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CONVERGENCE OF PARALLEL DIAGONAL ITERATION OF RUNGE-KUTTA METHODS FOR DELAY DIFFERENTIAL EQUATIONS *[1)]

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Abstract

Implicit Runge-Kutta method is highly accurate and stable for stiff initial value problem. But the iteration technique used to solve implicit Runge-Kutta method requires lots of computational efforts. In this paper, we extend the Parallel Diagonal Iterated Runge-Kutta(PDIRK) methods to delay differential equations(DDEs). We give the convergence region of PDIRK methods, and analyze the speed of convergence in three parts for the P-stability region of the Runge-Kutta corrector method. Finally, we analysis the speed-up factor through a numerical experiment. The results show that the PDIRK methods to DDEs are efficient.

Mathematics subject classification: 65F20.
Key words: Runge-Kutta method, Parallel iteration, Delay differential equation.

1. Introduction

We consider here a stiff initial value problem (IVP) method that is highly accurate and stable. This method is used as a corrector method, which achieves convergence by using parallel iteration techniques. In the selection of a suitable corrector method, we are automatically led to the classical implicit Runge-Kutta methods such as the Radau IIA methods. These methods fulfill the requirements of accuracy and stability and belong to the family of best correctors for stiff problems. For the iteration technique we select the PDIRK (Parallel Diagonally Implicit RK) approach developed in [1] that solves the RK corrector by diagonally implicit iteration using s processors for ODEs, where s being the number of stages of the corrector.

In this paper, we use a so-called step-parallel method. Here, a step-parallel method is understood to be a method that computes solution values at different points on the-axis simultaneously. Such methods are usually based on the iterative solution of an implicit step-by-step method. A further level of parallelism for ODE was introduced in [2,3,4] by making use of the PDIRK iteration technique. The conventional approach of iteration is that it iterates until convergence at a particular point is achieved, before advancing to the next point along the t-axis, while step-parallel methods already start the iteration process at the next point before the iteration at the preceding point converged. In the literature, we consider a step-parallel iteration of Runge-Kutta method for solving initial value problems (IVPs) of delay differential equations (DDEs):

$$y'(t) = f(y(t), y(t-\tau)), \quad t \geq 0, \quad y(t) = g(t), \quad t \leq 0, \tag{1.1}$$

where f, g denote given functions and are both sufficiently smooth. τ is a given constant with $\tau > 0$.

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2. The Iteration Scheme

Consider initial value problems of ordinary differential equations(ODEs):

$$y'(t) = f(y(t)), \quad t \geq 0, \quad y(0) = y_0. \tag{2.1}$$

To avoid the tensor product in our formulations, we consider equation(1.1) and (2.1) as scalar equations. Using the General Linear Method notation of Butcher, the Runge-Kutta corrector formula for the ODE (2.1) reads(cf.[5, 6]) as follows:

$$Y_n = EY_{n-1} + hAF(Y_n), \qquad n = 1, 2, \cdots, N, \tag{2.2}$$

here, $h(= t_n - t_{n-1})$ denotes the step-size, the matrix $A = (a_{ij})$ contains the RK parameters, and $F(Y_n)$ contains the derivative values $(f(Y_{n,i}))$, where $Y_{n,i}$, $i = 1, 2, \cdots, s$, denote components of the stage vector Y_n. In this paper we assume that (2.2) possesses s implicit stages and that the last stage corresponds to the step point. The first $s-1$ stage components represent numerical approximations at the intermediate points $t_{n-1} + c_i h$, $i = 1, 2, \cdots, s-1$, where $c = (c_i) = Ae$, $c_s = 1$, e being the vector with unit entries. We define $Y_0 = y_0 e$. The matrix E in (2.2) is of the form:

$$E = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Applying the method (2.2) to (1.1), we obtain the Runge-Kutta correction formula for DDEs:

$$Y_n = EY_{n-1} + hAF(Y_n, \gamma_n), \qquad n = 1, 2, \cdots, N. \tag{2.3}$$

Where $F(Y_n, \gamma_n)$ contains the derivative values $(f(Y_{n,i}, \gamma_{n,i}))$. If $\tau = mh$ with integer m, we let [6]

$$\gamma_n = \begin{cases} g(et_{n-1} + ch - e\tau), & n \leq m, \\ Y_{n-m}, & n > m. \end{cases} \tag{2.4}$$

We approximate the solution Y_n of (2.3),(2.4) by successive iterates $Y_n^{(j)}$ satisfying the iteration scheme:

$$\begin{aligned} & Y_n^{(0)} \ \textit{to be defined by the predictor formula}, \\ & Y_n^{(j)} - hDF(Y_n^{(j)}, \gamma_n) = EY_{n-1}^{(j)} + h[A - D]F(Y_n^{(j-1)}, \gamma_n), \qquad j = 1, 2, \cdots, T, \\ & \gamma_n = \begin{cases} g(et_{n-1} + ch - e\tau), & n \leq m, \\ Y_{n-m}^{(k(n)-m)}, & n > m, \end{cases} \\ & Y_n^{(j)} = Y_n^{(k(n))}, \qquad j > k(n), \qquad n = 1, 2, \cdots, N. \end{aligned} \tag{2.5}$$

The number of iterations $k(n)$ performed at the point t_n is defined by the condition that for $j = k(n)$, the $Y_n^{(j)}$s numerically satisfy the corrector equation (2.3),(2.4). The $k(n)$ depends on t_n (see(3.34)). But in a theoretical analysis, however, it seems not feasible to allow the parameter $k(n)$ to be an arbitrary function of n, so while deriving convergence results, $k(n)$ is taken as a constant. The matrix $D = (d_i)$ is assumed to be a diagonal matrix with s positive diagonal entries, so the formula (2.5) possesses parallelism across the method because of the diagonal structure of the matrix D. We also call the method (2.5) PDIRK method for DDEs.

Introducing the step index $i = n + j$, and writing the correction formula (2.5) as

$$Y_n^{(i-n)} - hDF(Y_n^{(i-n)}, Y_{n-m}^{(k)}) = EY_{n-1}^{(i-n)} + h[A - D]F(Y_n^{(i-n-1)}, Y_{n-m}^{(k)}). \tag{2.6}$$

If $N=\omega m+w$ with integers $\omega\geq 0$ and $w=1,2,\cdots,m$, the corresponding computational scheme can be implemented in accordance with the following scheme

$$\begin{array}{l}
\text{FOR } j:=1 \text{ TO } m\\
\quad \text{FOR } v:=0 \text{ TO } k\\
\quad\quad \text{LET } Y_{j-m}^{(v)}=g(et_{j-m-1}+ch)\\
\text{FOR } l:=0 \text{ TO } \omega\\
\quad \text{FOR } i:=lk+2 \text{ TO } (l+1)k+1\\
\quad\quad \text{FOR } n:=i-k \text{ TO } \min\{lm+(i-k),(l+1)m\}\\
\quad\quad\quad \text{CALL } \textit{correction subroutine}\\
\text{FOR } i:=(\omega+1)+2 \text{ TO } (\omega+1)k+w\\
\quad \text{FOR } n:=i-k \text{ TO } \min\{i,N\}\\
\quad\quad \text{CALL } \textit{correction subroutine}
\end{array} \tag{2.7}$$

The correction subroutine is defined by

$$\begin{array}{l}
\text{IF } i=n \text{ THEN } \textit{compute } Y_n^{(i-n)}=Y_n^{(0)} \textit{ by means of the predictor formula}\\
\quad \text{ELSE IF } n=0 \text{ THEN } Y_n^{(i-n)}=g(et_{n-1}+ch)\\
\quad\quad \text{ELSE } \textit{compute } Y_n^{(i-n)} \textit{ by means of the correction formula}
\end{array}$$

According to the computational scheme (2.7), we find that the method (2.5) also possesses parallelism across the step; the sequential costs are $N_{seq}=(\omega+1)k+w$ on sm processors (for fixed k). Indeed, if we restrict our considerations to predictor formulas that are equally expensive as the correction formula and if we assume that the iteration process at the end points of the integration subinterval is stopped only if iteration at preceding step points has converged, then the total sequential computational costs are now given by $N_{seq}=\sum_{i=1}^{\omega}k(im)+k(N)+w$.

3. Stability and Convergence

The region of convergence for the iteration method is discussed for the familiar basic test equation

$$y'(t)=\lambda y(t)+\mu y(t-\tau),\quad t>0,\quad y(t)=g(t),\quad t\leq 0. \tag{3.1}$$

With respect to this test equation, the stability properties of the predictor-corrector pair and convergence properties of the iteration process determine the stability properties of the iteration (2.5). Assuming that the underlying corrector (2.3),(2.4) is highly stable.

There are several possibilities in defining a predictor formula for PDIRK method (2.5). An imperialistically convenient choice is to define $Y_n^{(0)}$ as (3.2),here formula (3.2) is obtained by applying a backward differentiation formula(BDF) to the preceding iterates $Y_{n-1}^{(0)}$ and $Y_{n-m}^{(k)}$

$$Y_n^{(0)}-hDF(Y_n^{(0)},Y_{n-m}^{(k(n-m))})=E^*Y_{n-1}^{(0)}. \tag{3.2}$$

So that we can achieve predictor order $s-1$.If $E^*=E$,then this formula is only second-order accurate.

3.1. Region of Convergence of The Correction Formula

In this section, we derive the region of convergence for the recursion(2.5), which is applied to the test equation (3.1). Let us define the stage vector iteration errors

$$\varepsilon_n^{(j)}=Y_n^{(j)}-Y_n,\quad n=1,2,\cdots,N. \tag{3.3}$$

Subtracting (2.3),(2.4) from (2.5), we obtain the liner recursion

$$\varepsilon_n^{(j)} = K\varepsilon_{n-1}^{(j)} + Z\varepsilon_n^{(j-1)} + \beta M\varepsilon_{n-m}^{(j-1)}, \tag{3.4}$$

where the matrices K, Z and M are given by

$$K = [I - \alpha D]^{-1}E,$$

$$Z = \alpha[I - \alpha D]^{-1}[A - D],$$

$$M = [I - \alpha D]^{-1}A,$$

We study the convergence of the iteration error vectors

$$\varepsilon^{(j)} = (\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \cdots, \varepsilon_n^{(j)})^T. \tag{3.5}$$

In particular, we are interested in the speed of convergence of the error vectors as function of n. From the recursion (3.4), we can get

$$\varepsilon^{(k)} = Q(\alpha, \beta; k)\varepsilon^{(0)}, \tag{3.6}$$

where, the n-by-n block iteration matrix $Q(\alpha, \beta, k)$ is given by

$$Q(\alpha, \beta; k) = [I - \beta(I - H^k)(L - M_1)^{-1}M_2]^{-1}H^k \tag{3.7}$$

and

$$L = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -K & I & 0 & \cdots & 0 & 0 \\ 0 & -K & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & -K & I \end{pmatrix}, \tag{3.8}$$

$$M_1 = \begin{pmatrix} Z & 0 & 0 & \cdots & 0 & 0 \\ 0 & Z & 0 & \cdots & 0 & 0 \\ 0 & 0 & Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z & 0 \\ 0 & 0 & 0 & \cdots & 0 & Z \end{pmatrix}, \tag{3.9}$$

$$M_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ M & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & M & 0 & \cdots & 0 \end{pmatrix}, \tag{3.10}$$

$$H = L^{-1}M_1 = \begin{pmatrix} Z & 0 & 0 & \cdots & 0 & 0 \\ KZ & Z & 0 & \cdots & 0 & 0 \\ K^2Z & KZ & Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K^{n-2}Z & K^{n-3}Z & K^{n-4}Z & \cdots & Z & 0 \\ K^{n-1}Z & K^{n-2}Z & K^{n-3}Z & \cdots & KZ & Z \end{pmatrix}, \tag{3.11}$$

If $\rho(H) < 1$,we get $\lim_{k\to\infty} \varepsilon^{(k)} = (I-\beta[L-M_1]^{-1}M_2)^{-1} \cdot \lim_{k\to\infty} H^k\epsilon^{(0)} = 0$ and $\rho(H) = \rho(L^{-1}M_1) = \rho(Z) < 1$.This observation leads us to the convergence region $C(n,m) = \{(\alpha,\beta) \in C^2 : \rho(Z) < 1\}$, where $\rho(\bullet)$denotes the spectral radius function. Assuming that the underlying corrector(2.3), (2.4) is P-stable [7,8] (with respect to the basic test equation), that is, if S_p be the $P-$stability region of the underlying corrector (2.3),(2.4), we have

$$\Sigma_* = \{(\alpha,\beta) \in C^2 : Re(\alpha) + \mid \beta \mid < 0\} \sqsubseteq S_p, \qquad \alpha = h\lambda, \qquad \beta = h\mu. \tag{3.12}$$

If the corrector methods (2.3),(2.4) are based on the Radau IIA method, we know these methods are P-stable. We can also find that the region $\{(\alpha,\beta) \in C^2 : Re(\alpha) < 0$ is contained in $C(n,m)$, so the region Σ_* is also contained in $C(n,m)$. Hence, the Radau IIA based method (2.5) may be considered as "P-convergent".

3.2. Speed of Convergence

In order to get an insight into the convergence properties as a function of k and n, we need an estimate for the speed of convergence for the iteration process. In this paper, we adopt a definition as given in [9], where the average speed (or rate) of convergence for the recursion (3.6) is given by

$$R(n,k,\alpha,\beta) = -\log(\sqrt[k]{\|Q(\alpha,\beta;k)\|}). \tag{3.13}$$

Here we discuss the speed of convergence in three parts for the region Σ_* (3.2), in the first part non-stiff speed of convergence at the origin, in the second part stiff speed of convergence at infinity, and finally the speed of convergence at any intermediate point of Σ_*. In the region Σ_*, $\beta \to \infty$ implied $(\alpha,\beta) \to (\infty,\infty)$. So the stiff problem have three case as following:$I)(\alpha,\beta)$ satisfied the conditions $\alpha \to \infty$, β fixed; $II)$ $(\alpha,\beta) \to (\infty,\infty)$ and $\lim_{\beta\to\infty} \frac{\beta}{\alpha} = 0$; $III)$ $(\alpha,\beta) \to (\infty,\infty)$ and $\lim_{\beta\to\infty} \frac{\beta}{\alpha} = \theta, 0 < |\theta| < 1$.

At the origin, the matrix $Q(\alpha,\beta;k)$ can be approximated by

$$Q(\alpha,\beta;k) = \alpha^k[I-\beta J_1]^{-1}L_1^k + O((\alpha+\beta)^{k+1}), \tag{3.14}$$

where

$$L_1 = \begin{pmatrix} A-D & 0 & 0 & \cdots & 0 \\ S & A-D & 0 & \cdots & 0 \\ S & S & A-D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S & S & S & \cdots & A-D \end{pmatrix}, \tag{3.15}$$

$$S = E[A-D], \tag{3.16}$$

$$J_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ A & 0 & \cdots & 0 & 0 & \cdots & 0 \\ EA & A & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ EA & EA & \cdots & A & 0 & \cdots & 0 \end{pmatrix}, \tag{3.17}$$

and at infinity, we obtain

$$Q(\alpha,\beta;k) = [I + \frac{\beta}{\alpha}J_2]L_2^k + O(\alpha^{-1}), \tag{3.18}$$

$$L_2 = \begin{pmatrix} I-D^{-1}A & 0 & 0 & \cdots & 0 \\ 0 & I-D^{-1}A & 0 & \cdots & 0 \\ 0 & 0 & I-D^{-1}A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I-D^{-1}A \end{pmatrix}, \tag{3.19}$$

$$J_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & \cdots & & & \cdots & \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ I-[I-D^{-1}A]^k & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & I-[I-D^{-1}A]^k & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & \cdots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I-[I-D^{-1}A]^k & 0 & \cdots & 0 \end{pmatrix}. \tag{3.20}$$

In the following subsections, the maximum norm is used in the definition of $R(n,k,\alpha,\beta)$.
3.2.1 Convergence of nonstiff error components
At the origin, from (3.14) it follows that

$$R(n,k,\alpha,\beta) = -\log\left(\sqrt[k]{\|[\alpha^k[I-\beta J_1]^{-1}L_1^k\|} + O(|\alpha+\beta|^2)\right). \tag{3.21}$$

Theorem 3.1. *(a) For fixed values of n, the non-stiff speed of convergence of the method (2.5) satisfies the asymptotic relation*

$$R(n,k,\alpha,\beta) = -\log(|\alpha|\rho(A-D)) - O(k^{-1}\log(k)) \tag{3.22}$$

as $j \to \infty$ and $(\alpha,\beta) \to (0,0)$.
(b) If the matrix A and D satisfy the conditions $a_{ss} < d_s < 1$ and $a_{sj} > 0$ $(j = 1,2,\cdots,s)$, then for fixed k, the speed of convergence is given by,

$$R(n,k,\alpha,\beta) = -\log(n|\alpha|) - \log\left((1-d_s)\sqrt[k]{\frac{1-2a_{ss}+d_s}{k!(1-d_s)}} + O(n^{-1})\right) \tag{3.23}$$

as $n \to \infty$ and $(\alpha,\beta) \to (0,0)$.

Proof. The proof of part (a) is immediately follows from the asymptotic formula for the norm of powers of matrices (see e.g. [9]). About Assertion (3.23), from (3.14) and (3.21) it follows that

$$R(n,k,\alpha,\beta) = -\log|\alpha| - \log\left(\sqrt[k]{\|L_1^k\|_\infty} + O(|\alpha+\beta|)\right). \tag{3.24}$$

The following assertion (3.23) can be proved along the same lines of two theorems given in [2, 3] separately. It can be verified that

$$\|[\alpha L_1^k\|_\infty = \|Q_1^k\|_\infty + O(n^{j-1}) \quad \text{as} \quad n \to \infty, \tag{3.25}$$

where

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ S_1 & 0 & 0 & \cdots & 0 & 0 \\ S_1 & S_1 & 0 & \cdots & 0 & 0 \\ S_1 & S_1 & S_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_1 & S_1 & S_1 & \cdots & S_1 & 0 \end{pmatrix}, \tag{3.26}$$

$$S_1 = \alpha E[A-D]. \tag{3.27}$$

Also it can be verified easily that

$$\|Q_1^k\|_\infty = \frac{1}{k!}n^k\|S_1^k\|_\infty + O(n^{k-1}) \quad \text{as} \quad n \to \infty. \tag{3.28}$$

By observing that S_1 satisfies the recursion $S_1^k = |(1-d_s)\alpha|^{k-1}S_1$, and using the assumptions $a_{ss} < d_s < 1$ and $a_{sk} > 0$, we get

$$\|S_1^k\|_\infty = (1 - 2a_{ss} + d_s)(1 - d_s)^{k-1}|\alpha|^k. \tag{3.29}$$

From (3.18),(3.17),(3.16) and (3.13), we immediately get (3.15).

3.2.2 Convergence of stiff error components

At the infinity in the region Σ_*, from(3.21) it follows that

$$R(n,k,\alpha,\beta) = -\log(\sqrt[k]{\|[I - \frac{\beta}{\alpha}J_2]L_2^k\|_\infty} + O(\alpha^{-1})). \tag{3.30}$$

Theorem 3.2. *a) For fixed values of n, if $(\alpha,\beta) \in \Sigma*$ and $(\alpha,\beta) \to (\infty,\infty)$, then the speed of convergence is given by*

$$R(n,k,\alpha,\beta) = -\log(\rho(I - D^{-1}A)) - O(k^{-1}\log(k)) \quad as \quad k \to \infty. \tag{3.31}$$

b) For fixed values of k, if the (α,β) satisfy the conditions I) and II) or III), then the speed of convergence is given separately by

$$R(n,k,\alpha,\beta) = -\frac{1}{k}\log\|(I - D^{-1}A)^k\|_\infty, \quad as \quad n \to \infty, \tag{3.32}$$

or

$$R(n,k,\alpha,\beta) = -\frac{1}{k}(\log[1 + |\theta|\|I - (I - D^{-1}A)^k\|_\infty] + \log\|(I - D^{-1}A)^k\|_\infty) + O(\frac{1}{k}\log(|\theta|)), \quad as \quad n \to \infty. \tag{3.33}$$

Proof. At infinity, formulas (3.20) and (3.21) immediately follow from (3.12) and the asymptotic formula for the norm of powers of matrices [9].

About (3.33), from (3.18) we know that:

$$\begin{aligned} R(n,k,\alpha,\beta) &= -\log\sqrt[k]{\|[I - \theta J_2 + O(\theta)]L_2^k\|_\infty}) \\ &= -\frac{1}{k}(\log[1 + \theta\|I - (I - D^{-1}A)^k)\|_\infty] + \log\|(I - D^{-1}A)^k\|_\infty) \\ &\quad + O(\frac{1}{k}\log(|\theta|)) \end{aligned} \tag{3.34}$$

We remark that the stiff convergence speed of the PDIRK method (2.5) is not depend on n. When $|\theta| = 0$, this case can be considered as the case of ODEs, the convergence speed is determined by the values of $-\frac{1}{k}\log\|(I - D^{-1}A)^k\|_\infty$, but when $|\theta| \neq 0$, from (3.33) we can find that the delay has some influence on the speed of convergence for large values of n, that is the delay will decrease the convergence speed. Table 1 lists the values for a few values of k and $|\theta|$.

In an actual implementation, we must choose suitable matrices D. When $j \to \infty$, theorem 3.1 shows that $Q(\alpha,\beta,k)$ is already small for the nonstiff components, hence the possibility is the minimization of $Q(\alpha,\beta,k)$ for the highly stiff components (large values $|\alpha|$ of and $|\beta|$). According to the convergence region $C(n,m) = \{(\alpha,\beta \in C^2 : \rho(Z) < 1)\}$ and (3.31) of theorem 3.2, D is chosen such that the very stiff components in the iteration error are strongly damped. That is, D was chosen such that D minimizes $\rho(Z(-\infty)) = \rho(I - D^{-1}A)$, with $Z(z) = z(I - zD)^{-1}(A - D)$. In [1] diagonal matrices with $\rho(I - D^{-1}A) \approx 0$ were given for the two, three and four stage Radau IIA methods.

Table 1. Stiff $R(n,k,\alpha,\beta)$ values for the three-stage IIA corrector

$\lvert\theta\rvert$	k=1	k=2	k=4	k=8	k=16	k=32	k=∞
0	-0.62	-0.26	1.54	1.68	2.08	2.16	2.33
0.4	-0.95	-0.37	1.51	1.66	2.07	2.16	2.33
0.8	-1.14	-0.44	1.48	1.64	2.08	2.15	2.33

3.2.3 Convergence at Intermediate Values of (α,β)

The preceding subsections indicate that the stiff and nonstiff speeds of convergence of the method (2.5) are quite satisfactory, even for larger values of n. However, as soon as we move away from the origin or from infinity, the speed of convergence deteriorates. For the sake of simplicity, here, we suppose α and β as real, and the underlying corrector is the three-stage Radau IIA method iterated by means of the matrix $D = D_3$ as defined in [1]. In this case, the minimal rate of convergence was always found on the boundary of $\Sigma*$. Hence, for the test equation (3.1) we list the values of $\min\{R(n,k,\alpha,\beta) : \alpha + |\beta| = 0\}$ for $m = 4$ and for a few values of n in table 2.

Let the iteration error associated with $Y_i^{(k)}$, $i = 1, 2, \cdots, n$, be of magnitude $10^{-\Delta(k)}$(that is, the iterates $Y_i^{(k)}$ and the corrector solutions $Y_i^{(k)}, i \le n$, differ by $\Delta(k)$decimal digits). Then, taking logarithms to base 10, the number of iterations k needs to achieve this at most

$$k \approx \frac{\Delta(k) - \Delta(0)}{R(n,k,\alpha,\beta)} \tag{3.35}$$

Table 2 also lists the values of $k = k_\Delta(n)$ given by (3.35) with $\Delta(k) - \Delta(0) = 10$. Table 2 shows that the number of n varies from 4 to 5, we can find that the delay has some influence on the speed of convergence, and for large values of n, the speed of convergence only becomes positive if the numbers of iterations $k = k(n)$ are relatively large.

Finally, we remark that by means of the values of $k = k_\Delta(n)$ we can compute to get an estimate of the speed-up factor of the PDIRK method (2.5) with respect to the method introduced by [10]:

$$\begin{aligned} &Y_n^{(j)} - hDF(Y_n^{(j)}, \gamma_n) = EY_{n-1}^{(k(n))} + h[A - D]F(Y_n^{(j-1)}, \gamma_n) \qquad j = 1, 2, \cdots, k(n), \\ &\gamma_n = \begin{cases} g(et_{n-1} + ch - e\tau), & n \le m, \\ Y_{n-m}^{(k(n)-m)}, & n > m, \end{cases} \end{aligned} \tag{3.36}$$

By the method (3.36), the only possibility is to compute first the iterates Y_1^j, $j = 0, 1, 2, \cdots, k(1)$, next the iterates Y_2^j, $j = 0, 1, 2, \cdots, k(2)$, etc. Obviously, this method does not allow for parallelism across the steps. If the predictor formula defining $Y_n^{(0)}$ requires the same sequential costs as the correction formula in (3.36), then the sequential computational complexity of the method (3.36) is given by $N_{seq} = \sum_n k(n)$.

Setting $n = N$, the number of sequential iterations of the methods (2.5) and the method (3.36) are respectively given by $N_{seq} = \sum_{i=1}^{\omega} k_\Delta(im) + k_\Delta(N) + w$ and $N_{seq} = Nk_\Delta(1)$, resulting in the speed-up factor $S(N) = Nk_\Delta(1)[\sum_{i=1}^{\omega} k_\Delta(im) + k_\Delta(N) + w]^{-1}$.

Table 2. Value of $\min\{R(n,k,\alpha,\beta) : \alpha + |\beta| < 0\}$ and j_Δ for the three-stage IIA corrector

k	n=1	n=2	n=4	n=5	n=6	n=8
1	-0.62	-0.62	-0.62	-1.01	-1.05	-1.22
2	-0.26	-0.26	-0.26	-0.79	-0.83	-0.96
4	0.14	0.11	-0.05	-0.26	-0.31	-0.53
8	0.29	0.23	0.10	-0.10	-0.16	-0.39
16	0.48	0.40	0.22	-0.01	-0.04	-0.21
32	0.72	0.67	0.38	0.19	0.15	-0.11
44	0.81	0.78	0.56	0.35	0.31	0.24
$k_\Delta(n)$	20	23	30	34	37	43
S(n)	1	1.6	2.4	1.6	1.8	2.1

4. Numerical Experiment

The PDIRK method (2.5) described above was applied by using the three-stage Radau IIA corrector equation and the predictor formula (3.2). Since the number $k(n)$ of outer iterations needed to solve the corrector equation will strongly depend on n, we applied a dynamic iteration strategy with stopping criterium (cf.[2])

$$\Delta_n^{(j)} = \frac{\|e_s^T E(Y_n^{(j-1)} - Y_n^{(j)})\|_1}{\|e_s^T E Y_n^{(j-1)}\|_1} \leq Tol_{corr} \tag{4.1}$$

In our experiment, we set $Tol_{corr} = 10^{-12}$, and the calculations were performed using 15-digits arithmetic. The accuracy is given by the number of correct digits Δ and obtained by writing the maximum norm of the absolute error at the endpoint in the form $10^{-\Delta}$.

4.1. Comparison of The PDIRK (2.5) and The Method (3.36)

Our test problem is that of Hairer, $N\phi sett$ and Wanner [6]:

$$\begin{aligned} y'(t) &= (1.4 - y(t-1))y(t), \quad 0 < t \leq 2, \\ y(t) &= 0, \quad -1 \leq t \leq 0, \quad y(0) = 0.1. \end{aligned} \tag{4.2}$$

In our test, we compare results obtained by the PDIRK method (2.5) and the method (3.36). We apply the PDIRK method in unlimited-of-Processors mode and in one-processor mode (by which we generate the method (3.36)). The sequential computational complexity is measured by total number $N_{seq} = \sum_{i=1}^{\omega} k_\Delta(im) + k_\Delta(N) + w$ of sequential implicit systems to be solved during the integration process. Furthermore, we define the average number of iterations per step and the average number of sequential iterations per step by $k^* := N^{-1}\sum_n k(n)$ and $k^*_{seq} := N^{-1}N_{seq}$, respectively. For the method (3.36), we obviously have $N_{seq} = \sum_n k(n)$ and $k^* = k^*_{seq} = N^{-1}N_{seq}$. The ratio of the values of k^*_{seq} for the PDIRK method and the method (3.36) determines the speed-up factor $S(N)$. Table 3 presents multi-processor results for the PDIRK method (2.5) and the speed-up factor $S(N)$. From these results, we conclude that the PDIRK method becomes more efficient as the number of step points increases and the speed-up factors $S(N)$ are in good agreement with the theoretical speed-up factors listed in table 2.

Table 3. Results for the problem (4.2)

$h=\frac{1}{m}$	N	Δ	k^*	N_{seq}	k^*_{seq}	$S(N)$
1	2	6.17	16.0	16	16.0	1.00
1/2	4	6.90	18.8	23	17.9	1.05
1/4	8	7.88	23.2	42	11.4	2.04
1/8	16	9.20	25.7	58	9.10	2.83

5. Conclusion

In this paper, we firstly give a class of PDIRK method (2.5) for solving DDEs. It can be considered as an extension of the PDIRK method for solving ODEs. Secondly, the region of convergence was presented and the speed of convergence was separately analyzed for tow cases: at the region (nonstiff) and at infinity (stiff) of the region of the $P-$stability of the RK corrector method. It was found that the diagonal matrix D of the PDIRK methods for DDEs can be chosen in the same way as the PDIRK methods for ODEs[1]. But for the stiff IVPs of DDEs, from (3.33) of theorem3.2 we can find that the stiff speed of convergence of the method (2.5) depends on the value of $|\theta|$. We must increase the numbers of iteration to get the satisfactory value of the DDEs of the case III).

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