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RECONSTRUCTION OF SCATTERED FIELD FROM FAR-FIELD BY REGULARIZATION *1)

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Abstract

In this paper, we consider an inverse scattering problem for an obstacle $D \subset \mathcal{R}^2$ with Robin boundary condition. By applying the point source, we give a regularizing method to recover the scattered field from the far-field pattern. Numerical implementations are also presented.

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1. Introduction

Let $D \subset \mathcal{R}^2$ be a simply connected domain with C^2 boundary ∂D. The scattering of time-harmonic acoustic plane wave for the obstacle D with impedance boundary condition is modeled by an exterior boundary value problem for the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx\cdot d}$, $d \in \Omega = \{\xi \in \mathcal{R}^2 : |\xi| = 1\}$, the total wave field $u = u^i + u^s \in H^1_{loc}(\mathcal{R}^2 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathcal{R}^2 \setminus \overline{D} \\ \frac{\partial u}{\partial \nu} + ik\sigma(x)u = 0, & \text{on } \partial D \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{\sqrt{r}}\right), & r = |x| \longrightarrow \infty \end{cases} \tag{1.1}$$

where ν is the unit normal vector of ∂D directed into the exterior of D. $u^s(x)$ is the scattered wave corresponding to the incident wave $u^i(x)$. Assume that $0 < \sigma(x) \in C(\partial D)$, then by the result in [4], we know that there exists a unique solution for the forward scattering problem (1.1).

For the incident field $u^i(x) = e^{ikx\cdot d}$, the far-field pattern $u^\infty(d,\theta)$ of scattered wave $u^s(x)$ can be defined by

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u^\infty(d,\theta) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \longrightarrow \infty,$$

where $\theta \in \Omega$.

For a direct scattering problem, it aims to determine the scattered field $u^s(x)$ as well as its far-field pattern, provided the scatterer and the incident waves are known. The inverse

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scattering problems cover different ranges of scattering. For example, the determination of ∂D with unknown $\sigma(x)$ from the far-field pattern $u^\infty(d,\theta)$ for $\theta, d \in \Omega$ has been investigated in [1]. For some special cases of this kind of problem corresponding to $\sigma(x) = 0$ and $\sigma(x) = \infty$ respectively, we refer to [2], [3], [4], [6], [7], [8].

From the standard scattering theory, we know that there exists a one-to-one correspondence between $u^\infty(d,\theta)$ and $u^s(x)$ for any given $d \in \Omega$. It is known as early as in 40's that the far-field pattern uniquely determines the analytic scattered field in the exterior of the scatterer(Rellich lemma). However, the map $u^s|_{\partial B} \to u^\infty$ is generally compact in any reasonable function space, where the cycle $B(0,R)$ contains the scatterer D within it. This fact indicates that the determination of scattered field from its far-field pattern is ill-posed. To the author's knowledge, although the inverse scattering problems of determining scatterer have been researched thoroughly, the reconstruction of scattered field from its far-field pattern receive little attention, especially in the treatment of ill-posedness and the numerical implementation. The importance of this problem is due to the fact that construction of the near field data of scattered wave from far-field pattern is a crucial step, when one considers the inverse scattering problem by the Dirichlet-to-Neumann map([1]). An important development in this issue can be found in [9], where the stability of recovering $u^s(x)$ from $u^\infty(d,\theta)$ is established by applying the point-source method.

This paper deals with the numerical schemes of determining $u^s(x)$ from the error data of u^∞. Due to the ill-posedness, some regularizing technique should be introduced. More precisely, for known error data $u^\infty_\delta(d,\theta)$ of far-filed pattern with the error level $\delta > 0$, i.e.,

$$\|u^\infty_\delta(d,\cdot) - u^\infty(d,\cdot)\|_{L^2(\Omega)} \le \delta,$$

our task is to recover $u^s(x)$ approximately from $u^\infty_\delta(d,\theta)$. Motivated by the basic idea in [9] for treating the inverse scattering problem with the Dirichlet boundary, we establish a reciprocity principle for the scattering problem with the Robin boundary firstly, then we propose a regularizing scheme to reconstruct the near-field from $u^\infty(d,\theta)$. Numerical results illustrating the inversion scheme are also given. Our numerics show that the near field data is very sensitive to the far-field pattern when one applies the point source method to recover near field, even though the stability of this method has been proved theoretically in [9].

Our paper is organized as follows:

- Section 2: Relation between near-field and far-field
- Section 3: Regularization method
- Section 4: Numerical implementations

2. Relation between Near-Field and Far-Field

In this section, we state the relation between far-field and near-field. This relation is led from the potential theory.

Denote by $\Phi(x,y)$ the fundamental solution to 2-D Helmholtz equation, i.e.,

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$$

with $H_0^{(1)}$ the Hankel function and define

$$(\mathbf{K}'\psi)(x) = 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)} \psi(y) ds(y), \quad x \in \partial D,$$

$$(\mathbf{S}\psi)(x) = 2\int_{\partial D} \Phi(x,y)\psi(y)ds(y), \quad x \in \partial D$$

for density function $\psi(x) \in C(\partial D)$. We have

Lemma 2.1. *Assume that $-k^2$ is not the Dirichlet inner eigenvalue for the Laplace operator. If the density function $\psi(x) \in C(\partial D)$ solves*

$$\psi(x) - (\mathbf{K}'\psi)(x) - ik\sigma(x)(\mathbf{S}\psi)(x) = -2f(x) \tag{2.1}$$

with a known function

$$f(x) = -\frac{\partial u^i}{\partial \nu(x)} - ik\sigma(x)u^i, \quad x \in \partial D,$$

then the scattered field and its far-field pattern can be represented as

$$u^s(x) = \int_{\partial D} \Phi(x,y)\psi(y)ds(y), \quad x \in R^2 \setminus \overline{D}, \tag{2.2}$$

$$u^\infty(d,\theta) = e^{\pi i/4}\int_{\partial D} e^{-ik(\theta,y)}\psi(y)ds(y), \quad \theta \in \Omega. \tag{2.3}$$

This lemma may be found in [7], which implies two things. Firstly, for known scatterer, $u^s(x)$ and $u^\infty(d,\theta)$ can be generated from (2.1) to (2.3). This fact will be used in our numerical implementation to generate the inversion input data $u^\infty_\delta(d,\theta)$. For the numerical solution to (2.1), we refer the readers to [7]. Secondly, reconstruction of $u^s(x)$ from $u^\infty(d,\theta)$ is ill-posed due to the fact that one should solve an integral equation of the first kind to get the density function. Therefore, determination of $u^s(x)$ from the noisy data $u^\infty_\delta(d,\theta)$ should apply some regularization. One possible way is to get $\psi(x)$ from (2.3) by regularization and then determine $u^s(x)$ from (2.2). However, the kernel function $\psi(x)$ depends on the far-field pattern in this case. Here we propose another regularizing scheme of recovering $u^s(x)$. In this scheme, we determine a regularizing operator $\mathbf{A}_\varepsilon$ independent of the far-field pattern, which can be used to construct $u^s(x)$ for different far-fields. For given scatterer D, denote by a domain G satisfying $\overline{D} \subset G$. For $z \in R^2 \setminus \overline{G}$ and $\xi \in \Omega$, define

$$(Hg)(z,x) = \int_\Omega e^{ikx\cdot\xi}g(z,\xi)ds(\xi), \quad x \in \partial G, z \in R^2 \setminus \overline{G}$$

for $g(z,\xi)$. For any fixed $z \in R^2 \setminus \overline{G}$, denote by $g_\varepsilon(z,\cdot)$ the minimum norm solution to the ill-posed equation

$$(Hg)(z,\cdot) = \Phi(\cdot,z) \tag{2.4}$$

under the constraint

$$\|(Hg)(z,\cdot) - \Phi(\cdot,z)\|_{L^2(\partial G)} \le \varepsilon, \tag{2.5}$$

then the following result holds.

Lemma 2.2. *For $g_\varepsilon(z,\cdot)$ defined above,*

(1) there exists a unique minimum norm solution $g_\epsilon(z,\cdot) \in L^2(\Omega)$,
(2) $g_\epsilon(z,\cdot) \in L^2(\Omega)$ depends weakly continuously on $z \in R^2 \setminus \overline{G}$,
(3) $\|g_\epsilon(z,\cdot)\|_{L^2(\Omega)}$ depends continuously on ϵ,
(4) $\|g_\epsilon(z,\cdot)\|_{L^2(\Omega)}$ is unbounded as $\varepsilon \to 0$.

Proof. The first 3 conclusions can be found in [9] in the cases of $\sigma(x) = 0$ and $\sigma(x) = \infty$, or can be obtained by the standard argument in [5]. We prove (4) here. If it is not true, then there exists some constant $M > 0$ such that $\|g_\epsilon(z,\cdot)\|_{L^2(\Omega)} \leq M$ for all $\epsilon > 0$. From the definition of $g_\epsilon(z,\cdot)$, there exists a function sequence $\{g_n(z,\cdot)\}_{n=1}^\infty$ such that

$$\|(Hg_n)(z,\cdot) - \Phi(\cdot,z)\|_{L^2(\partial G)} \leq \frac{1}{n} \tag{2.6}$$

for all $n = 1, 2, \cdots$. On the other hand, since $(Hg_n)(z,\cdot) - \Phi(\cdot,z)$ solves the Helmholtz equation within G, we know for any compact set $M_0 \subset G$ it holds

$$\|(Hg_n)(z,\cdot) - \Phi(\cdot,z)\|_{C^3(M_0)} \leq C_1 \|(Hg_n)(z,\cdot) - \Phi(\cdot,z)\|_{L^2(\partial G)} \leq \frac{C_1}{n}, \tag{2.7}$$

which implies $\{(Hg_n)(z,\cdot) - \Phi(\cdot,z)\}_{n=1}^\infty$ is a bounded subset in $C^3(M_0)$ and therefore is compact in $C^2(M_0)$, hence there exists some convergent subsequence $\{(Hg_{n_k})(z,\cdot) - \Phi(\cdot,z)\}_{k=1}^\infty$. Furthermore, it follows from

$$\|(Hg_{n_k})(z,\cdot) - \Phi(\cdot,z)\|_{C^3(M_0)} \leq \frac{C_1}{n_k}$$

that

$$(Hg_{n_k})(z,\cdot) \to \Phi(\cdot,z) \tag{2.8}$$

as $k \to \infty$ in $C^2(M_0)$.

On the other hand, since we assume $\{g_{n_k}(z,\cdot)\}_{k=1}^\infty$ is bounded in $L^2(\Omega)$, there exists some function $g_0(z,\cdot) \in L^2(\Omega)$ such that some subsequence of $\{g_{n_k}(z,\cdot)\}_{k=1}^\infty$ converges weakly to $g_0(z,\cdot)$ in $L^2(\Omega)$. We still denote by $\{g_{n_k}\}_{k=1}^\infty$ this subsequence. Since the operator Hg can be considered as an inner product in $L^2(\Omega)$, we get $(Hg_{n_k}))(z,\cdot) \to (Hg_0)(z,\cdot)$ in $C(G)$. Finally we get

$$\Phi(\cdot,z) = (Hg_0)(z,\cdot)$$

in $C^2(M_0)$. Since both $\Phi(\cdot,z)$ and $(Hg_0)(z,\cdot)$ are the analytic function in compact set M_0, by the uniqueness of continuation for analytic function, we get $\Phi(\cdot,z) = (Hg_0)(z,\cdot)$ in R^2, this is a contradiction, since $\Phi(x,z)$ is singular at $x = z$, while $(Hg_0)(z,x)$ is analytic for all $x \in R^2$.

For $g_\varepsilon(z,\cdot)$ generated above, define an operator $\mathbf{A}_\varepsilon$ by

$$(\mathbf{A}_\varepsilon \phi)(z) = \frac{1}{\gamma_2}\int_\Omega g_\varepsilon(z,\xi)\phi(-\xi)ds(\xi), z \in R^2 \setminus \overline{G} \tag{2.9}$$

with the constant

$$\gamma_2 = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \tag{2.10}$$

for $\phi(x) \in L^2(\Omega)$. From the property of $g_\varepsilon(z,\cdot)$ given in Lemma 2.2, we know $\mathbf{A}_\varepsilon : L^2(\Omega) \to C(R^2 \setminus \overline{G})$. By virtue of this operator, we can generate an approximation of scattered field from the noisy data $u_\delta^\infty(d,\theta)$. For this purpose, we need a fundamental result called as the reciprocity principle for the point-source. This is a generalization of reciprocity principle for the scatterer with sound-soft or sound-hard boundary, see [10] and [11] for these two special cases respectively.

For an incident plane wave $u^i(x,d) = e^{ikx\cdot d}$ along incident direction $d \in \Omega$, denote by $u^s(\cdot,d)$ the scattered wave from scatterer D and $u^\infty(\hat{x},d), \hat{x} \in \Omega$, the corresponding far-field pattern. Moreover, for the point-source $\Phi(x,z)$ with source-point z, we denote by $\Phi^s(\cdot,z)$ and $\Phi^\infty(\cdot,z)$ the scattered field and far-field pattern respectively. Then the following reciprocity principle is true.

Lemma 2.3. *For an obstacle D with impedance boundary, the far-field patterns corresponding to incident plane wave and point-source satisfy*

$$u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x}), \quad \hat{x}, d \in \Omega, \tag{2.11}$$

$$\Phi^\infty(\hat{x}, z) = \gamma_2 u^s(z, -\hat{x}), \quad \hat{x} \in \Omega, z \in R^2 \setminus \overline{D}. \tag{2.12}$$

Proof. We only give the proof for the mixed reciprocity principle (2.12). For the proof of (2.11), it can be found in [4].

Firstly, since both $u^s(x,z)$ and $\Phi^s(x,z)$ are the radiating solution to the Helmholtz equation, then by the Green's formula, we have

$$u^s(x,z) = \int_{\partial D} \left[u^s(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y,z)}{\partial \nu(y)} \Phi(x,y) \right] ds(y), \tag{2.13}$$

$$\Phi^s(x,z) = \int_{\partial D} \left[\Phi^s(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} \Phi(x,y) \right] ds(y) \tag{2.14}$$

for $x, z \in R^2 \setminus \overline{D}$. On the other hand, for any fixed $y \in \partial D$, it is well-known that the fundamental solution has the asymptotic

$$\Phi(x,y) = \gamma_2 \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right) \right\}, \quad \frac{\partial \Phi(x,y)}{\partial \nu(y)} = \gamma_2 \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} + O\left(\frac{1}{|x|}\right) \right\}$$

due to the asymptotic for $H_0^{(1)}(z)$ as $z \to \infty$ and $|x - y| = |x| - \langle \hat{x}, y \rangle + O(|x|^{-1})$ as $|x| \to \infty$. By inserting these expressions to (2.14), we get

$$\Phi^s(x,z) = \gamma_2 \frac{e^{ik|x|}}{\sqrt{|x|}} \times \left[\int_{\partial D} \left(\Phi^s(y,z) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} e^{-ik\hat{x}\cdot y} \right) ds(y) + O\left(\frac{1}{|x|}\right) \right] \tag{2.15}$$

as $|x| \to \infty$, which yields

$$\begin{aligned} \Phi^\infty(\hat{x}, z) &= \gamma_2 \int_{\partial D} \left[\Phi^s(y,z) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} e^{-ik\hat{x}\cdot y} \right] ds(y) \\ &= \gamma_2 \int_{\partial D} \left[\Phi^s(y,z) \frac{\partial u^i(y,-\hat{x})}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u^i(y,-\hat{x}) \right] ds(y) \end{aligned} \tag{2.16}$$

from the definition of far-field pattern. On the other hand, for any fixed z, the radiation condition for Φ^s and u^s implies

$$\begin{cases} \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} = ik\Phi^s(y,z) + O\left(\frac{1}{|y|}\right) \\ \frac{\partial u^s(y,-\hat{x})}{\partial \nu(y)} = iku^s(y,-\hat{x}) + O\left(\frac{1}{|y|}\right) \end{cases} \tag{2.17}$$

as $|y| \to \infty$. For a large cycle $B(0,R)$, applying the Green formula in $B(0,R) \setminus \overline{D}$ and the above radiation condition tell us

$$\begin{aligned} &\int_{\partial D} \left[\Phi^s(y,z) \frac{\partial u^s(y,-\hat{x})}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u^s(y,-\hat{x}) \right] ds(y) \\ &= \lim_{R\to\infty} \int_{\partial B(0,R)} \left[\Phi^s(y,z) \frac{\partial u^s(y,-\hat{x})}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u^s(y,-\hat{x}) \right] ds(y) \\ &= \lim_{R\to\infty} \int_{\partial B(0,R)} \left[\Phi^s(y,z) O\left(\frac{1}{|y|}\right) - u^s(y,-\hat{x}) O\left(\frac{1}{|y|}\right) \right] ds(y) = 0 \end{aligned}$$

due to $u^s(y,z), \Phi^s(y,z) \to 0$ in $y \in \partial B(0,R)$ as $R \to \infty$. Now applying this relation to (2.16) gets

$$\Phi^\infty(\hat{x},z) = \gamma_2 \int_{\partial D} \left[\Phi^s(y,z)\frac{\partial u(y,-\hat{x})}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u(y,-\hat{x})\right] ds(y) \tag{2.18}$$

since $u = u^i + u^s$. Noticing that $u^i(\cdot,z)$ solves the Helmholtz equation in D, the Green's formula says

$$0 = \int_{\partial D} \left[u^i(y,z)\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^i(y,z)}{\partial \nu(y)}\Phi(x,y)\right] ds(y). \tag{2.19}$$

Combining this equality with (2.13) says

$$u^s(z,-\hat{x}) = \int_{\partial D} \left[u(y,-\hat{x})\frac{\partial \Phi(y,z)}{\partial \nu(y)} - \frac{\partial u(y,-\hat{x})}{\partial \nu(y)}\Phi(y,z)\right] ds(y). \tag{2.20}$$

Now, in the sound-soft case, we have

$$\gamma_2 u^s(z,-\hat{x}) = \gamma_2 \int_{\partial D} \Phi^s(y,z)\frac{\partial u(y,-\hat{x})}{\partial \nu(y)} ds(y) = \Phi^\infty(\hat{x},z)$$

from (2.18) and (2.20) due to $\Phi^s(y,z) + \Phi(y,z) = 0$ and $u(y,-\hat{x}) = 0$ for $y \in \partial D$. If ∂D is the impedance boundary which implies

$$\frac{\partial(\Phi^s(y,z) + \Phi(y,z))}{\partial \nu(y)} + i\sigma(y)(\Phi^s(y,z) + \Phi(y,z)) = 0,$$

$$\frac{u(y,-\hat{x})}{\partial \nu(y)} + i\sigma(y)u(y,-\hat{x}) = 0$$

for some impedance coefficient $\sigma(y)$ in ∂D. It also generates

$$\gamma_2 u^s(z,-\hat{x}) = -\gamma_2 \int_{\partial D} u(y,-\hat{x}) \left[i\sigma(y)\Phi^s(y,z) + \frac{\Phi^s(y,z)}{\nu(y)}\right] ds(y) = \Phi^s(\hat{x},z)$$

from (2.18) and (2.20), the proof is complete.

Now we can state the regularizing property of operator $\mathbf{A}_\varepsilon$. That is,

Theorem 2.1. *Assume that we have chosen G such that $\overline{D} \subset G$ for given scatterer D. Corresponding to an incident plane wave u^i with direction $d \in \Omega$, if we approximate the scattered wave field u^s from the disturbed far-field pattern u^∞_δ with error bound*

$$\|u^\infty - u^\infty_\delta\|_{L^2(\partial G)} \le \delta \tag{2.21}$$

by $(\mathbf{A}_\varepsilon u^\infty_\delta)(z)$ for $z \in R^2 \setminus \overline{G}$, then it holds

$$|u^s(z) - (\mathbf{A}_\varepsilon u^\infty_\delta)(z)| \le c\epsilon + \frac{1}{\gamma_m}\|g_\epsilon(z,\cdot)\|_{L^2(\Omega)}\,\delta \tag{2.22}$$

with some constant c depending on D and on the bound c_∞ for the norm of the scattering $S: C^1(\partial D) \to C(\Omega), u^i \to u^\infty$.

Remark 2.1. This result is the same as Theorem 2 in [9]. However, since the scatterer D is known in our problem, we need not assume that D should lie in some compact set M in G, hence the proof is a little different.

Proof. We denote by $v^i(z,\cdot)$ the incident Herglotz wave function with density $g_\epsilon(z,\cdot)$ for $z \in R^2 \setminus \overline{G}$. According to the proof for Theorem 2 in [9], the main idea of proof contains two points:

(1). Prove $\left\|\Phi(\cdot,z) - v^i(z,\cdot)\right\|_{C^1(\overline{D})} \le C_1\epsilon$,

(2). Apply the reciprocity principle.

Since we have proved that the reciprocity principle is also true in the case of impedance boundary, we only prove (1).

Firstly, from the definition of density function $g_\epsilon(z,\cdot)$, we know

$$\left\|v^i(z,\cdot) - \Phi(\cdot,z)\right\|_{L^2(\partial G)} \le \epsilon.$$

Now following the way in the previous lemma, we know it holds that

$$\left\|v^i(z,\cdot) - \Phi(\cdot,z)\right\|_{C^2(M)} \le C_1\epsilon$$

for any compact set $M \subset \overline{D} \subset \overline{G}$, which implies

$$\left\|v^i(z,\cdot) - \Phi(\cdot,z)\right\|_{C(\partial M)} \le C_1\epsilon, \quad \left\|\frac{\partial}{\partial\nu(\cdot)}[v^i(z,\cdot) - \Phi(\cdot,z)]\right\|_{C(\partial M)} \le C_1\epsilon.$$

Since $v^i(z,\cdot) - \Phi(\cdot,z)$ solves the Helmholtz equation in D for any $z \in R^2 \setminus \overline{D}$, we solve the Cauchy problem for the Helmholtz equation in $D \setminus \overline{M}$ for any bounded domain D, then we get

$$\left\|v^i(z,\cdot) - \Phi(\cdot,z)\right\|_{C^1(D\setminus\overline{M})} \le C\epsilon,$$

so we have proved (1).

Now we can complete the proof for this theorem following the way in [9].

3. Regularization Method

From the above theorem, $\mathbf{A}_\varepsilon$ can be considered as a regularizing operator for determining $u^s(x)$ approximately. That is, for given regularizing parameter $\varepsilon > 0$, we can construct $u^s(x)$ from the noisy data u_δ^∞ by $\mathbf{A}_\varepsilon u_\delta^\infty$. The error is given in (2.22). Of course, it depends on both ε and δ. Although $\|g_\varepsilon(\cdot,z)\|$ is unbounded as $\varepsilon \to 0$(Lemma 2.2, (4)), it can be shown that we can choose $\varepsilon = \varepsilon(\delta)$ such that

$$\varepsilon(\delta) \to 0, \quad \left\|g_{\varepsilon(\delta)}(\cdot,z)\right\|\delta \to 0$$

as $\delta \to 0$, see [9]. Therefore $\mathbf{A}_\varepsilon$ is a regularizing scheme for solving the near-field from the noisy data u_δ^∞. We omit the details here and turn our attention to the numerics.

The main task in our inversion scheme is to determine $g_\varepsilon(z,\cdot)$. Without loss of generality, we assume that the regularizing parameter ε is chosen small enough such that

$$\|\Phi(\cdot,z)\|_{L^2(\partial G)} > \varepsilon, \tag{3.1}$$

otherwise we can take $g_\varepsilon(z,\xi) = 0$ as the minimum norm solution. It is easy to know that $\|\Phi(\cdot,z)\|_{L^2(\partial G)}$ is independent of z from the property of Hankle function $H_0^{(1)}$ if we take ∂G as a cycle centered at 0. According to the property of minimum norm solution([5]), $\phi_0(\cdot) = g_\varepsilon(z,\cdot)$ can be solved from

$$\alpha\phi_0(\xi) + (H^*H\phi_0)(\xi) = (H^*\Phi)(\xi,z), \quad \xi \in \Omega, \tag{3.2}$$

where $\alpha = \alpha(\varepsilon)$ is the zero point of function

$$G(\alpha) = \|H\phi_\alpha - \Phi\|^2_{L^2(\partial G)} - \varepsilon^2, \tag{3.3}$$

where $\phi_\alpha(\xi)$ solves

$$\alpha\phi_\alpha(\xi)+(H^*H\phi_\alpha)(\xi)=(H^*\Phi)(\xi,z),\quad \xi\in\Omega. \tag{3.4}$$

From (3.3) and (3.4), we can solve $\alpha=\alpha(\varepsilon)$ for any given $\varepsilon>0$, then $g_\varepsilon(z,\cdot)$ can be obtained from (3.2). Now we give the solvability of (3.3) and the upper bound of $\alpha(\varepsilon)$.

Theorem 3.1. *For the regularizing parameter ε satisfying (3.1), there exists a unique solution $\alpha=\alpha(\varepsilon)$ to (3.3). Moreover, the zero point α of function $G(\alpha)$ satisfies*

$$0<\alpha<\frac{\|H\|^2\varepsilon}{\|\Phi(\cdot,z)\|-\varepsilon} \tag{3.5}$$

Proof. It is obvious that $G(\alpha)$ is continuous for $\alpha\in(0,\infty)$. On one hand, it follows that

$$\lim_{\alpha\to 0}G(\alpha)=-\varepsilon^2\le 0,$$

$$\lim_{\alpha\to\infty}G(\alpha)=\|\Phi(\cdot,z)\|^2-\varepsilon^2>0$$

due to the property of regularizing solution([5], Chapter 16) and (3.1). On the other hand, since ϕ_α solves (3.4), we know $\frac{d\phi_\alpha}{d\alpha}$ satisfies

$$\alpha\frac{d\phi_\alpha}{d\alpha}+H^*H\frac{d\phi_\alpha}{d\alpha}=-\phi_\alpha. \tag{3.6}$$

Hence we know by simple computation that

$$\begin{aligned}G'(\alpha)&=2\Re\langle H\frac{d\phi_\alpha}{d\alpha},H\phi_\alpha-\Phi\rangle=2\Re\langle\frac{d\phi_\alpha}{d\alpha},H^*(H\phi_\alpha-\Phi)\rangle\\&=2\alpha\Re\langle\frac{d\phi_\alpha}{d\alpha},\alpha\frac{d\phi_\alpha}{d\alpha}+H^*H\frac{d\phi_\alpha}{d\alpha}\rangle\\&=2\alpha^2\left\|\frac{d\phi_\alpha}{d\alpha}\right\|^2+2\alpha\left\|H\frac{d\phi_\alpha}{d\alpha}\right\|^2\ge 0.\end{aligned} \tag{3.7}$$

Therefore, there exists a unique solution to (3.3). The estimate (3.5) can be obtained from

$$\begin{aligned}\|\Phi(\cdot,z)\|-\varepsilon&=\|\Phi(\cdot,z)\|-\|H\Phi_\alpha-\Phi\|\\&\le\|H\Phi_\alpha\|=\frac{1}{\alpha}\|HH^*\Phi-HH^*H\Phi_\alpha\|\\&=\|HH^*(\Phi-H\Phi_\alpha)\|\le\frac{\|H\|^2\varepsilon}{\alpha}\end{aligned} \tag{3.8}$$

due to (3.3) and (3.4).

The approximation to $\alpha=\alpha(\varepsilon)$ can be obtained by Newton's iteration, while the estimate (3.5) can be considered as a restriction for the choice of initial α. From the definition of operator H, we know that

$$\begin{aligned}\|(Hg)(\cdot,z)\|^2_{L^2(\partial G)}&\le\int_{\partial G}\left[\int_\Omega|e^{ikx\cdot\xi}|^2ds(\xi)\int_\Omega|g(z,\xi)|^2ds(\xi)\right]ds(x)\\&\le\int_{\partial G}mes(\Omega)\int_\Omega|g(z,\xi)|^2ds(\xi)ds(x)\\&=mes(\Omega)\|g(z,\cdot)\|^2_{L^2(\Omega)}mes(\partial G),\end{aligned} \tag{3.9}$$

from which we get the estimate

$$\|H\|^2\le mes(\Omega)mes(\partial G). \tag{3.10}$$

Since

$$G'(\alpha) = -2\alpha\Re\langle \frac{d\phi_\alpha}{d\alpha}, \phi_\alpha \rangle, \tag{3.11}$$

now $\alpha = \alpha(\varepsilon)$ can be obtained by Newton's iteration method in principle.

Now we give the numerics for solving (3.3) and (3.4). Since the map H maps the complex-valued space $L^2(\Omega)$ to a complex-valued space $L^2(\partial G)$, its adjoint operator is

$$(H^*\phi)(\xi) = \int_{\partial G} \phi(x) e^{-ikx.\xi} ds(x)$$

for any complex value function $\phi(x) \in L^2(\partial G)$. So for any fixed $z \in R^2 \setminus \overline{G}$, (3.4) and (3.6) can be rewritten explicitly as

$$\alpha\phi_\alpha(z,\xi) + \int_\Omega K(\xi,\eta)\phi_\alpha(z,\eta)d\eta = \frac{i}{4}\int_{\partial G} e^{-ikx.\xi} H_0^{(1)}(k|x-z|)ds(x), \tag{3.12}$$

$$\alpha\frac{d\phi_\alpha(z,\xi)}{d\alpha} + \int_\Omega K(\xi,\eta)\frac{d\phi_\alpha(z,\eta)}{d\alpha}d\eta = -\phi_\alpha(z,\xi) \tag{3.13}$$

with the kernel function

$$K(\xi,\eta) = \int_{\partial G} e^{-ikx.(\xi-\eta)} ds(x). \tag{3.14}$$

Since the construction of the left-hand side for the above equations is the same, once we have solved ϕ_α from the first one, $\frac{d\phi_\alpha}{d\alpha}$ can be obtained from the second equation by the same procedure. For known function $F(\xi)$ with $\xi = (\xi_1, \xi_2) = (\cos t, \sin t) \in \Omega$ for $t \in [0, 2\pi]$, if we divide the cycle Ω into $2N$ subintervals by the points

$$\xi^j = (\xi_1^j, \xi_2^j) = (\cos t_j, \sin t_j)$$

with $t_j = \pi/N$ for $j = 0, 1, \cdots, 2N-1$, then the numerical solution to

$$\alpha\phi(\xi) + \int_\Omega K(\xi,\eta)\phi(\eta)ds(\eta) = F(\xi) \tag{3.15}$$

can be solved from the linear equations

$$\alpha\phi(\xi^i) + \frac{\pi}{N}\sum_{j=0}^{2N-1} K(\xi^i,\xi^j)\phi(\xi^j) = F(\xi^i) \tag{3.16}$$

for $i = 0, 1, 2, \cdots, 2N-1$, where the kernel $K(\xi^i, \xi^j)$ is

$$K(\xi^i,\xi^j) = \int_0^{2\pi} e^{-ikx(t)\cdot(\xi^i-\xi^j)}|x'(t)|dt \tag{3.17}$$

and $x(t) = (x_1(t), x_2(t))$ is the parameterization of ∂G for $t \in [0, 2\pi]$. After obtaining the numerical solution to $g_\varepsilon(z,\cdot)$, $u^s(z)$ from the noisy data u_δ^∞ can be obtained by

$$u^s(z) = \frac{1}{\gamma_2}\frac{\pi}{N}\sum_{j=0}^{2N-1} g_\varepsilon(z,\xi_j)u_\delta^\infty(d,-\xi^j) \tag{3.18}$$

for any fixed $z \in R^2 \setminus \overline{G}$.

4. Numerical Implementations

Here we consider the numerics for the model

$$\partial D=\{x:x=(x_1,x_2)=(1.2\cos t,1.2\sin t),t\in[0,2\pi]\}$$

with the boundary impedance

$$\sigma(x)=\frac{2+x_1x_2}{(3+x_2)^2},$$

where we take the incident direction $d=(1.0,0.0)$ and the wave number $k=1.0$. By the method proposed in [7] (also see Lemma 2.1), we can obtain the scattered wave and its far-field pattern as follows, where the scattered field is calculated by

$$u^s(z)=\frac{\pi}{n}\sum_{j=0}^{2n-1}\Phi(z,y_j)\psi(y_j)|\xi'(t_j)| \tag{4.1}$$

for any $z\in R^2\setminus\overline{D}$ due to (2.2). In our numerics, we take

$$\partial G=\{x:x(t)=(1.5\cos t,1.5\sin t)\},\quad \partial Z=\{z:z(t)=1.15\times(1.5\cos t,1.5\sin t)\}$$

for $t\in[0,2\pi]$, where $z\in\partial Z$ and $\partial D,\partial G,\partial Z$ are divided into $2n-$subintervals by points with $t=t_j=\frac{j\pi}{n}$ for $j=0,1,\cdots,2n-1$. For $n=16$, the scattered wave $u^s(z)$ and its far-field $u^\infty(\hat{x})$ at $z_j=z(t_j)$ and $\hat{x}_j=(\cos t_j,\sin t_j)$ for $t_j=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ are listed in Tab.0.

Tab.0 The Scattered Field and Far-field for $n=16$

t_j	$u^s(z(t_j))$	$u^\infty(\hat{x}(t_j))$
0	(-.02692822, -1.02767500)	(-.94524530, .65232880)
$\pi/2$	(.66626600, -.46901030)	(-1.27866100, -.50480560)
π	(.65511720, -.53359220)	(-1.04255700, -.53527310)
$3\pi/2$	(.34976210, -.41178460)	(-.97433700, -.24571610)

Now using the far-field pattern $u^\infty(\hat{x})$ generated above as our inversion input, the inversion results applying the point source method are listed in Tab.1, where the regularizing parameters

$$\alpha_1=1.450226E-06,\varepsilon_1=1.040109E-06.$$

The relative error $err(t_j)$ is defined as

$$err(t_j)=\frac{|u^s(z(t_j))-iu^s(z(t_j))|}{|u^s(z(t_j))|}\times 100\%.$$

Tab.1 A Comparison of Scattered Field and Inversions for $n=16$

t_j	$exactu^s(z(t_j))$	$inverseu^s(z(t_j))$	$err(t_j)$
0	(-2.692822E-02,-1.027675)	(-2.803457E-02,-1.027045)	0.12%
$\pi/2$	(6.662660E-01,-4.690103E-01)	(6.657581E-01,-4.684763E-01)	0.09%
π	(6.551172E-01,-5.335922E-01)	(6.543543E-01,-5.333374E-01)	0.09%
$3\pi/2$	(3.497621E-01,-4.117846E-01)	(3.492075E-01,-4.110203E-01)	0.17%

Tab. 1 gives the inversion results at 4 special points. The whole picture of our inversion is given in Figure 1, which shows that the inversions at all points are quite satisfactory with the exact input data. In fact, our numerical results show that the smallest relative error $err(t_j)$ is 0.07% arising in $j=10$, while the largest one is 0.18% with $j=25$.

Now let us test the stability of the inversion scheme. We hope that if some noisy is added into the exact far-field data, then the inversion results are still stable. Here we use the far-field

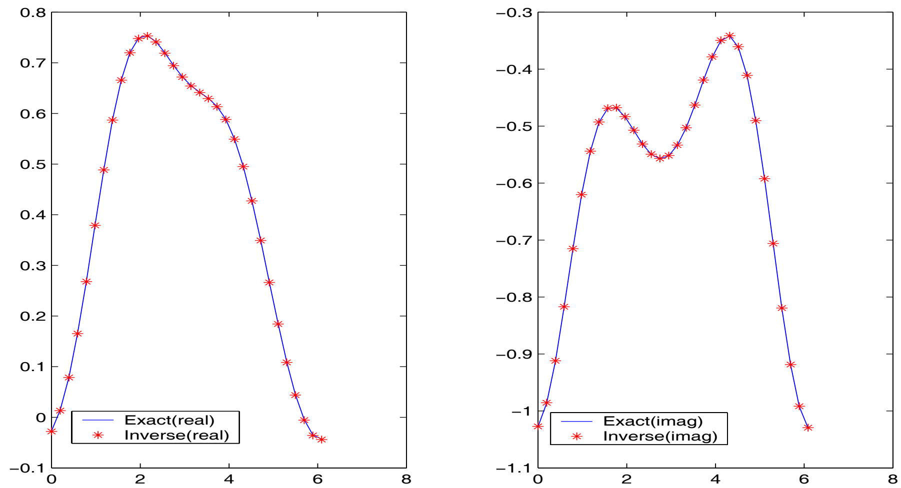

Figure 1: Recovery of Scattered Wave with Exact Data for $n = 16$

obtained by density function method with $n = 16$ as the exact data (see [7]) and then generate the noisy data by

$$u_\delta^\infty(\hat{x}(t_j)) = (1+\delta)u^\infty(\hat{x}(t_j)) \tag{4.2}$$

with some input error level $\delta \in (-1, 1)$. Applying these noisy data as inversion input in testing our inversion scheme, the inversion results for $n = 16, \delta = 0.1$ at special points are given in Tab.2, while the whole inversion results are shown in Figure 2, where the best relative error is 9.80% for $j = 25$, and the worst one is 9.94% for $j = 31$.

Tab.2 A Comparison of Scattered Field and Inversions for $n = 16, \delta = 0.1$

t_j	$exactu^s(z(t_j))$	$inverseu^s(z(t_j))$	$err(t_j)$
0	(-2.692822E-02,-1.027675)	(-3.083766E-02,-1.129750)	9.94%
$\pi/2$	(6.662660E-01,-4.690103E-01)	(7.323351E-01,-5.153170E-01)	9.90%
π	(6.551172E-01,-5.335922E-01)	(7.197876E-01,-5.866771E-01)	9.90%
$3\pi/2$	(3.497621E-01,-4.117846E-01)	(3.841274E-01,-4.521179E-01)	9.81%

The noisy data generated in the form (4.2) means that the error distribution has the same way at all measured points, which is reasonable for a given measurement instrument. Considering $\delta = 0.1$ means that the relative error for far-field pattern is 10%, the above results show our inversions contain also 10% relative error approximately. Therefore, we can conclude that our inversion scheme is stable under the perturbation of input data $u^\infty(\hat{x})$, provided that the noise is distributed in a uniform way at all measured points.

We can also test the numerics for the noisy data generated by

$$u_\delta^\infty(\hat{x}(t_j)) = (1+(-1)^j\delta)u^\infty(\hat{x}(t_j)), \tag{4.3}$$

which means the noise always oscillates at the measured points. The inversion results for $n = 16, \delta = 0.01$ are shown in Tab.3 and Fig.3. The maximum relative error is 26.7% in this case. Noticing (4.3) is the worst case of error distribution, we think that the inversions are still acceptable.

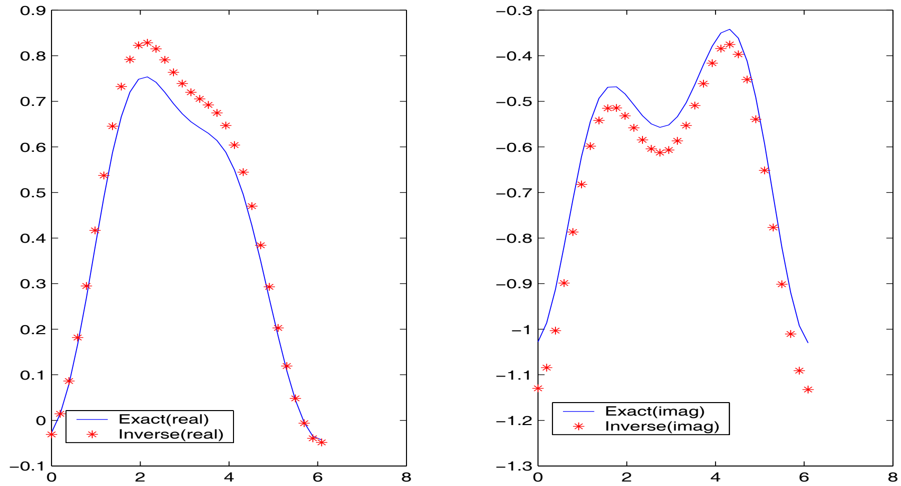

Figure 2: Recovery of u^s with $n = 16$ and Non-oscillatory Noisy Data($\delta = 0.1$)

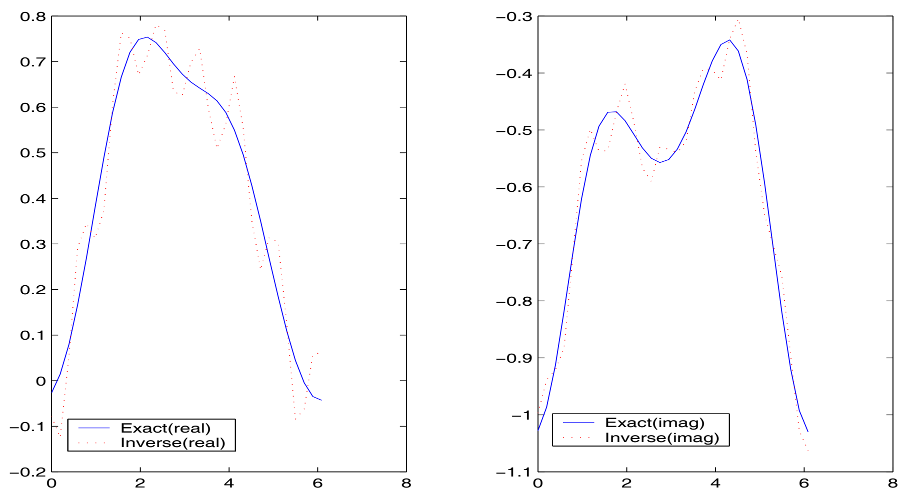

Figure 3: Recovery of u^s with $n = 16$ and Oscillatory Noisy Data($\delta = 0.01$)

Tab.3 A Comparison of Scattered Field and Inversions for $n = 16, \delta = 0.01$

t_j	$exactu^s(z(t_j))$	$inverseu^s(z(t_j))$	$err(t_j)$
0	(-2.692822E-02,-1.027675)	(-7.828331E-02,-9.970956E-01)	5.81%
$\pi/2$	(6.662660E-01,-4.690103E-01)	(7.631726E-01,-5.363312E-01)	14.4%
π	(6.551172E-01,-5.335922E-01)	(6.983678E-01,-5.399587E-01)	5.17%
$3\pi/2$	(3.497621E-01,-4.117846E-01)	(2.426164E-01,-3.676062E-01)	21.4%

Now we apply the exact input data and change the regularization parameter $\varepsilon > 0$ for the recovery of scattered wave, since the regularization parameter can not be determined exactly in practice. In our problems, there are two regularizing parameters: one is ε for recovery of scattered wave, the other is α for determining minimum norm density function for given ε. Here we only consider the influence of variation of ε. Consider the same problem with $n = 16$. In our numerics, the final regularization parameter ε is determined by varying $\varepsilon > 0$ in some interval $(c_0, r_0\|\Phi(\cdot,z)\|)$ for different step with some $r_0 \in (0,1)$, since ε should satisfy $\varepsilon < \|\Phi(\cdot,z)\|$ in Theorem 3.1. So we can construct the perturbation on ε by choosing different up bound and different decreasing step for ε. In the previous examples, we determine the final ε by considering $\varepsilon = \|\Phi(\cdot,z)\|/(5*10^m)$ for $m = 1,2,3,4,5$ respectively. Now we consider a different ε by considering $\varepsilon = \|\Phi(\cdot,z)\|/(2*10^m)$ for $m = 3,4,5,6$ respectively. The final regularization parameter is

$$\alpha_2 = 3.554832E-06, \varepsilon_2 = 2.600273E-07.$$

The inversion results, as well as the results for $\varepsilon = \varepsilon_1 = 1.040109E-06$ which have been given in Tab.1, are listed in Tab.4.

Tab.4 A Comparison of Scattered Field and Inversions for Different ε

t_j	$inverseu^s(z(t_j)), \varepsilon = \varepsilon_1$	$inverseu^s(z(t_j)), \varepsilon = \varepsilon_2$
0	(-2.803457E-02,-1.027045)	(-2.803618E-02,-1.027048)
$\pi/2$	(6.657581E-01,-4.684763E-01)	(6.657591E-01,-4.684784E-01)
π	(6.543543E-01,-5.333374E-01)	(6.543523E-01,-5.333388E-01)
$3\pi/2$	(3.492075E-01,-4.110203E-01)	(3.492105E-01,-4.110112E-01)

Noticing that $\varepsilon_1 \approx 5\varepsilon_2$, the inversion results corresponding to these different ε are quite stable. We do not give the whole picture corresponding to Tab.4, since it is almost the same as Figure 1.

It is worthwhile of pointing out that, in our numerical implementation, the regularizing parameters ε, α are obtained by searching automatically so that $G(\alpha)$ is small enough, rather than by the Newton iteration, since we have found in practice that the gradient $G'(\alpha)$ becomes so small that the iteration procedure stops before the accuracy for α is reached. Perhaps we can avoid this difficulty by applying the double precision variables in the computing programs. On the other hand, the above numerical results seem reasonable. Firstly, we can prove that the regularizing solution g_ε converges if we take $\alpha = O(\varepsilon^2)$. Secondly, if both α and ε are too small, then the truncated error in the computations will cover the accuracy in solving (3.4) and (3.5). So the inversion results are hard to be improved. Finally, we mention that one may also use some other approaches to find a reasonable regularization parameter α, for example, the methods proposed in [12], [13].

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