s are symmetrically distributed around $\boldsymbol{z}_i$.

We may always select $n \geq m$ sampling points. However, this alone is not sufficient to guarantee that problem (2.1) has a unique solution. Towards this end, we introduce an **Angle condition**: The sum of any two adjacent angles in $\mathcal{T}_h$ is no more than π.

Theorem 1. *The angle condition implies a unique solution of (2.1) when $n \geq m$.*

Proof. See [7].

In practice, the least-squares fitting is performed with scaling, since it is the relative position of those sampling points that counts. For simplicity, we use a simple example of uniform mesh to illustrate the idea and keep in mind that the least-squares procedure is applicable to arbitrary and anisotropic meshes.

Example 1. Linear element with uniform triangular mesh of the Union-Jake pattern, see Figure 1(a). We fit

$$p_2(x,y) = (1, x, y, x^2, xy, y^2)(a_1, \cdots, a_6)^T$$

with respect to the nine nodal values on eight triangles that share a common vertex. By scaling, we denote

$$\vec{e} = (1,1,1,1,1,1,1,1,1)^T, \quad \vec{\xi} = (0,1,1,0,-1,-1,-1,0,1)^T,$$

$$\vec{\eta} = (0,0,1,1,1,0,-1,-1,-1)^T, \quad A = (\vec{e}, \vec{\xi}, \vec{\eta}, \vec{\xi^2}, \vec{\xi\eta}, \vec{\eta^2}).$$

Performing the discrete least-squares fitting, we obtain the recovered gradient at the patch center

$$\nabla p_2(0,0) = a_1 = (A^T A)^{-1} A^T \vec{b} = \frac{1}{6h}\begin{pmatrix} u_1 - u_5 + u_2 - u_4 + u_8 - u_6 \\ u_3 - u_7 + u_2 - u_8 + u_4 - u_6 \end{pmatrix}, \tag{2.4}$$

where $\vec{b}^T = (u_0, u_1, \cdots, u_8)$, $u_0 = u(0,0)$, $u_1 = u(h,0)$, $u_2 = u(h,h), \ldots$. An alternate strategy is to fit

$$q_2(x,y) = (1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2)(a_1, \cdots, a_9)^T$$

with respect to the nine nodal values. This is the same as interpolation, which results in

$$\nabla q_2(0,0) = \frac{1}{2h}\begin{pmatrix} u_1 - u_5 \\ u_3 - u_7 \end{pmatrix}. \tag{2.5}$$

Fitting

$$\tilde{q}_2(x,y) = (1, x, y, x^2, xy, y^2, x^2y, xy^2)(a_1, \cdots, a_8)^T$$

with respect to the nine nodal values on the patch also produces (2.5).

Remark 1. The recovery operator would be the same as the above example for bilinear element on uniform square mesh when we perform the fitting on the nine nodal values of the four squares that share a common vertex. Figure 1(b) demonstrate the resulted coefficients $\vec{C}_j$s, which are the same as given by (2.4). This reveals an important property: G_h is nodal dependent, not patch dependent.

Condition number for different polynomial bases. When higher-order polynomials are involved, we recommend using orthogonal polynomial basis functions for the least-squares fitting in order to avoid ill conditioning. We explain this point in the one dimensional setting.

Consider the standard least-squares fitting problem: Given data set (t_j, b_j) with t_j, $j = 1, 2, \ldots, n$, distinct numbers in $(0,1)$, find $p \in \mathcal{P}_k$, such that

$$\sum_{j=1}^{n} (p(t_j) - b_j)^2 = \min_{q \in \mathcal{P}_k} \sum_{j=1}^{n} (q(t_j) - b_j)^2$$

If we use the conventional basis functions $1, t, t^2, \ldots, t^k$, the coefficients of p are given by $(A^T A)^{-1} A^T \vec{b}$ with

$$A = \begin{pmatrix} 1 & t_1 & \cdots & t_1^k \\ 1 & t_2 & \cdots & t_2^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^k \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The (l, m) entry of $A^T A$ is,

$$\sum_{j=1}^{n} t_j^{l+m-2} \approx n \int_0^1 t^{l+m-2} dt = \frac{n}{l+m-1},$$

when n is sufficiently large and t_j's are properly distributed. Therefore, $A^T A$ behaves like the Hilbert matrix and has a condition number $O(10^k)$.

A better choice for the basis function would be the Legendre polynomials on the unit interval in which case the coefficients of p is given by $(B^T B)^{-1} B^T \vec{b}$ with

$$B = \begin{pmatrix} 1 & L_1(t_1) & \cdots & L_k(t_1) \\ 1 & L_1(t_2) & \cdots & L_k(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & L_1(t_n) & \cdots & L_k(t_n) \end{pmatrix}$$

The (l, m) entry of $B^T B$ is

$$\sum_{j=1}^{n} L_{l-1}(t_j) L_{m-1}(t_j) \approx n \int_0^1 L_{l-1}(t) L_{m-1}(t) dt = \begin{cases} n, & m = l = 1 \\ \frac{2n}{2m-1} \delta_{lm}, & m > 1, \end{cases}$$

when n is sufficiently large and t_j's are properly distributed. Therefore, $B^T B$ behaves like the diagonal matrix: $n\mathrm{diag}(1, 2/3, \cdots 2/(2k+1))$.

3. Properties of the Gradient Recovery Operator

In this section, we assume that the problem (2.1) has a unique solution. We denote ω_z, an element patch associated with node $\boldsymbol{z}$; and denote ω_τ, the union of element patches associated with three vertices of τ.

Proposition 1. *G_h is polynomial preserving in the sense that $G_h u_I = \nabla u$ for any $u \in \mathcal{P}_{k+1}$, where $u_I \in S_{h,k}$ is the Lagrange interpolation of u. (See [14] for details.)*

Proposition 2. *G_h is a bounded operator in the sense that there exists a constant C, independent of h, such that*

$$|G_h v(\boldsymbol{z})| \le C |v|_{W^1_\infty(\omega_z)}, \quad \text{or} \quad \|G_h v\|_{L_2(\tau)} \le C |v|_{H^1(\omega_\tau)}, \quad \forall v \in S_{h,k}. \tag{3.1}$$

Proof. Let $\bar{v}$ be the average of $v \in S_{h,k}$ on the patch ω_z. Then by (2.3),

$$G_h v(\boldsymbol{z}) = G_h (v - \bar{v})(\boldsymbol{z}) = \frac{1}{h} G(v - \bar{v})(\boldsymbol{z}).$$

Here G is the scaled recovery operator on a reference patch. Therefore, G is bounded independent of h. By the inverse inequality,

$$|G_h v_h(z)| \le \frac{\|G\|}{h} \|v - \bar{v}\|_{L_\infty(\omega_z)} \le C |v|_{W^1_\infty(\omega_z)}.$$

Proposition 3. *If sampling nodes $\boldsymbol{z}_{ij}$ are symmetrically distributed around the assembly node $\boldsymbol{z}_i = \boldsymbol{z}_{i1}$, then the coefficients $\vec{C}_j$ are distributed anti-symmetrically, i.e., there is a permutation $(j_2, \cdots, j_n)$ of $(2, \cdots, n)$, such that*

$$\vec{C}_1 = \vec{0}, \quad \vec{C}_2 + \vec{C}_{j_2} = \vec{0}, \quad \cdots, \quad \vec{C}_n + \vec{C}_{j_n} = \vec{0}.$$

Proof. Let $v \in S_{h,k}$ be even and consider $G_h v$. By the least-squares fitting procedure, the fitting polynomial p_{k+1} is invariant under anti-nodal transform $\boldsymbol{z} \to \boldsymbol{z}_i - (\boldsymbol{z} - \boldsymbol{z}_i)$ if $\boldsymbol{z}_{ij}$ are symmetrically distributed around $\boldsymbol{z}_i = \boldsymbol{z}_{i1}$. In other words, we have $p_{k+1}(\boldsymbol{z}_i - (\boldsymbol{z} - \boldsymbol{z}_i)) = p_{k+1}(\boldsymbol{z})$. Consequently,

$$\nabla p_{k+1}(\boldsymbol{z}_i - (\boldsymbol{z} - \boldsymbol{z}_i)) = -\nabla p_{k+1}(\boldsymbol{z}),$$

and therefore, $G_h v(\boldsymbol{z}_i) = \nabla p_{k+1}(\boldsymbol{z}_i) = \vec{0}$.

Now we set $v = N_i$, the nodal shape function at $\boldsymbol{z}_i$, then $G_h v(\boldsymbol{z}_i) = \vec{C}_1 v(\boldsymbol{z}_i) = \vec{C}_1 = \vec{0}$, since v is even under symmetrically distributed nodes.

Next, we choose $v = N_2 + N_{j_2}$, where N_2 and N_{j_2} are nodal shape functions at $\boldsymbol{z}_{i2}$ and $\boldsymbol{z}_{ij_2}$, respectively, with $\boldsymbol{z}_{ij_2} - \boldsymbol{z}_i = \boldsymbol{z}_i - \boldsymbol{z}_{i2}$. This is possible since the sampling points are symmetrically distributed around $\boldsymbol{z}_i$. Clearly v is even and hance

$$G_h v(\boldsymbol{z}_i) = \vec{C}_2 v(\boldsymbol{z}_{i2}) + \vec{C}_{j_2} v(\boldsymbol{z}_{ij_2}) = \vec{C}_2 + \vec{C}_{j_2} = \vec{0}.$$

The rest can be proved similarly.

All above properties are valid for $d = 1, 2, 3$, and anisotropic meshes.

Example 2. We consider the uniform triangulation of the regular pattern with mesh size H in the x-direction, and h in the y-direction, see Figure 2. Consider linear finite element space S_H, the recovered gradient is given by

$$G_H v(0,0) = \begin{pmatrix} [2(v_1 - v_4) + v_2 - v_3 + v_6 - v_5)]/(6H) \\ [2(v_3 - v_6) + v_2 - v_1 + v_4 - v_5]/(6h) \end{pmatrix},$$

where

$$v_0 = (0,0), \quad v_1 = (H,0), \quad v_2 = v(H,h), \quad v_3 = v(0,h), \quad v_4 = v(-H,0), \quad \ldots$$

Given $v \in S_H$, we have

$$\frac{v_1 - v_4}{H} = \frac{v_1 - v_0}{H} + \frac{v_0 - v_4}{H} = \frac{\partial v}{\partial x}|_{\tau_6 \cup \tau_1} + \frac{\partial v}{\partial x}|_{\tau_3 \cup \tau_4}.$$

Similar results hold for other terms in $G_H v$. With some simple manipulation, we can verify that

$$G_H v(0,0) = \frac{1}{6} \sum_{j=1}^{6} \nabla v(\tau_j). \tag{3.2}$$

Therefore, G_H is a bounded operator in the sense that both inequalities in (3.1) have bounding constant $C = 1$. Clearly, C is not only independent of H, but also independent of h or the mesh aspect ratio.

Next we consider another anisotropic mesh as depicted in Figure 3. The horizontal edge length is H and the coefficients for G_H are marked. Again, it is straightforward to verify (3.2).

In both cases, some coefficients of G_H after scaling are unbounded at the anisotropic limit. They have a factor H/h in case (a), and $\cot\theta$ in case (b). Nevertheless, the resulting gradient recovery operator is still bounded with bounding constant $C = 1$.

Arbitrary mesh via perturbation. We are able to prove that the bounding constant equals 1 for some uniform meshes. Based on this, we can establish the bounded-ness of the recovery operator for meshes that distorted by a measure of ϵ.

The least-squares fitting involves a coordinate matrix A as demonstrated in Example 1. Now we decompose $A = A_0 + \epsilon A_1$, where A_0 is the coordinate matrix associated with a uniform or symmetry mesh. We see that A is perturbed from A_0 by a measure of ϵ. Here it is reasonable to assume that A_1 is a "well behaved" matrix. Then we are able to show that if A_0 has a full rank, then A_1 has a full rank for sufficiently small ϵ. Indeed, after some simple algebraic manipulation, we can show that

$$(A^T A)^{-1} A^T = (A_0^T A_0)^{-1} A_0^T + \epsilon E,$$

where it is possible to trace the dependent of E with respect to A_0 and A_1. Now, if the recovery operator associated with A_0 is given by

$$G_h^{\circ} v(\boldsymbol{z}_i) = \sum_{j=1}^{n} \vec{C}_j^{\circ} v(\boldsymbol{z}_{ij}), \quad \sum_{j=1}^{n} \vec{C}_j^{\circ} = \vec{0};$$

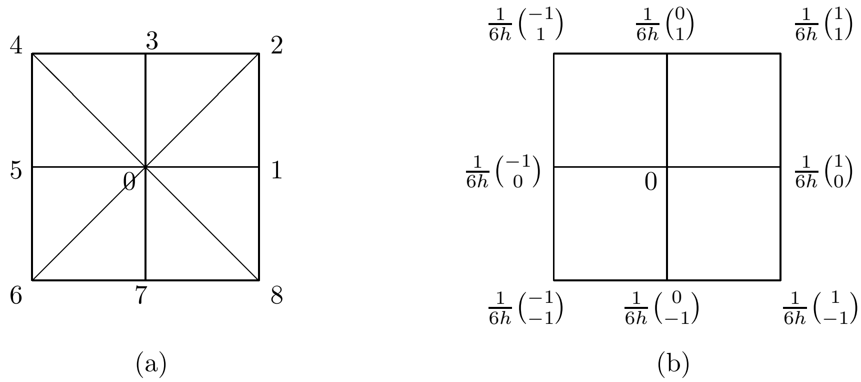


Figure 1: PPR is patch independent

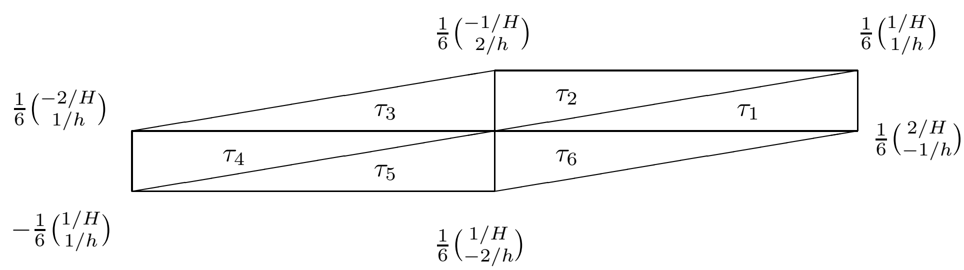


Figure 2: Anisotropic mesh (a)

then the recovery operator associated with A can be expressed as

$$G_h v(\boldsymbol{z}_i) = \sum_{j=1}^{n} (\vec{C}_j^{\circ} + \epsilon \vec{C}_j^{1}) v(\boldsymbol{z}_{ij}), \quad \sum_{j=1}^{n} \vec{C}_j^{1} = \vec{0}.$$

Therefore, the bounded-ness of G_h can be obtained from that of G_h° provided ϵ is sufficiently small.

Ultra-convergence for even-order elements. We shall prove an ultra-convergence result for the recovery operator G_h under translation invariant meshes.

As mentioned earlier, the selection of sampling points can be very flexible. As an example, we consider quadratic element under triangular mesh of the regular pattern. In order to recover the gradient at a diagonal edge center, we may choose sampling points as depicted in Figure 7(b), or choose only those diagonal edge centers as shown in Figure 7(a). If all sampling points $\boldsymbol{z}_{ij}$ are selected the same type as the assembly point $\boldsymbol{z}_i$ (Figure 7(a)), then the recovered gradient can be represented as a finite quotient in the following manner:

$$G_h u_h(\boldsymbol{z}) = \sum_{|\nu| \le M} \sum_{i} \vec{C}_{\nu,h}^{(i)} u_h(\boldsymbol{z} + \nu h l_i). \tag{3.3}$$

Now for these specially selected sampling points, we sketch a proof for ultra-convergence of order $2k$ ($k \ge 1$) elements on uniform triangular mesh of the regular pattern.

To fix the idea and to simplify matters, we consider a second-order elliptic equation with

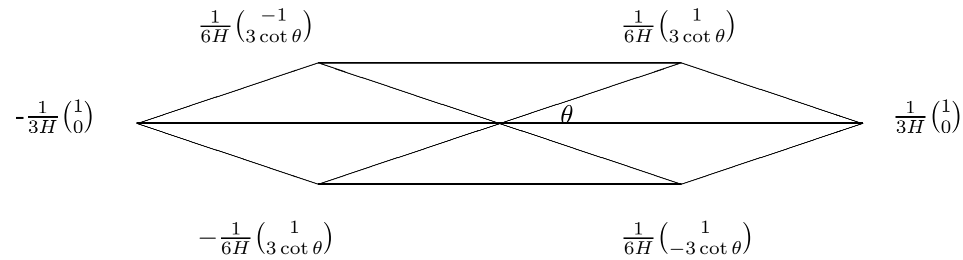


Figure 3: Anisotropic mesh (b)

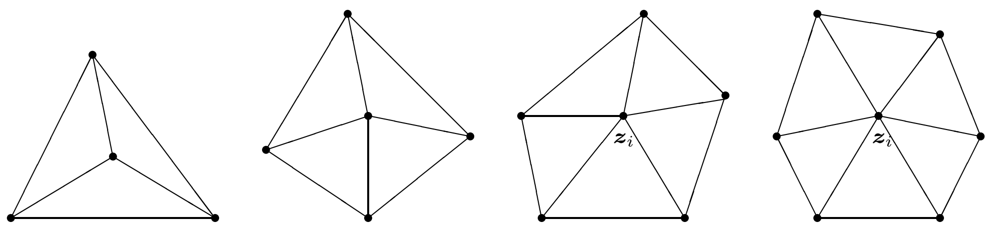


Figure 4: Internal patches

homogeneous Dirichlet boundary condition on a polygonal domain: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega [(A\nabla u + \boldsymbol{b}u)\cdot\nabla v + cuv] = f(v), \quad \forall v \in H_0^1(\Omega). \tag{3.4}$$

We assume that all the coefficient functions are smooth, A is a 2×2 symmetric positive definite matrix, $f(\cdot)$ is a linear functional, and the bilinear form is continuous and satisfies the inf-sup condition on $H^1(\Omega)$. These conditions insure that (3.4) has a unique solution.

We consider even-order finite element space. The finite element solution $u_h \in S^0_{h,2k}(\Omega)$ satisfies

$$a(u_h, v_h) = f(v_h) \quad \forall v_h \in S^0_{h,2k}(\Omega). \tag{3.5}$$

To insure a unique solution for (3.5), we further assume the inf-sup condition of $a(\cdot,\cdot)$ be satisfied on $S^0_{h,2k}(\Omega)$.

Set $w = G_h u$ and $w_h = G_h u_h$. Observe that the dual operator G_h^* is well defined in an interior sub-domain $D \subset \Omega$. Therefore, for any $v \in S^0_{h,2k}(D_0)$ with $D_0 \subset\subset D$ and $\mathrm{dist}(\partial D, D_0) \geq Mh$, we have

$$a(w - w_h, v) = a(u - u_h, G_h^* v) = 0,$$

Note that $G_h^* v \in S^0_{h,2k}(D)^d$ and $w_h \in S_{h,2k}$. By the theory for translation invariant meshes under maximum norm, we have the optimal convergent rate (see, e.g., [9])

$$(w - w_h)(\boldsymbol{z}) = O(h^{2k+1}).$$

If $\boldsymbol{z}$ is a symmetry point with $O(1)$ symmetry, the superconvergence occurs, and we actually have

$$G_h(u - u_h)(\boldsymbol{z}) = (w - w_h)(\boldsymbol{z}) = O(h^{2k+2}).$$

Now the question is whether w has the required regularity. At the first glance, $w = G_h u \in S^d_{h,2k}$. However, if we view it from the translation expression (3.3), we then have the required regularity as long as u is sufficiently smooth. Therefore, we have the following:

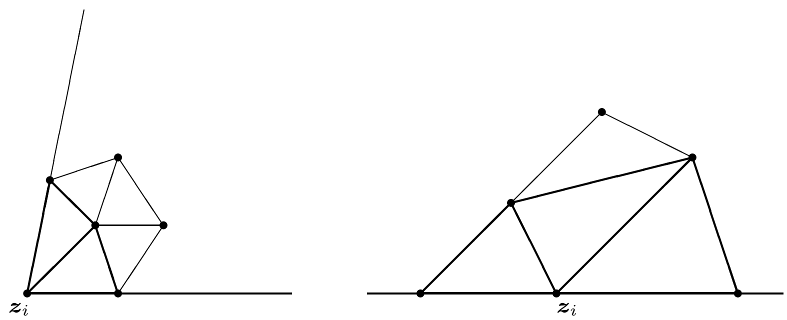


Figure 5: Most common boundary patches

Figure 6: Illegal boundary patches

Theorem 2. *Consider even-order finite element approximation for the second-order elliptic equation under uniform triangular mesh of the regular pattern. The recovered gradient by PPR is ultra-convergent at an interior nodal point.*

A rigorous proof follows the above argument with some interior analysis.

Corollary. *Under the same condition as in Theorem 2, PPR has a cubic convergence rate for quadratic element, locally or globally, as long as the regularity permits.*

Proof. By Theorem 2, PPR has a 4th-order convergent rate at interior nodes, which include vertices and edge centers. Interpolating at nodal points by the quadratic basis functions, we then obtain a global piece-wisely continuous gradient field with 3dr-order convergent rate by standard approximation theory.

Remark 2. In practice, the recovered gradient for quadratic element at an edge center is usually obtained by averaging the values from the two related vertex recoveries. The reader is referred to [14, 16] for details.

Unstructured grids by the Delaunay triangulation. There are two important ingredients in an automatic mesh generation code based on the Delaunay triangulation.

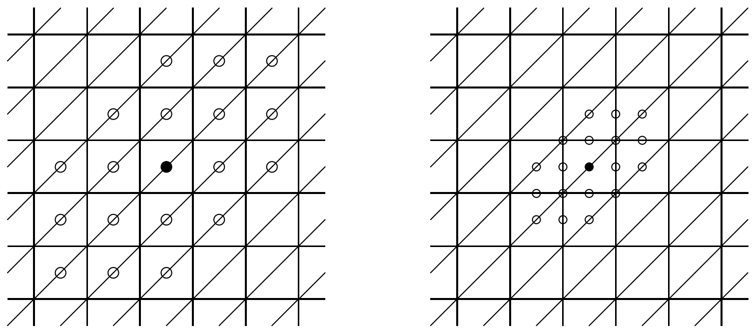
i) *Lagrange smoothing.* It places (iteratively) each node near a mesh symmetry center.

ii) *swap diagonal.* It changes the direction of some diagonal edges to near parallel directions for adjacent element edges and makes as many nodes as possible have six triangles attached.

A sophisticated automatic mesh generator makes every two adjacent triangles form an $O(h^{1+\alpha})$ ($\alpha > 0$) parallelogram, except a small portion of elements (including boundary elements). Therefore, we introduce the following mesh condition:

Definition. *The triangulation $\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$ is said to satisfy Condition (α, σ) if there exist positive constants α and σ such that: Every two adjacent triangles inside $\mathcal{T}_{1,h}$ form an $O(h^{1+\alpha})$ parallelogram and*

$$\bar{\Omega}_{1,h} \cup \bar{\Omega}_{2,h} = \bar{\Omega}, \quad |\Omega_{2,h}| = O(h^\sigma), \quad \bar{\Omega}_{i,h} \equiv \bigcup_{\tau \in \mathcal{T}_{i,h}} \bar{\tau}, \quad i = 1, 2.$$



(a) (b)

Figure 7: Sampling points for quadratic element

Theorem 3. *Consider an interior patch $\omega_z \subset\subset \Omega_d \subset \Omega_{1,h}$ with $d = dist(\omega_z, \partial\Omega_d) \geq Kh$ for some constant $K > 0$. Let $u \in W^2_\infty(\Omega) \cap W^3_\infty(\Omega_d)$ be the solution of (3.4), let u_h be the linear finite element approximation, and let G_h be a recovery operator defined by PPR. Then we have*

$$|(\nabla u - G_h u_h)(z)| \leq C(h^2\|u\|_{3,\infty,\omega_z} + d^{-1}h^2 \ln\frac{1}{h}\|u\|_{2,\infty,\Omega} + h^{1+\alpha}\ln\frac{d}{h}|u|_{2,\infty,\Omega_d}),$$

where C is a constant independent of u and h.

Proof. Let $u_I \in S^0_{h,1}(\Omega)$ be the linear interpolation of u. We have proved in [12],

$$|u_h - u_I|_{1,\infty,\omega_z} \leq C(d^{-1}h^2 \ln\frac{1}{h}\|u\|_{2,\infty,\Omega} + h^{1+\alpha}\ln\frac{d}{h}|u|_{2,\infty,\Omega_d}). \tag{3.6}$$

Now we decompose

$$(\nabla u - G_h u_h)(z) = (\nabla u - G_h u_I)(z) + G_h(u_I - u_h)(z). \tag{3.7}$$

The beauty of the polynomial preserving is that the first term on the right hand side is superconvergent unconditionally, i.e.,

$$|(\nabla u - G_h u_I)(z)| \leq Ch^2\|u\|_{W^3_\infty(\omega_z)}. \tag{3.8}$$

By the bounded-ness property and (3.6), we have

$$|G_h(u_I - u_h)(z)| \leq C|u_h - u_I|_{1,\infty,\omega_z} \leq C(d^{-1}h^2 \ln\frac{1}{h}\|u\|_{2,\infty,\Omega} + h^{1+\alpha}\ln\frac{d}{h}|u|_{2,\infty,\Omega_d}). \tag{3.9}$$

The conclusion follows by applying (3.8) and (3.9) to (3.7).

Remark 3. Choose $d = h^{1-\alpha}$ in Theorem 3, we see that the recovered gradient is superconvergent point-wisely with order $O(h^{1+\alpha})$, which is h^α better than the optimal global rate for the gradient approximation. Our numerical tests supported this theoretical result. When Delaunay triangulation are used, PPR always provides some order of superconvergence. For more details, see [14, 7].

Remark 4. Based on the superconvergence property

$$|(\nabla u - G_h u_h)(\boldsymbol{z})| \leq Ch^{1+\alpha},$$

we can design a point-wise error estimator

$$\eta_z = |(G_h u_h - \nabla u_h)(\boldsymbol{z})|,$$

which is asymptotically exact. In fact, it is straightforward to verify that

$$\frac{\eta_z}{|\nabla(u - u_h)(z)|} = 1 + O(h^\alpha),$$

under the non-saturation condition

$$|\nabla(u - u_h)(z)| \geq Ch.$$

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