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A SUMMATION FORMULA FOR ESTIMATING THE SURFACE AREAS OF ELLIPSOIDS *

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Abstract

In this paper a summation formula for finding the surface area of general ellipsoids is derived.

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1. Introduction

Very tedious methods are employed at present to obtain the surface area of general ellipsoidal objects as no direct analytical formula is available. In the present work, we derive a simple summation formula for the purpose. Using it in a simple computer program, the surface area of any ellipsoid can be evaluated.

2. Derivation of the Summation Formula for the Surface Area of General Ellipsoid

The general ellipsoid is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

where a is the semi-major axis, b is the semi-intermediate axis and c is the semi-minor axis. It is symmetrical with respect to each coordinate plane and the origin. If $b = c$, then the ellipsoid (1) is called a prolate spheroid and its surface area is given by [7]

$$S_p = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) sin^{-1}(e) \tag{2}$$

where $b^2 = a^2(1 - e^2)$. Similarly, when $a = b$, we get the oblate spheroid and its surfrace area is given by [7]

$$S_0 = 2\pi a^2 + \pi \left(\frac{c^2}{e}\right) \ln \left(\frac{1+e}{1-e}\right) \tag{3}$$

where $c^2 = a^2(1 - e^2)$.
Solving for z from (1) gives

$$f(x, y) := z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{4}$$

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Similarly solving for y and x,

$$g(x,z) := y = \pm b\sqrt{1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}} \tag{5}$$

$$h(x,y) := x = \pm a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}} \tag{6}$$

Observe that the orthogonal projection of (1) in the xy plane is the elliptic disc $R_1 := \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$. From equations (4), (5) and (6) we find that $f(x,y), g(x,z)$ and $h(x,y)$ are positive in the upper half of the xy plane and the partial derivatives of these functions are continuous in its domain. Hence to evaluate the surface area one can use any one of the following formulae [8]:

$$S = \iint_{R_1} \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\; dxdy \tag{7}$$

$$= \iint_{R_2} \sqrt{(g_x(x,z))^2 + (g_y(x,z))^2 + 1}\; dxdz \tag{8}$$

$$= \iint_{R_3} \sqrt{(h_x(y,z))^2 + (h_y(y,z))^2 + 1}\; dydz \tag{9}$$

From (4)we find that $f_x(x,y) = -\frac{c^2 x}{a^2 z}$ and $f_y(x,y) = -\frac{c^2 y}{b^2 z}$. Also in the region R_1, x varies from $-a$ to a , and y varies from $-\frac{b}{a}\sqrt{a^2 - x^2}$ to $\frac{b}{a}\sqrt{a^2 - x^2}$. Inserting the values of $f_x(x,y)$ and $f_y(x,y)$ in equation (7) and simplifying one gets,

$$S = 8\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\frac{1 - \frac{x^2}{a^2}\left(1 - \frac{c^2}{a^2}\right) - \frac{y^2}{b^2}\left(1 - \frac{c^2}{b^2}\right)}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\; dxdy \tag{10}$$

The transformation $y = b\sqrt{1 - \frac{x^2}{a^2}}\;\sin\theta$ reduces the integral (10) in the following form:

$$S = \frac{8b}{a}\int_0^a \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^2\theta + \frac{c^2}{b^2}\sin^2\theta - \frac{c^2}{a^2}\right)(a^2 - x^2) + c^2}\; dxd\theta \tag{11}$$

Letting $x = a\sin\phi$ in equation(11) one gets,

$$\begin{aligned} S = 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} &\left[a^2\left(b^2\cos^2\theta + c^2\sin^2\theta\right)\right. \\ &\left. - \left(a^2\left(b^2\cos^2\theta + c^2\sin^2\theta\right) - b^2c^2\right)\sin^2\phi\right]^{\frac{1}{2}}\cos\phi d\theta d\phi \end{aligned} \tag{12}$$

Similarly, from equations (8) and (9), we get the following integrals.

$$\begin{aligned} S = 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} &\left[b^2\left(c^2\cos^2\theta + a^2\sin^2\theta\right)\right. \\ &\left. - \left(b^2\left(c^2\cos^2\theta + a^2\sin^2\theta\right) - c^2a^2\right)\sin^2\phi\right]^{\frac{1}{2}}\cos\phi d\theta d\phi \end{aligned} \tag{13}$$

$$\begin{aligned} S = 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} &\left[c^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\right. \\ &\left. - \left(c^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right) - a^2b^2\right)\sin^2\phi\right]^{\frac{1}{2}}\cos\phi d\theta d\phi \end{aligned} \tag{14}$$

Observe that equations (12), (13) and (14) gives the same surface area.
Defining,

$$\begin{aligned} A&: = a^2b^2\cos^2\theta + a^2c^2\sin^2\theta \\ B&: = b^2c^2\cos^2\theta + b^2a^2\sin^2\theta \\ C&: = c^2a^2\cos^2\theta + c^2b^2\sin^2\theta \\ D&: = A - b^2c^2 \\ E&: = c^2a^2 - B \\ F&: = a^2b^2 - C \\ t&: = \sin\phi \end{aligned} \tag{15}$$

Then we find that

$$(a^2-b^2)c^2 \le D \le (a^2-c^2)b^2 \tag{16}$$

$$(c^2-b^2)a^2 \le E \le (a^2-b^2)c^2 \tag{17}$$

$$(a^2-c^2)b^2 \le F \le (b^2-c^2)a^2 \tag{18}$$

Since $a \ge b \ge c$, from equations (16), (17) and (18), it follows that $D \ge 0$, $F \ge 0$ and E takes positive and negative values. Applying the transformations (15) in equations (12), (13) and (14) and simplifying, we get the following integrals.

$$S = 8\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{A - Dt^2}\, dtd\theta \tag{19}$$

$$S = 8\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{B + Et^2}\, dtd\theta \tag{20}$$

$$S = 8\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{C + Ft^2}\, dtd\theta \tag{21}$$

As a special case, if $a = b = c$, then $D = E = F = 0$ and hence equations (19), (20) and (21) reduces to, $S = 4\pi a^2$,which is the surface area of the sphere. If any two of a, b and c are different, then $D > 0$ and $F > 0$. Hence equation (19) can be written as

$$S = 8\int_0^{\frac{\pi}{2}}\sqrt{D}\int_0^1 \sqrt{\frac{A}{D} - t^2}\, dtd\theta = 2bc\pi + 4\int_0^{\frac{\pi}{2}}\sqrt{A}\left(\frac{sin^{-1}\left(\sqrt{\frac{D}{A}}\right)}{\sqrt{\frac{D}{A}}}\right) d\theta \tag{22}$$

Similarly, from equation (21), we find that

$$S = 2ab\pi + 4\int_0^{\frac{\pi}{2}}\sqrt{C}\left(\frac{sinh^{-1}\left(\sqrt{\frac{F}{C}}\right)}{\sqrt{\frac{F}{C}}}\right) d\theta \tag{23}$$

Observe that equations (22) and (23) give the same surface area of the ellipsoid (1). Similarly the same surface area can be got from equation (20) also, but since E can be positive and negative the equation obtained for S will not be as simple as equation (22) or (23).
To derive the summation formula, divide the interval $[0, 2\pi]$ into N equal subintervals. For each

positive integer k define the following:

$$A_k = \sqrt{a^2b^2cos^2\left(\frac{2\pi k}{N}\right) + a^2c^2sin^2\left(\frac{2\pi k}{N}\right)}$$
$$C_k = \sqrt{a^2c^2cos^2\left(\frac{2\pi k}{N}\right) + b^2c^2sin^2\left(\frac{2\pi k}{N}\right)}$$
$$D_k = \sqrt{a^2b^2cos^2\left(\frac{2\pi k}{N}\right) + a^2c^2sin^2\left(\frac{2\pi k}{N}\right) - b^2c^2}$$
$$F_k = \sqrt{a^2b^2 - \left(a^2c^2cos^2\left(\frac{2\pi k}{N}\right) + b^2c^2sin^2\left(\frac{2\pi k}{N}\right)\right)} \tag{24}$$
$$B_k = \frac{D_k}{A_k}, W_k = \frac{A_k}{B_k}, E_k = \frac{F_k}{C_k}, H_k = \frac{C_k}{E_k}$$

We note that equations (22) and (23) are contour integrals over the ellipses. Hence, in the light of equations (24) we find that good approximations to equations (22) and (23) are respectively

$$S = 2bc\pi + \frac{2\pi}{N}\sum_{k=0}^{k=N} W_k sin^{-1}(B_k) \tag{25}$$

and

$$S = 2ab\pi + \frac{2\pi}{N}\sum_{k=0}^{k=N} H_k sinh^{-1}(E_k) \tag{26}$$

For the special case when $b = c$ equation (25) coincides with equation (2). Also when $a = b$, equation (26) coincides with equation (3).

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