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ASYMPTOTIC STABILITY PROPERTIES OF $\theta - METHODS$ FOR THE MULTI-PANTOGRAPH DELAY DIFFERENTIAL EQUATION *1)

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Abstract

This paper deals with the asymptotic stability analysis of $\theta - methods$ for multi-pantograph delay differential equation

$$\begin{cases} u'(t) = \lambda u(t) + \sum_{i=1}^{l} \mu_i u(q_i t), \quad 0 < q_l < q_{l-1} < \cdots < q_1 < 1, \\ u(0) = u_0. \end{cases}$$

Here $\lambda, \mu_1, \mu_2, \cdots, \mu_l, u_0 \in C$.

In recent years stability properties of numerical methods for this kind of equation has been studied by numerous authors. Many papers are concerned with meshes with fixed stepsize. In general the developed techniques give rise to non-ordinary recurrence relation. In this work, instead, we study constrained variable stpesize schemes, suggested by theoretical and computational reasons, which lead to a non-stationary difference equation. A general theorem is presented which can be used to obtain the characterization of the stability regions of $\theta - methods$.

Mathematics subject classification: 65H10.
Key words: $\theta - methods$, Asymptotic stability, Multi-pantograph delay differential equation.

1. Introduction

Delay differential equations (DDEs) have a wide range of application in applied sciences. Recent studies in diverse fields biology, economy, control and electrodynamics (see for examples[1, 11]) have shown that DDEs play an important role in explaining many different phenomena. In particular they turn out to be fundamental when ODEs-based model fail. DDEs have been studied by many authors who have investigated both their analytical and numerical aspects [2][4][8][12].

The general functional differential equation is given by

$$u'(t) = f(t, u(t), u(\alpha_1(t)), u(\alpha_2(t)), \cdots, u(\alpha_l(t))).$$

A classical case that is the subject of a lot of papers is the following:

$$\alpha_i(t) = t - \tau_i, i = 1, 2, \ldots,$$

where τ_i is a positive constant [6] [7][10]. Another interesting case which is far different from the previous is that the pantograph equation:

$$\begin{cases} u'(t) = f(t, u(t), u(q_1 t), u(q_2 t), \cdots, u(q_l t)), t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

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where f is a given function and $0 < q_l < q_{l-1} < \cdots < q_1 < 1$, whereas $u(t)$ is unknown for $t > 0$.

There are many applications for (1.1), for instance, in number theory, in electrodynamics and in the collection of current by the pantograph of an electric locomotive, in nonlinear dynamical systems [5][11].

From a numerical point of view, it is important to study the potential of numerical methods in preserving the qualitative behavior of the analytical solutions. In paper [3] A. bellen and N. Guglielmi investigate the stability properties of $\theta - method$ when it is applied to the following pantograph test equation:

$$\begin{cases} u'(t) = \lambda u(t) + \mu u(qt), t > 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $\lambda, \mu, u_0 \in C$ and $0 < q < 1$.

In this paper, we study the stability properties of $\theta - methods$ when they are applied to the multi-pantograph test equation:

$$\begin{cases} u'(t) = \lambda u(t) + \sum\limits_{i=1}^{l} \mu_i u(q_i t), \quad 0 < q_l < q_{l-1} < \cdots < q_1 < 1, \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Here $\lambda, \mu_1, \mu_2, \cdots, \mu_l, u_0 \in C$.

In section 2 we provided the discretization scheme by applying $\theta - methods$, whose stepsize increase geometrically, to the pantograph equation (1.3).

In section 3 we recall the results concerning the asymptotic stability for the analytical solution of (1.3) and introduce the numerical stability framework. We present the results concerning the stability analysis of $\theta - methods$.

In section 4 we give some numerical experiments to show the asymptotic stability and convergencethe of $\theta - methods$.

2. θ-methods

A. Bellen, N. Guglielmi and L. Torelli described in detail the discretization scheme and constrained global mesh in [3]. We quote their description in the present paper.

Since we are interested in the asymptotic behavior of numerical solution of Eq.(1.3), we suppose to have the numerical solution available till the point $T_0 > 0$.

Firstly we build a primary mesh based on the following relation:

$$T_k = \frac{1}{q_1} T_{k-1}, k = 1, 2, \ldots .$$

In this way we define the primary intervals

$$H_k := T_k - T_{k-1} = \frac{1 - q_1}{q_1^k} T_0, k = 1, 2, \ldots . \tag{2.1}$$

Observe that the sequence increases exponentially. So we define the global mesh H by partitioning every primary interval into a fixed number m of subintervals of the same size. We set

$$h_{n+1} = \frac{H_{[n/m]+1}}{m} = \frac{1}{m} \frac{1 - q_1}{q_1^{[n/m]+1}}, n = 0, 1, 2, \ldots . \tag{2.2}$$

Here $[n/m]$ denotes integer part of n/m.

From (2.2) we have that

$$q_1 h_n = h_{n-m}, n > m.$$

Here for simplicity (but without any loss of generality), we have assumed $t_0 = T_0 = 1$. With $k = n \ mod \ m$, we are now in a position to define the grid points of constrained global mesh H,

$$t_n := T_{[n/m]} + k h_n, n = 1, 2, \ldots . \tag{2.3}$$

More directly, the global mesh point are defined by the recursion formulation

$$t_n := q_1^{-1} t_{n-m}, n > m. \tag{2.4}$$

Since $0 < q_l < q_{l-1} < \cdots < q_1 < 1$, there exists $s_i \in N$, $i = 1, 2, \ldots, l$, such that

$$q_1^{s_i+1} \le q_i \le q_1^{s_i}, i = 1, 2, \ldots, l. \tag{2.5}$$

Furthermore

$$t_{n-(s_i+1)m} \le q_i t_n \le t_{n-s_i m}. \tag{2.6}$$

Let

$$\begin{aligned} \delta_i &= \frac{q_i t_n - t_{n-(s_i+1)m}}{t_{n-s_i m} - t_{n-(s_i+1)m}} = \frac{q_i t_n - q_1^{s_i+1} t_n}{q_1^{s_i} t_n - q_1^{s_i+1} t_n} \\ &= \frac{q_i - q_1^{s_i+1}}{q_1^{s_i} - q_1^{s_i+1}} \end{aligned} \tag{2.7a}$$

and

$$\gamma_i = 1 - \delta_i, i = 1, 2, \ldots, l. \tag{2.7b}$$

Then δ_i, γ_i, $i = 1, 2, \ldots, l$ are constants and $0 \le \delta_i \le 1$, $0 \le \gamma_i \le 1$.

Now we consider the adaptation of $\theta - method$ to (1.1).

$$\begin{aligned} u_{n+1} &= u_n + h_{n+1} \left(\theta f(t_{n+1}, u_{n+1}, u^h(q_1 t_{n+1}), \cdots, u^h(q_l t_{n+1})) \right. \\ &\quad \left. +(1-\theta) f(t_n, u_n, u^h(q_1 t_n), \cdots, u^h(q_l t_n))\right). \end{aligned} \tag{2.8}$$

Here $u^h(t)$ is the continuous extension of discrete numerical solution u_n and

$$u^h(q_i t_n) = \delta_i u_{n-s_i m} + \gamma_i u_{n-(s_i+1)m}, i = 1, 2, \ldots, l. \tag{2.9}$$

Applying (2.8) and (2.9) to (1.3), we obtain that

$$\begin{aligned} u_{n+1} &= u_n + \theta h_{n+1} \left(\lambda u_{n+1} + \sum_{i=1}^{l} \mu_i (\delta_i u_{n+1-s_i m} + \gamma_i u_{n+1-(s_i+1)m})\right) \\ &\quad +(1-\theta) h_{n+1} \left(\lambda u_n + \sum_{i=1}^{l} \mu_i (\delta_i u_{n-s_i m} + \gamma_i u_{n-(s_i+1)m})\right). \end{aligned} \tag{2.10}$$

From (2.10) we have

$$\begin{aligned} u_{n+1} &= \frac{1 + (1-\theta)\lambda h_{n+1}}{1 - \theta\lambda h_{n+1}} u_n + \sum_{i=1}^{l} \left(\frac{\theta \mu_i \delta_i h_{n+1}}{1 - \theta\lambda h_{n+1}} u_{n+1-s_i m} + \frac{(1-\theta)\mu_i \delta_i h_{n+1}}{1 - \theta\lambda h_{n+1}} u_{n-s_i m} \right) \\ &\quad + \sum_{i=1}^{l} \left(\frac{\theta \mu_i \gamma_i h_{n+1}}{1 - \theta\lambda h_{n+1}} u_{n+1-(s_i+1)m} + \frac{(1-\theta)\mu_i \gamma_i h_{n+1}}{1 - \theta\lambda h_{n+1}} u_{n-(s_i+1)m} \right). \end{aligned} \tag{2.11}$$

For simplicity, we only consider the case that $n+1-s_1 m > n - s_1 m > n+1-s_2 m > n - s_2 m > \cdots > n+1-s_l m > n - s_l m$. In other case the proof is analogous. Let $U_n = (u_n, u_{n-1}, \cdots, u_{n-(s_l+1)m})^T$. From process (2.11) we have that

$$U_{n+1} = A_n U_n, \tag{2.12}$$

where $A_n = (a_{ij}^n)$ is $((s_l+1)m+1) \times ((s_l+1)m+1)$ matrix and

$$a_{ij}^n = \begin{cases} \dfrac{1+(1-\theta)\lambda h_{n+1}}{1-\theta\lambda h_{n+1}}, & i=1, j=1, \\ \dfrac{\theta\mu_k\delta_k h_{n+1}}{1-\theta\lambda h_{n+1}}, & i=1, j=s_k m, k=1,\ldots,l, \\ \dfrac{(1-\theta)\mu_k\delta_k h_{n+1}}{1-\theta\lambda h_{n+1}}, & i=1, j=s_k m+1, k=1,\ldots,l, \\ \dfrac{\theta\mu_k\gamma_k h_{n+1}}{1-\theta\lambda h_{n+1}}, & i=1, j=(s_k+1)m, k=1,\ldots,l, \\ \dfrac{(1-\theta)\mu_k\gamma_k h_{n+1}}{1-\theta\lambda h_{n+1}}, & i=1, j=(s_k+1)m+1, k=1,\ldots,l, \\ 1, & i=j+1, j=1,2,\ldots,(s_l+1)m, \\ 0, & others. \end{cases} \tag{2.13}$$

3. Stability Analysis

We shall assess the stability of process (2.8) by analyzing their stability behavior of the numerical solution of test problem (1.3). Using the method of paper [9], we can prove that the solution $u(t)$ of equation (1.3) tends to zero as $t \to \infty$, for all $u_0 \in C$ and all $0 < q_l < q_{l-1} < \cdots < q_1 < 1$, if

$$Re\lambda < 0, \sum_{k=1}^{l} |\mu_k| < |\lambda|. \tag{3.1}$$

We define

$$S = \left\{ (\lambda, \mu_1, \mu_2, \cdots, \mu_l) \in C^{l+1} \,\middle|\, Re\lambda < 0, \sum_{k=1}^{l} |\mu_k| < |\lambda| \right\}. \tag{3.2}$$

Definition 3.1. *Let $0 < q_l < q_{l-1} < \cdots < q_1 < 1$ and $H = \{t_0, t_1, \cdots, t_n, \cdots\}$ an assigned mesh. The numerical method is called asymptotic stability at $(\lambda, \mu_1, \mu_2, \cdots, \mu_l)$ if any application of the method to the problem (1.3) generates numerical approximations u_n that tend to zero as $n \to \infty$. The subset $S(H) \subseteq C^{l+1}$ consisting of all pairs $(\lambda, \mu_1, \mu_2, \cdots, \mu_l)$ at which the numerical method is asymptotically stable is called its H-stability region.*

Definition 3.2. *A numerical method is said to be H-stable if*

$$S(H) \supseteq S,$$

for any $t_0 \in R^+, m \in Z^+, 0 < q_l < q_{l-1} < \cdots < q_1 < 1$ and corresponding constrained mesh H defined by (2.3).

In the following part, we denote the determinant of any matrix M by $det(M)$, the set of eigenvalues by $\sigma(M)$ and the spectral-radius by $\rho(M)$, let I_s denote $s \times s$ identity matrix and $e = (1, 1, \cdots, 1)^T \in R^s$.

In order to derive the conditions for the H-stability of $\theta - method$ (2.8), we give the following lemmas.

Lemma 3.1. *Let the matrix A_n be given by (2.13). Then there exists a matrix $A = (a_{ij})$ such that $A_n \to A$, as $n \to \infty$.*

Proof. Since

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1+(1-\theta)\lambda h_{n+1}}{1-\theta\lambda h_{n+1}}=-\frac{1-\theta}{\theta}, && \\
&\lim_{n\to\infty}\frac{\theta\mu_k\delta_k h_{n+1}}{1-\theta\lambda h_{n+1}}=-\frac{\mu_k\delta_k}{\lambda}, && k=1,\dots,l,\\
&\lim_{n\to\infty}\frac{(1-\theta)\mu_k\delta_k h_{n+1}}{1-\theta\lambda h_{n+1}}=-\frac{\mu_k\delta_k}{\lambda}\frac{1-\theta}{\theta}, && k=1,\dots,l,\\
&\lim_{n\to\infty}\frac{\theta\mu_k\gamma_k h_{n+1}}{1-\theta\lambda h_{n+1}}=-\frac{\mu_k\gamma_k}{\lambda}, && k=1,\dots,l,\\
&\lim_{n\to\infty}\frac{(1-\theta)\mu_k\gamma_k h_{n+1}}{1-\theta\lambda h_{n+1}}=-\frac{\mu_k\gamma_k}{\lambda}\frac{1-\theta}{\theta}, && k=1,\dots,l,
\end{aligned}
\tag{3.3}
$$

let

$$
a_{ij}=\begin{cases}
-\dfrac{1-\theta}{\theta}, & i=1,j=1,\\
-\dfrac{\mu_k\delta_k}{\lambda}, & i=1,j=s_km,k=1,\dots,l,\\
-\dfrac{\mu_k\delta_k}{\lambda}\dfrac{1-\theta}{\theta}, & i=1,j=s_km+1,k=1,\dots,l,\\
-\dfrac{\mu_k\gamma_k}{\lambda}, & i=1,j=(s_k+1)m,k=1,\dots,l,\\
-\dfrac{\mu_k\gamma_k}{\lambda}\dfrac{1-\theta}{\theta}, & i=1,j=(s_k+1)m+1,k=1,\dots,l,\\
1, & i=j+1,j=1,2,\dots,(s_l+1)m,\\
0, & others.
\end{cases}
\tag{3.4}
$$

The lemma 3.1 can be proved.

Lemma 3.2. *Let matrix A be given by (3.4). If $\theta>\frac{1}{2}$ and $\sum_{i=1}^{l}|\mu_i|<|\lambda|$, then $\rho(A)<1$.*

Proof. It is easy to deduce that the characteristic polynomial of matrix A is

$$
\begin{aligned}
p(x)\ &= x^{(s_l+1)m+1}+\tfrac{1-\theta}{\theta}x^{(s_l+1)m}+\sum_{i=1}^{l}\left(\tfrac{\mu_i\delta_i}{\lambda}x^{(s_l-s_i+1)m+1}+\tfrac{\mu_i\delta_i}{\lambda}\tfrac{1-\theta}{\theta}x^{(s_l-s_i+1)m}\right)\\
&\quad+\sum_{i=1}^{l}\left(\tfrac{\mu_i\gamma_i}{\lambda}x^{(s_l-s_i)m+1}+\tfrac{\mu_i\gamma_i}{\lambda}\tfrac{1-\theta}{\theta}x^{(s_l-s_i)m}\right)\\
&=(x+\tfrac{1-\theta}{\theta})\left(x^{(s_l+1)m}+\sum_{i=1}^{l}\tfrac{\mu_i\delta_i}{\lambda}x^{(s_l-s_i+1)m}+\sum_{i=1}^{l}\tfrac{\mu_i\gamma_i}{\lambda}x^{(s_l-s_i)m}\right).
\end{aligned}
\tag{3.5}
$$

It is immediate to observe that one simple root of $p(x)$ is given by

$$x_1=\frac{\theta-1}{\theta}, \tag{3.6}$$

if $\theta>\frac{1}{2}$, then $|x_1|<1$ and the remaining roots satisfy the equation

$$x^{(s_l+1)m}+\sum_{i=1}^{l}\frac{\mu_i\delta_i}{\lambda}x^{(s_l-s_i+1)m}+\sum_{i=1}^{l}\frac{\mu_i\gamma_i}{\lambda}x^{(s_l-s_i)m}=0. \tag{3.7}$$

Let

$$g(x)=x^{(s_l+1)m}+\sum_{i=1}^{l}\frac{\mu_i\delta_i}{\lambda}x^{(s_l-s_i+1)m}+\sum_{i=1}^{l}\frac{\mu_i\gamma_i}{\lambda}x^{(s_l-s_i)m}$$

and

$$f(x) = x^{(s_l+1)m}.$$

From $\sum_{i=1}^{l} |\mu_i| < |\lambda|$, $\delta_i \geq 0, \gamma_i \geq 0$ and $\delta_i + \gamma_i = 1$, $i = 1, 2, \ldots, l$, if $|x| = 1$, then

$$|f(x) - g(x)| \leq \sum_{i=1}^{l} \frac{|\mu_i|\delta_i}{|\lambda|} + \frac{|\mu_i|\gamma_i}{|\lambda|} = \sum_{i=1}^{l} \frac{|\mu_i|}{|\lambda|} < 1 = |f(x)|. \tag{3.8}$$

By Rouché Theorem, we have that $f(x)$ and $g(x)$ have the same number of zeros inside the unit circle. It is observed that $f(x)$ has $(s_l + 1)m$ zeros inside the unit circle. So $g(x)$ also has $(s_l + 1)m$ zeros inside the unit circle. Hence all roots of characteristic polynomial of matrix A have modulus less than 1, which means that $\rho(A) = c < 1$. The proof is completed.

In following part we shall give the sufficient and necessary condition which assure that $\theta - method$ are $H - stable$.

Theorem 3.1. *The $\theta -$ methods are H-stable, if and only if $\theta > \frac{1}{2}$.*

Proof. The equation (2.12) can be written as

$$U_{n+1} = (A + A_n - A)U_n. \tag{3.9}$$

From $\rho(A) = c < 1$, for $\frac{1-c}{4}$ there exists a norm $\|\cdot\|_*$ such that

$$\|A\|_* < c + \frac{1-c}{4}. \tag{3.10}$$

According to Lemma 3.1, for sufficient large N, if $n > N$, then

$$\|A_n - A\|_* < \frac{1-c}{4}. \tag{3.11}$$

From (3.10) and (3.11), we can obtain that

$$\|A_n\|_* \leq \|A_n - A\|_* + \|A\|_* < \frac{1-c}{4} + \frac{1+3c}{4} = \frac{1+c}{2} < 1, \tag{3.12}$$

if $n > N$.

This means that

$$\lim_{n\to\infty} U_n = 0.$$

The sufficient condition is proved.

In the following part we give the proof of necessary conditions. Suppose that

$$\lim_{n\to\infty} u_n = 0. \tag{3.13}$$

We focus attention to the case that $\mu_i = 0, i = 1, 2, \ldots, l$, and $Re\lambda < 0$. Clearly, these satisfy condition (3.1). For this particular case, Eq.(2.11) reads

$$u_{n+1} = (\beta_1 + \beta_2(n))u_n,$$

where

$$\beta_1 = -\frac{1-\theta}{\theta}, \beta_2(n) = \frac{1}{\theta(1 - \theta\lambda h_{n+1})}.$$

We choose λ in the way that

$$1 + (1-\theta)\lambda h_{n+1} \neq 0, \forall n \geq 0,$$

which means that $\beta_1 + \beta_2(n) \neq 0$ for any $n \geq 0$. We firstly consider the case $\theta < \frac{1}{2}$. Observe that

$$\theta < \frac{1}{2} \Longleftrightarrow |\beta_1| > 1.$$

Since $\lim_{n\to\infty} \beta_2(n) = 0$, there exists $\bar{n}$ such that

$$\theta < \frac{1}{2} \Rightarrow |u_{n+1}| > |u_n|, \forall n > \bar{n},$$

which contradicts (3.13).

Consider the case $\theta = \frac{1}{2}$. In this case it holds

$$u_{n+1} = (-1 + \beta_2(n))u_n.$$

Since

$$\beta_2(n) = O(q_1^{[n/m]+1}),$$

u_n is asymptotically bounded, but does not vanish. The necessary condition is proved.

4. Numerical Test

In this section we give several numerical examples to illustrate the properties of the methods (2.8) and all of them we performed on the computer using Matlab 6.0 with double precision.

Example 4.1. We consider the following equation

$$u^{'}(t) = -3u(t) + 0.4u(0.4t) + u(0.1t), u(0) = 2.7. \quad (4.1)$$

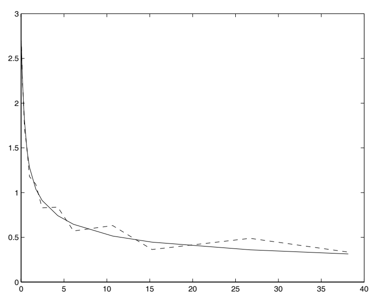

Fig.1

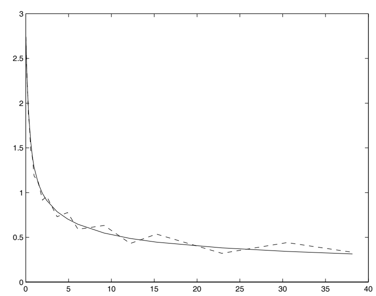

Fig.2

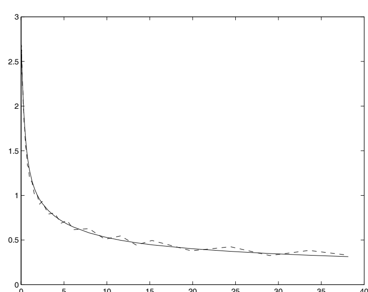

Fig.3

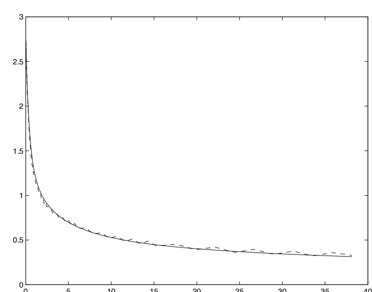

Fig.4

We obtain the exact solution of equation (4.1) using Dirichlet series (see [9]). Using the methods (2.8) with $\theta = 0.8$ and letting $m = 2, 3, 5, 10$, we give numerical solutions of equation (4.1). In figures 1-4, the solid line and dashed line denote exact solution and numerical solution, respectively. From Figures 1-4, it can be easily observed that $u_n \to u(t_n)$ as $m \to \infty$ and $u_n \to \infty$, as $n \to 0$.

Example 4.2. We consider the equation

$$u^{'}(t) = -u(t) + \mu_1(t)u(0.5t) + \mu_2(t)u(0.25t), u(0) = 1. \quad (4.2)$$

Here $\mu_1(t) = -e^{-0.5t}sin(0.5t)$, $\mu_2(t) = -2e^{-0.75t}cos(0.5t)sin(0.25t)$. It can be seen that the exact solution of equation (4.2) is $u(t) = e^{-t}cos(t)$. Using the method (2.8) with $\theta = 0.7$, we obtain the numerical solution for $m = 2, 4, 8, 10$, respectively. From these numerical solution, we also found that $u_n \to u(t_n)$, as $m \to \infty$. In fig 5-8 we give the exact solution and numerical solutions corresponding.

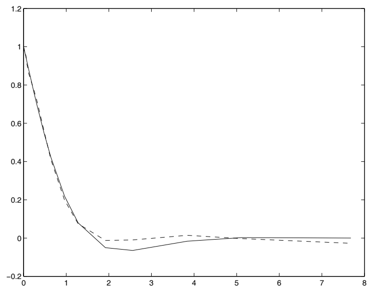

Fig.5

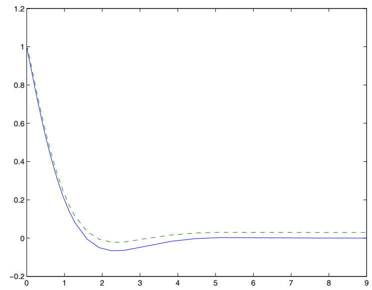

Fig.6

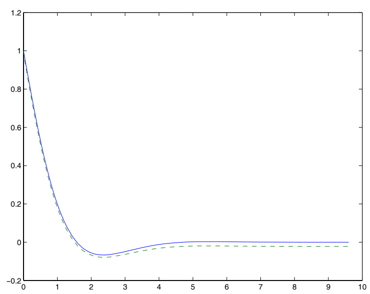

Fig.7

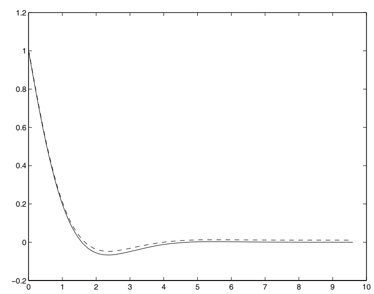

Fig.8

These numerical experiment show that the method (2.8) is asymptotic stability and convergence.

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