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MODIFIED LEGENDRE RATIONAL SPECTRAL METHOD FOR THE WHOLE LINE *1)

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Abstract

A mutually orthogonal system of rational functions on the whole line is introduced. Some approximation results are established. As an example of applications, a modified Legendre rational spectral scheme is given for the Dirac equation. Its numerical solution keeps the same conservation as the genuine solution. This feature not only leads to reasonable numerical simulation of nonlinear waves, but also simplifies the analysis. The convergence of the proposed scheme is proved. Numerical results demonstrate the efficiency of this new approach and coincide with the analysis well.

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1. Introduction

In sciences and engineerings, we often need to solve some problems in unbounded domain numerically, such as fluid flows in an infinite strip, nonlinear wave equations in quantum mechanics and so on. One of numerical methods for such problems is to use spectral approximations associated with certain orthogonal systems of polynomials in unbounded domains, such as the Hermite and the Laguerre approximations, see, Funaro and Kavian [8], Maday, Pernaud-Thomas and Vandeven [23], Guo [9], Guo and Shen [17] and Shen [25]. The next is to reform the original problems in unbounded domains and then use the Jacobi approximation to resolve the resulting singular problems in bounded domains numerically, see, Guo [10-13]. Another effective method is based on rational approximations. Boyd [5,6] and Christov [7] provided some spectral schemes for linear problems on infinite intervals by using certain mutually orthogonal systems of rational functions. Recently, Guo, Shen and Wang [18,19], Guo and Wang [21], and Wang and Guo [27] developed various rational approximations on infinite intervals. The rational spectral methods have several advantages. For instance, their weights are much weaker than the Hermite and Laguerre spectral methods and so it is not needed to reform the original problems usually. Moreover they are easier to be used for exterior problems than the Jacobi spectral methods. However, the non-uniform weights in the standard rational approximations may bring in some difficulties in actual computation in some applications. In particular, for the numerical simulations of hyperbolic systems, non-parabolic dissipative systems and nonlinear waves, such as the Schödinger equation, the Korteweg-de Vries equation and the Dirac equation etc.. Indeed the solutions of these systems satisfy some conservations which play important roles in theoretical analysis and numerical simulation. But the appearance of the non-uniform weights may destroy the corresponding conservations for the numerical solutions. This fact decreases the exactness of numerical experiments, and makes the numerical analysis

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complicated. To remedy this deficiency, Guo and Shen [17] proposed a modified Legendre rational approximation on the half line with the weight $\chi(x) \equiv 1$. The purpose of this paper is to develop a modified Legendre rational approximation on the whole line and its applications to numerical solutions of nonlinear wave equations. In this case, the numerical solutions keep the same conservations as in continuous cases. Meanwhile, the corresponding numerical analysis is simplified essentially.

This paper is organized as follows. In the next section, we introduce a mutually orthogonal system of rational functions on the whole line with the weight $\chi(x) \equiv 1$, and discuss its basic properties. We also recall some basic results on the Jacobi approximation, which will be used in the sequel. Then we study the modified Legendre rational approximation in Section 3, and the corresponding interpolation approximation in Section 4. Some approximation results are established, which form the mathematical foundation of the modified Legendre rational spectral method on the whole line. Section 5 is for some applications of this new approach. We take the Dirac equation on the whole line as an example to show how to use this method for nonlinear wave equations. The convergence of the proposed scheme is proved. Some numerical results are presented in the final section, which demonstrate the efficiency of this new approach, and coincide with the analysis well. It is easy to generalize the results of this paper to other nonlinear problems in multiple-dimensions.

2. Modified Legendre Rational Functions and Some Basic Results on Jacobi Approximation

2.1. Modified Legendre Rational Functions

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and $\chi(x)$ be certain weight function in the usual sense. Denote by $(u,v)_\chi$ and $||v||_\chi$ the inner product and the norm of the weighted space $L^2_\chi(\Lambda)$ respectively, i.e.,

$$(u,v)_\chi = \int_\Lambda u(x)v(x)\chi(x)dx, \quad ||v||_\chi = (v,v)^{\frac{1}{2}}_\chi.$$

Further let $\partial_x v(x) = \frac{\partial}{\partial x}v(x)$, etc.. For any non-negative integer m,

$$H^m_\chi(\Lambda) = \{v \mid \partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m\}.$$

The inner product, the semi-norm and the norm of $H^m_\chi(\Lambda)$ are given by

$$(u,v)_{m,\chi} = \sum_{k=0}^{m} (\partial^k_x u, \partial^k_x v)_\chi,$$
$$|v|_{m,\chi} = ||\partial^m_x v||_\chi, \quad ||v||_{m,\chi} = (v,v)^{\frac{1}{2}}_{m,\chi},$$

respectively. For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ with the norm $||v||_{r,\chi}$ by space interpolation. If $\chi(x) \equiv 1$, then we denote $H^r_\chi(\Lambda), |v|_{r,\chi}, ||v||_{r,\chi}, ||v||_\chi$ and $(u,v)_\chi$ by $H^r(\Lambda), |v|_r$, $||v||_r$, $||v||$ and (u,v), respectively. In addition, $||v||_\infty = ||v||_{L^\infty(\Lambda)}$.

Let $L_l(y)$ be the Legendre polynomial of degree l, $l = 0,1,2\cdots$. They are the eigenfunctions of the singular Sturm-Liouville problem

$$\partial_y((1-y^2)\partial_y L_l(y)) + l(l+1)L_l(y) = 0, \ l = 0,1,2\cdots, \quad (2.1)$$

and satisfy the following recurrence relations

$$L_{l+1}(y) = \frac{2l+1}{l+1} y L_l(y) - \frac{l}{l+1} L_{l-1}(y), \ l \ge 1, \quad (2.2)$$

$$(2l+1)L_l(y) = \partial_y L_{l+1}(y) - \partial_y L_{l-1}(y), \ l \ge 1. \quad (2.3)$$

Besides

$$L_l(1) = 1, \quad L_l(-1) = (-1)^l, \quad \partial_y L_l(1) = \frac{1}{2} l(l+1), \quad \partial_y L_l(-1) = (-1)^{l+1}\frac{1}{2} l(l+1).$$

Let $I = (-1, 1)$. The set of Legendre polynomials is mutually orthogonal on I, namely,

$$\int_I L_l(y)L_m(y)dy = (l + \frac{1}{2})^{-1}\delta_{l,m} \tag{2.4}$$

where $\delta_{l,m}$ is the Kronecker function. By virtue of (2.1) and (2.4),

$$\int_I \partial_y L_l(y)\partial_y L_m(y)(1-y^2)dy = l(l+1)(l+\frac{1}{2})^{-1}\delta_{l,m}. \tag{2.5}$$

The modified Legendre rational function of degree l is defined by

$$R_l(x) = (x^2+1)^{-\frac{3}{4}} L_l(\frac{x}{\sqrt{x^2+1}}), \ l = 0, 1, 2, \cdots.$$

Let $y = \frac{x}{\sqrt{x^2+1}}$. Clearly,

$$\frac{dy}{dx} = (x^2+1)^{-\frac{3}{2}}, \quad \frac{dx}{dy} = (1-y^2)^{-\frac{3}{2}}.$$

Thus by (2.1),

$$(x^2+1)^{\frac{3}{4}}\partial_x((x^2+1)^{\frac{1}{2}}\partial_x((x^2+1)^{\frac{3}{4}}R_l(x))) + l(l+1)R_l(x) = 0, \quad l = 0, 1, 2\cdots. \tag{2.6}$$

Due to (2.2) and (2.3), the modified Legendre rational functions satisfy the recurrence relations

$$R_{l+1}(x) = \frac{2l+1}{l+1}\frac{x}{\sqrt{x^2+1}}R_l(x) - \frac{l}{l+1}R_{l-1}(x), \quad l \geq 1, \tag{2.7}$$

and

$$(2l+1)R_l(x) = (x^2+1)^{\frac{3}{4}}(\partial_x((x^2+1)^{\frac{3}{4}}R_{l+1}(x)) - \partial_x((x^2+1)^{\frac{3}{4}}R_{l-1}(x))), \quad l \geq 1. \tag{2.8}$$

It can be checked that

$$\begin{aligned} &\lim_{x\to\infty}(x^2+1)^{\frac{3}{4}}R_l(x) = 1, \quad \lim_{x\to-\infty}(x^2+1)^{\frac{3}{4}}R_l(x) = (-1)^l, \\ &\lim_{x\to\infty}(x^2+1)^{\frac{3}{2}}\partial_x((x^2+1)^{\frac{3}{4}}R_l(x)) = \frac{1}{2}l(l+1), \\ &\lim_{x\to-\infty}(x^2+1)^{\frac{3}{2}}\partial_x((x^2+1)^{\frac{3}{4}}R_l(x)) = (-1)^{l+1}\frac{1}{2}l(l+1). \end{aligned} \tag{2.9}$$

The set $\{R_l(x)\}$ is the $L^2(\Lambda)-$orthogonal system, i.e.,

$$\int_\Lambda R_l(x)R_m(x)dx = (l+\frac{1}{2})^{-1}\delta_{l,m}. \tag{2.10}$$

Let $\omega_1(x) = (x^2+1)^{\frac{1}{2}}$. By virtue of (2.6) and (2.10), the set $\{\partial_x((x^2+1)^{\frac{3}{4}}R_l(x))\}$ is a mutually orthogonal system in the space $L^2_{\omega_1}(\Lambda)$, namely,

$$\int_I \partial_x((x^2+1)^{\frac{3}{4}}R_l(x))\partial_x((x^2+1)^{\frac{3}{4}}R_m(x))\omega_1(x)dx = l(l+1)(l+\frac{1}{2})^{-1}\delta_{l,m}. \tag{2.11}$$

The modified Legendre rational expansion of a function $v \in L^2(\Lambda)$ is

$$v(x) = \sum_{l=0}^{\infty}\hat{v}_l R_l(x)$$

with

$$\hat{v}_l = (l+\frac{1}{2})\int_\Lambda v(x)R_l(x)dx, \quad l = 0, 1, 2, \cdots.$$

Let N be any positive integer, and

$$\mathcal{R}_N = \text{span}\{R_0, R_1, \cdots, R_N\}.$$

Denote by c a generic positive constant independent of any function and N. We now establish several inverse inequalities and embedding inequalities which will be used in the sequel.

Theorem 2.1. *For any $\phi \in \mathcal{R}_N$ and $1 \le p \le q \le \infty$,*

$$||\phi||_{L^q} \le cN^{\lambda(p)(\frac{1}{p}-\frac{1}{q})}||\phi||_{L^p}$$

where $\lambda(p) = \frac{3}{2}p - 1$ for $p \ge \frac{4}{3}$, and $\lambda(p) = 1$ otherwise.

Proof. Let $y \in I$ and $x = \frac{y}{\sqrt{1-y^2}}$. Denote by $\mathcal{P}_N$ the set of all algebraic polynomials of degree at most N. For any $\phi \in \mathcal{R}_N$, we set $\psi(y) = (1-y^2)^{-\frac{3}{4}}\phi(\frac{y}{\sqrt{1-y^2}})$. Clearly $\psi(y) \in \mathcal{P}_N$. Let $\chi^{(\alpha,\beta)}(y) = (1-y)^\alpha(1+y)^\beta$, $\alpha,\ \beta > -1$. By an inverse inequality in $\mathcal{P}_N$ (see Theorem 2.1 of Guo [15]), for any $\psi \in \mathcal{P}_N$ and $1 \le p \le q \le \infty$,

$$(\int_I |\psi(y)|^q \chi^{(\alpha,\beta)}(y)dy)^{\frac{1}{q}} \le cN^{\sigma(\alpha,\beta)(\frac{1}{p}-\frac{1}{q})}(\int_I |\psi(y)|^p \chi^{(\alpha,\beta)}(y)dy)^{\frac{1}{p}}$$

where

$$\sigma(\alpha,\beta) = \begin{cases} 2\max(\alpha,\beta)+2, & \text{if } \max(\alpha,\beta) \ge -\frac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} ||\phi||^q_{L^q(\Lambda)} &= \int_I |\psi(y)|^q(1-y^2)^{\frac{3}{4}q-\frac{3}{2}}dy \le \int_I |\psi(y)|^q(1-y^2)^{\frac{3}{4}p-\frac{3}{2}}dy \\ &\le cN^{\sigma(\frac{3}{4}p-\frac{3}{2},\frac{3}{4}p-\frac{3}{2})(\frac{q}{p}-1)}(\int_I |\psi(y)|^p(1-y^2)^{\frac{3}{4}p-\frac{3}{2}}dy)^{\frac{q}{p}} \\ &= cN^{\sigma(\frac{3}{4}p-\frac{3}{2},\frac{3}{4}p-\frac{3}{2})(\frac{q}{p}-1)}||\phi||^q_{L^p(\Lambda)}. \end{aligned}$$

By a simple calculation for $\sigma(\frac{3}{4}p-\frac{3}{2},\frac{3}{4}p-\frac{3}{2})$, we reach the desired result.

Remark 2.1. By Theorem 2.1, for any $\phi \in \mathcal{R}_{\mathcal{N}}$ and $q \ge 1$,

$$||\phi||_{L^{2q}} \le cN^{1-\frac{1}{q}}||\phi||.$$

In particular,

$$||\phi||_\infty \le cN||\phi||.$$

Theorem 2.2. *For any $\phi \in \mathcal{R}_N$ and $r \ge 0$,*

$$|\phi|_r \le cN^r||(x^2+1)^{-\frac{r}{2}}\phi||.$$

Proof. Let $y \in I$, $\chi^{(\alpha,\beta)}(y)$, $\mathcal{P}_N$ and $\psi(y)$ be the same as in the proof of the last theorem. According to an inverse inequality (see Theorem 2.2 of Guo [15]), for any $\psi(y) \in \mathcal{P}_N$, non-negative integer r, and $\alpha,\ \beta > r-1$,

$$\int_I (\partial_y^r\psi(y))^2\chi^{(\alpha,\beta)}(y)dy \le cN^{2r}\int_I \psi^2(y)\chi^{(\alpha-r,\beta-r)}(y)dy. \tag{2.12}$$

By induction,

$$\partial_x^k\phi(x) = \sum_{j=0}^k (1-y^2)^{\frac{3}{4}+\frac{k}{2}+j}q_j(y)\partial_y^j\psi(y) \tag{2.13}$$

where $q_j(y)$ are some polynomials which are bounded uniformly on I. Hence we use (2.12) and (2.13) to obtain

$$\begin{aligned} |\phi|^2_k &\le c\sum_{j=0}^k \int_I (1-y^2)^{k+2j}(\partial_y^j\psi(y))^2dy \\ &\le c\sum_{j=0}^k (N^{2j}\int_I (1-y^2)^{k+j}\psi^2(y)dy) \\ &\le cN^{2k}\sum_{j=0}^k \int_\Lambda (x^2+1)^{-k-j}\phi^2(x)dx \\ &\le cN^{2k}||(x^2+1)^{-\frac{k}{2}}\phi||^2. \end{aligned}$$

The general result follows from the above and space interpolation.

2.2. Some Basic Results on Jacobi Approximation

For analyzing the Legendre rational approximation, we need some basic results on the Jacobi approximation.

Let $\chi^{(\alpha,\beta)}(y)$ be the same as before, and

$$L^2_{\chi^{(\alpha,\beta)}}(I) = \{v \mid v \text{ is measurable on } I \text{ and } ||v||_{L^2_{\chi^{(\alpha,\beta)}}(I)} < \infty\}$$

where

$$||v||_{L^2_{\chi^{(\alpha,\beta)}}(I)} = (\int_I v^2(y)\chi^{(\alpha,\beta)}(y)dy)^{\frac{1}{2}}.$$

For any non-negative integer r,

$$H^r_{\chi^{(\alpha,\beta)}}(I) = \{v \mid \partial_y^k v \in L^2_{\chi^{(\alpha,\beta)}}(I),\ 0 \le k \le r\}$$

with the norm

$$||v||_{r,\chi^{(\alpha,\beta)}} = (\sum_{k=0}^{r} ||\partial_x^k v||^2_{L^2_{\chi^{(\alpha,\beta)}}(I)})^{\frac{1}{2}}.$$

For any real $r > 0$, we define the space $H^r_{\chi^{(\alpha,\beta)}}(I)$ with the norm $||v||_{r,\chi^{(\alpha,\beta)}}$ by space interpolation as in Adams [1].

For technical reasons, Guo [14,15] introduced the space $H^r_{\chi^{(\alpha,\beta)},\widetilde{A}}(I)$. For any non-negative integer r,

$$H^r_{\chi^{(\alpha,\beta)},\widetilde{A}}(I) = \{v \mid v \text{ is measuable on } I \text{ and } ||v||_{r,\chi^{(\alpha,\beta)},\widetilde{A}} < \infty\}$$

where

$$||v||_{r,\chi^{(\alpha,\beta)},\widetilde{A}} = (\sum_{k=0}^{[\frac{r-1}{2}]} ||(1-y^2)^{\frac{r}{2}-k}\partial_y^{r-k} v||^2_{\chi^{(\alpha,\beta)}} + ||v||^2_{[\frac{r}{2}],\chi^{(\alpha,\beta)}})^{\frac{1}{2}}. \tag{2.14}$$

For any real $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},\widetilde{A}}(I)$ and its norm $||v||_{r,\chi^{(\alpha,\beta)},\widetilde{A}}$ are defined by space interpolation. Next, for any non-negative integer μ, we define

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(I) = \{v \mid \partial_y^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},\widetilde{A}}(I)\},$$

$$H^r_{\chi^{(\alpha,\beta)},**,\mu}(I) = \{v \mid v \in H^r_{\chi^{(\alpha,\beta)},*,k}(I), \quad 0 \le k \le \mu\}$$

with the norms

$$||v||_{r,\chi^{(\alpha,\beta)},*,\mu} = ||\partial_y^\mu v||_{r-\mu,\chi^{(\alpha,\beta)},\widetilde{A}},$$

$$||v||_{r,\chi^{(\alpha,\beta)},**,\mu} = (\sum_{k=0}^{\mu} ||v||^2_{r,\chi^{(\alpha,\beta)},*,k})^{\frac{1}{2}}.$$

For any real $\mu > 0$, we define the spaces $H^r_{\chi^{(\alpha,\beta)},*,\mu}(I)$, $H^r_{\chi^{(\alpha,\beta)},**,\mu}(I)$ and their norms by space interpolation.

For simplicity, we denote $H^r_{\chi^{(\alpha,\beta)},*,1}(I)$ by $H^r_{\chi^{(\alpha,\beta)},*}(I)$. It can be verified that

$$||v||^2_{r,\chi^{(\alpha,\beta)},*} = A^{(1)}_{r,\alpha,\beta}(v) + A^{(2)}_{r,\alpha,\beta}(v) \tag{2.15}$$

where

$$\begin{aligned} A^{(1)}_{r,\alpha,\beta}(v) &= \sum_{k=r-[\frac{r}{2}]+1}^{r} \int_I (\partial_y^k v(y))^2 (1-y^2)^{-r+2k-1}(1-y)^\alpha(1+y)^\beta dy, \\ A^{(2)}_{r,\alpha,\beta}(v) &= \sum_{k=1}^{[\frac{r+1}{2}]} \int_I (\partial_y^k v(y))^2 (1-y)^\alpha(1+y)^\beta dy. \end{aligned} \tag{2.16}$$

The orthogonal projection $\widetilde{P}_{N,\alpha,\beta}: L^2_{\chi^{(\alpha,\beta)}}(I) \to \mathcal{P}_N$ is a mapping such that for any $v \in L^2_{\chi^{(\alpha,\beta)}}(I)$,

$$\int_I (\widetilde{P}_{N,\alpha,\beta}v(y) - v(y))\psi(y)\chi^{(\alpha,\beta)}(y)dy = 0, \quad \forall \psi \in \mathcal{P}_N.$$

The following lemmas come from Theorems 2.3 and 2.4 of Guo [15].

Lemma 2.1. *For any* $v \in H^r_{\chi^{(\alpha,\beta)},\widetilde{A}}(I)$ *and* $r \geq 0$,

$$||\widetilde{P}_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leq cN^{-r}||v||_{r,\chi^{(\alpha,\beta)},\widetilde{A}}.$$

Lemma 2.2. *If* $\alpha + r > 1$ *or* $\beta + r > 1$, *then for any* $v \in H^r_{\chi^{(\alpha,\beta)},**,\mu}(I)$, $r \geq 1$ *and* $0 \leq \mu \leq r$,

$$||\widetilde{P}_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq cN^{\sigma(\mu,r)}||v||_{r,\chi^{(\alpha,\beta)},**,\mu}$$

where

$$\sigma(\mu, r) = 2\mu - r.$$

In particular, for any $\alpha = \beta > -1$, *the above result is valid with*

$$\sigma(\mu, r) = \begin{cases} 2\mu - r - \frac{1}{2}, & \text{for } \mu > 1, \\ \frac{3}{2}\mu - r, & \text{for } 0 \leq \mu \leq 1. \end{cases}$$

3. Modified Legendre Rational Approximation

This section is devoted to various orthogonal projections. The $L^2(\Lambda)$-orthogonal projection $P_N : L^2(\Lambda) \to \mathcal{R}_N$ is a mapping such that for any $v \in L^2(\Lambda)$,

$$(P_N v - v, \phi) = 0, \quad \forall \phi \in \mathcal{R}_N,$$

or equivalently,

$$P_N v(x) = \sum_{l=0}^{N} \widehat{v}_l R_l(x).$$

For technical reasons, we introduce the Hilbert space $H^r_A(\Lambda)$. For any integer $r \geq 0$,

$$H^r_A(\Lambda) = \{v \mid v \text{ is a measurable on } \Lambda \text{ and } ||v||_{r,A} < \infty\},$$

equipped with the norm

$$||v||_{r,A} = (\sum_{k=0}^{r} ||(x^2+1)^{\frac{r+k}{2}}\partial_x^k v||^2)^{\frac{1}{2}}.$$

For any real $r > 0$, we define the space $H^r_A(\Lambda)$ and its norm $||v||_{r,A}$ by space interpolation.

Theorem 3.1. *For any* $v \in H^r_A(\Lambda)$ *and* $r \geq 0$,

$$||P_N v - v|| \leq cN^{-r}||v||_{r,A}.$$

Proof. By the definition of P_N, for any $\phi \in \mathcal{R}_N$,

$$\int_\Lambda (P_N v(x) - v(x))\phi(x)dx = 0.$$

Let $y = \frac{x}{\sqrt{x^2+1}}$, $u(y) = (x^2+1)^{\frac{3}{4}}v(x)|_{x=\frac{y}{\sqrt{1-y^2}}}$, $u^*_N(y) = (x^2+1)^{\frac{3}{4}}P_N v(x)|_{x=\frac{y}{\sqrt{1-y^2}}}$ and $\psi(y) = (x^2+1)^{\frac{3}{4}}\phi(x)|_{x=\frac{y}{\sqrt{1-y^2}}}$. Then $\psi(y)$, $u^*_N(y) \in \mathcal{P}_N$ and

$$\int_I (u^*_N(y) - u(y))\psi(y)dy = 0, \quad \forall \psi \in \mathcal{P}_N.$$

Therefore, $u_N^*(y) = \widetilde{P}_{N,0,0}u(y)$. By virtue of Lemma 2.1, for $r \geq 0$,

$$||P_N v - v||^2 = ||u_N^* - u||^2_{L^2_{\chi^{(0,0)}}(I)} = ||\widetilde{P}_{N,0,0}u - u||^2 \leq cN^{-2r}||u||^2_{r,\chi^{(0,0)},\widetilde{A}}.$$

By induction,

$$\partial_y^k u(y) = \sum_{j=0}^{k} p_j(x)(x^2+1)^{k+\frac{j}{2}+\frac{3}{4}}\partial_x^j v(x) \tag{3.1}$$

where $p_j(x)$ are some rational functions which are bounded uniformly on Λ. Thus, we have from (2.14) that for any integer $r \geq 0$,

$$\begin{aligned}||u||^2_{r,\chi^{(0,0)},\widetilde{A}} &\leq c\sum_{k=0}^{[\frac{r-1}{2}]}\sum_{j=0}^{r-k}\int_\Lambda (x^2+1)^{r+j}(\partial_x^j v(x))^2dx + c\sum_{k=0}^{[\frac{r}{2}]}\sum_{j=0}^{k}\int_\Lambda (x^2+1)^{2k+j}(\partial_x^j v(x))^2dx\\ &\leq c\sum_{j=0}^{r}\int_\Lambda (x^2+1)^{r+j}(\partial_x^j v(x))^2dx + c\sum_{j=0}^{[\frac{r}{2}]}\int_\Lambda (x^2+1)^{r+j}(\partial_x^j v(x))^2dx \leq c||v||^2_{r,A}.\end{aligned}$$

The previous statements with space interpolation lead to the desired result.

We now introduce another Hilbert space $H_B^r(\Lambda)$. For any non-negative integer r,

$$H_B^r(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{r,B} < \infty\}$$

where

$$||v||_{r,B} = (\sum_{k=0}^{r}||(x^2+1)^{\frac{r+k+1}{2}}\partial_x^k v||^2)^{\frac{1}{2}}.$$

For any real $r > 0$, the space $H_B^r(\Lambda)$ and its norm $||v||_{r,B}$ are defined by space interpolation.

Theorem 3.2. *For any $v \in H_B^r(\Lambda)$, $r \geq 1$ and $0 \leq \mu \leq 1$,*

$$||P_N v - v||_\mu \leq cN^{\frac{3}{2}\mu - r}||v||_{r,B}.$$

Proof. Let y, $u(y)$ and $u_N^*(y)$ be the same as in the proof of Theorem 3.1. By Lemma 2.2,

$$\begin{aligned}|P_N v - v|_1^2 &= \int_\Lambda (\partial_x(P_N v(x) - v(x)))^2dx\\ &\leq c\int_I (1-y^2)y^2(u_N^*(y) - u(y))^2dy + c\int_I (1-y^2)^3(\partial_y(u_N^*(y) - u(y)))^2dy\\ &\leq c||\widetilde{P}_{N,0,0}u - u||^2 + c||\partial_y(\widetilde{P}_{N,0,0}u - u)||^2 \leq cN^{3-2r}||u||^2_{r,\chi^{(0,0)},**,1}\\ &= cN^{3-2r}(||u||^2_{r,\chi^{(0,0)},*,0} + ||u||^2_{r,\chi^{(0,0)},*,1})\\ &= cN^{3-2r}(||u||^2_{r,\chi^{(0,0)},\widetilde{A}} + ||\partial_y u||^2_{r-1,\chi^{(0,0)},\widetilde{A}}).\end{aligned} \tag{3.2}$$

The upper-bound of $||u||_{r,\chi^{(0,0)},\widetilde{A}}$ has been estimated in Theorem 3.1. We now estimate the term $||\partial_y u||^2_{r-1,\chi^{(0,0)},\widetilde{A}}$. By (3.1),

$$\begin{aligned}||\partial_y u||^2_{r-1,\chi^{(0,0)},\widetilde{A}} &\leq c(\sum_{k=0}^{[\frac{r}{2}-1]}\sum_{j=0}^{r-k}\int_\Lambda (x^2+1)^{r+j+1}(\partial_x^j v(x))^2dx\\ &+ \sum_{k=0}^{[\frac{r-1}{2}]}\sum_{j=0}^{k+1}\int_\Lambda (x^2+1)^{2k+j+2}(\partial_x^j v(x))^2dx) \leq c||v||^2_{r,B}\end{aligned}$$

Therefore by (3.2),

$$|P_N v - v|_{1,\omega} \leq cN^{\frac{3}{2}-r}||v||_{r,B}.$$

The above with Theorem 3.1 and space interpolation leads to the desired result.

When we apply the modified Legendre rational spectral method to nonlinear problems, we need to estimate the upper-bounds of various orthogonal projections. One of them is stated below.

Theorem 3.3. *For any* $v \in H_B^{\frac{3}{2}d}(\Lambda)$ *and* $1 \geq d > \frac{1}{2}$,

$$||P_N v||_\infty \leq c||v||_{\frac{3}{2}d,B}.$$

Proof. By the embedding theory and Theorem 3.2, for any $1 \geq d > \frac{1}{2}$,

$$\begin{aligned}||P_N v||_\infty &\leq ||v||_\infty + ||P_N v - v||_\infty \leq ||v||_d + ||P_N v - v||_d \\ &\leq ||v||_d + c||v||_{\frac{3}{2}d,B}.\end{aligned}$$

In order to analyze the modified Legendre rational interpolation, we need another orthogonal projection. To do this, we introduce the space

$$H^1_{\widehat{A}_0}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{r,\widehat{A}_0} < \infty\}$$

equipped with the norm

$$||v||_{1,\widehat{A}_0} = (||(x^2+1)\partial_x v||^2 + ||v||^2)^{\frac{1}{2}},$$

and

$$H^r_C(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{r,C} < \infty\}$$

with the norm

$$||v||_{r,C} = ||(x^2+1)^{\frac{3}{4}}\partial_x((x^2+1)^{\frac{3}{4}}v)||_{r-1,A}.$$

The $H^1_{\widehat{A}_0}(\Lambda)$–orthogonal projection $\widehat{P}^1_N : H^1_{\widehat{A}_0}(\Lambda) \to \mathcal{R}_N$ is a mapping such that

$$\int_\Lambda \partial_x(\widehat{P}^1_N v(x) - v(x))\partial_x\phi(x)(x^2+1)^2dx + \int_\Lambda(\widehat{P}^1_N v(x) - v(x))\phi(x)dx = 0, \quad \forall \phi \in \mathcal{R}_N.$$

Theorem 3.4. *For any* $v \in H^r_C(\Lambda)$ *and* $r \geq 1$,

$$||\widehat{P}^1_N v - v||_{1,\widehat{A}_0} \leq cN^{1-r}||v||_{r,C}.$$

Proof. Let $u(x) = (x^2+1)^{\frac{3}{4}}v(x)$ and

$$\phi(x) = (x^2+1)^{-\frac{3}{4}}(\int_0^x (z^2+1)^{-\frac{3}{4}}P_{N-1}((z^2+1)^{\frac{3}{4}}\partial_z u(z))dz + v(0)).$$

By the definition of P_{N-1}, there exists a polynomial $q_{N-1} \in \mathcal{P}_{N-1}$ such that

$$P_{N-1}((z^2+1)^{\frac{3}{4}}\partial_z u(z)) = (z^2+1)^{-\frac{3}{4}}q_{N-1}(\frac{z}{\sqrt{z^2+1}})$$

whence

$$\phi(x) = (x^2+1)^{-\frac{3}{4}}(\int_0^{\frac{x}{\sqrt{x^2+1}}} q_{N-1}(z)dz + v(0)).$$

Clearly $\phi \in \mathcal{R}_N$. By the Hardy inequality (see Hardy, Littlewood and Pólya [22]) and Theorem 3.1,

$$\begin{aligned}||\phi - v||^2 &= \int_\Lambda (x^2+1)^{-\frac{3}{2}}(\int_0^x (z^2+1)^{-\frac{3}{4}}(P_{N-1}((z^2+1)^{\frac{3}{4}}\partial_z u(z)) - (z^2+1)^{\frac{3}{4}}\partial_z u(z))dz)^2dx \\ &\leq \int_\Lambda (x^2+1)^{-1}(\int_0^x (z^2+1)^{-\frac{3}{4}}(P_{N-1}((z^2+1)^{\frac{3}{4}}\partial_z u(z)) - (z^2+1)^{\frac{3}{4}}\partial_z u(z))dz)^2dx \\ &\leq c\int_\Lambda (x^2+1)^{-\frac{3}{2}}(P_{N-1}((x^2+1)^{\frac{3}{4}}\partial_x u(x)) - (x^2+1)^{\frac{3}{4}}\partial_x u(x))^2dx \\ &\leq cN^{2-2r}||(x^2+1)^{\frac{3}{4}}\partial_x u||^2_{r-1,A} = cN^{2-2r}||(x^2+1)^{\frac{3}{4}}\partial_x((x^2+1)^{\frac{3}{4}}v)||^2_{r-1,A} \\ &= cN^{2-2r}||v||^2_{r,C}.\end{aligned}$$

Next, we have

$$\partial_x\phi(x) - \partial_x v(x) = F(x) + G(x)$$

where

$$F(x) = (x^2+1)^{-\frac{3}{2}}(P_{N-1}((x^2+1)^{\frac{3}{4}}\partial_x u(x)) - (x^2+1)^{\frac{3}{4}}\partial_x u(x)),$$

$$G(x) = -\frac{3}{2}x(x^2+1)^{-\frac{7}{4}}\int_0^x (z^2+1)^{-\frac{3}{4}}(P_{N-1}((z^2+1)^{\frac{3}{4}}\partial_z u(z)) - (z^2+1)^{\frac{3}{4}}\partial_z u(z))dz.$$

It can be verified that

$$\begin{aligned}||(x^2+1)F|| &\le ||P_{N-1}((x^2+1)^{\frac{3}{4}}\partial_x u) - (x^2+1)^{\frac{3}{4}}\partial_x u|| \\ &\le cN^{1-r}||(x^2+1)^{\frac{3}{4}}\partial_x u||_{r-1,A} = cN^{1-r}||v||_{r,C}.\end{aligned}$$

Again by the Hardy inequality,

$$||(x^2+1)G|| \le cN^{1-r}||v||_{r,C}.$$

Consequently,

$$\begin{aligned}||\widehat{P}_N^1 v - v||_{1,\widehat{A}_0} &\le ||\phi - v||_{1,\widehat{A}_0} \le ||(x^2+1)(F+G)|| + ||\phi - v|| \\ &\le cN^{1-r}||v||_{r,C}.\end{aligned}$$

4. Modified Legendre Rational Interpolation

We now consider the modified Legendre-Gauss rational interpolation. Let $\xi_{N,j}$ be the $N+1$ distinct real zeros of $R_{N+1}(x)$, $0 \le j \le N$. Indeed, we have

$$\xi_{N,j} = \frac{\sigma_{N,j}}{\sqrt{1-\sigma_{N,j}^2}}, \quad 0 \le j \le N, \tag{4.1}$$

where $\sigma_{N,j}$ are the roots of $L_{N+1}(x)$. We denote

$$\omega_{N,j} = (\xi_{N,j}^2+1)^{\frac{3}{2}}\rho_{N,j}, \quad 0 \le j \le N, \tag{4.2}$$

where $\rho_{N,j}$ are the weights of the Legendre-Gauss quadrature,

$$\rho_{N,j} = \frac{2}{(1-\sigma_{N,j}^2)(\partial_y L_{N+1}(\sigma_{N,j}))^2}, \quad 0 \le j \le N.$$

By virtue of (15.3.10) in Szegö [26],

$$\rho_{N,j} \sim \frac{2\pi}{(N+1)}(1-\sigma_{N,j}^2)^{\frac{1}{2}}.$$

Thus,

$$\omega_{N,j} \sim \frac{2\pi}{(N+1)}(\xi_{N,j}^2+1). \tag{4.3}$$

We define the discrete inner product and norm as follows,

$$(u,v)_N = \sum_{j=0}^{N} u(\xi_{N,j})v(\xi_{N,j})\omega_{N,j}, \quad ||v||_N = (v,v)_N^{\frac{1}{2}}.$$

Lemma 4.1. *For any* $\phi \in \mathcal{R}_i$, $\psi \in \mathcal{R}_j$ *with* $i+j \le 2N+1$,

$$(\phi,\psi) = (\phi,\psi)_N.$$

In particular,

$$||\phi|| = ||\phi||_N, \quad \forall \phi \in \mathcal{R}_N. \tag{4.4}$$

Proof. Let $y \in I$ and $\mathcal{P}_N$ be the same as before. Set

$$q(y) = (x^2+1)^{\frac{3}{4}}\phi(x)|_{x=\frac{y}{\sqrt{1-y^2}}}, \quad r(y) = (x^2+1)^{\frac{3}{4}}\psi(x)|_{x=\frac{y}{\sqrt{1-y^2}}}.$$

Then $q \in \mathcal{P}_i$ and $r \in \mathcal{P}_j$. By the property of the Legendre-Gauss quadrature,

$$(\phi,\psi) = \int_I q(y)r(y)dy = \sum_{j=0}^{N} q(\sigma_{N,j})r(\sigma_{N,j})\rho_{N,j}$$
$$= \sum_{j=0}^{N} \phi(\xi_{N,j})\psi(\xi_{N,j})\omega_{N,j} = (\phi,\psi)_N.$$

For any $v \in C(\Lambda)$, the modified Legendre-Gauss rational interpolant $\mathcal{I}_N v \in \mathcal{R}_N$ such that

$$\mathcal{I}_N v(\xi_{N,j}) = v(\xi_{N,j}), \quad 0 \le j \le N,$$

or equivalently,

$$(\mathcal{I}_N v - v, \phi)_N = 0, \quad \forall \phi \in \mathcal{R}_N.$$

The following theorem is related to the stability of rational interpolation.

Theorem 4.1. *For any* $v \in H^1_{\widehat{A}_0}(\Lambda)$,

$$||\mathcal{I}_N v|| \le c(||v|| + N^{-1}||(x^2+1)\partial_x v||).$$

Proof. By (4.3) and (4.4),

$$||\mathcal{I}_N v||^2 = ||\mathcal{I}_N v||_N^2 = \sum_{j=0}^{N} v^2(\xi_{N,j})\omega_{N,j}$$
$$\le cN^{-1}\sum_{j=0}^{N} v^2(\xi_{N,j})(\xi_{N,j}^2+1).$$

Let $x = \text{ctg}\theta$ $(0 \le \theta \le \pi)$ and $\widehat{v}(\theta) = v(\text{ctg}\theta)$. Then

$$||\mathcal{I}_N v||^2 \le cN^{-1}\sum_{j=0}^{N} \widehat{v}^2(\theta_{N,j})\sin^{-2}\theta_{N,j}.$$

According to (4.1) and Theorem 8.9.1 in Szegö [26],

$$\theta_{N,j} = \frac{1}{N+1}(j\pi + \mathcal{O}(1)), \ 0 \le j \le N \tag{4.5}$$

where $\mathcal{O}(1)$ is bounded uniformly for all $0 \le j \le N$. Now, let $a_0 = \frac{\mathcal{O}(1)}{N+1}$ and $a_1 = \frac{N\pi+\mathcal{O}(1)}{N+1}$. Then $\theta_{N,j} \in K_j \subset [a_0, a_1]$, K_j being of size $\frac{c}{N+1}$. Consequently,

$$||\mathcal{I}_N v||^2 \le cN^{-1}\sum_{j=0}^{N} \sup_{\theta \in K_j} |\widehat{v}(\theta)\sin^{-1}\theta|^2.$$

According to (13.7) of Bernardi and Maday [4], for any $f \in H^1(a,b)$,

$$\max_{a \le x \le b}|f(x)| \le c(\frac{1}{\sqrt{b-a}}||f||_{L^2(a,b)} + \sqrt{b-a}||\partial_x f||_{L^2(a,b)}). \tag{4.6}$$

Thus

$$
\begin{aligned}
||\mathcal{I}_N v||^2 &\le c\sum_{j=0}^{N}(||\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(K_j)} + N^{-2}||\partial_\theta(\widehat{v}(\theta)\sin^{-1}\theta)||^2_{L^2(K_j)})\\
&\le c(||\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)} + N^{-2}||\partial_\theta(\widehat{v}(\theta)\sin^{-1}\theta)||^2_{L^2(a_0,a_1)})\\
&\le c(||\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)} + N^{-2}||\partial_\theta\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)})\\
&+(\sup_{a_0\le\theta\le a_1}\frac{\cos^2\theta}{N^2\sin^2\theta})||\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)})\\
&\le c(||\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)} + N^{-2}||\partial_\theta\widehat{v}(\theta)\sin^{-1}\theta||^2_{L^2(0,\pi)}).
\end{aligned}
$$

It can be verified that

$$x^2+1=\sin^{-2}\theta,\quad dx=-\sin^{-2}\theta d\theta,\quad \partial_x v(x)=-\sin^2\theta\partial_\theta\widehat{v}(\theta).$$

Accordingly

$$||\mathcal{I}_N v||^2 \le c(||v||^2 + N^{-2}||(x^2+1)\partial_x v||^2).$$

We now state the main result of this section.

Theorem 4.2. *For any $v\in H^r_C(\Lambda)$ and $0\le\mu\le 1\le r$,*

$$||\mathcal{I}_N v - v||_\mu \le cN^{\mu+1-r}||v||_{r,C}.$$

Proof. Since $\mathcal{I}_N(\widehat{P}^1_N v)=\widehat{P}^1_N v$, we use Theorems 3.4 and 4.1 to obtain that

$$
\begin{aligned}
||\mathcal{I}_N v - \widehat{P}^1_N v|| &\le c(||\widehat{P}^1_N v - v|| + N^{-1}||(x^2+1)\partial_x(\widehat{P}^1_N v - v)||)\\
&\le cN^{1-r}||v||_{r,C}.
\end{aligned}
\tag{4.7}
$$

Using Theorem 3.4 again yields

$$||\mathcal{I}_N v - v|| \le ||\widehat{P}^1_N v - v|| + ||\mathcal{I}_N v - \widehat{P}^1_N v|| \le cN^{1-r}||v||_{r,C}. \tag{4.8}$$

Furthermore, by (4.7) and Theorems 2.2 and 3.4,

$$
\begin{aligned}
|\mathcal{I}_N v - v|_1 &\le |\widehat{P}^1_N v - v|_1 + |\mathcal{I}_N v - \widehat{P}^1_N v|_1\\
&\le |\widehat{P}^1_N v - v|_1 + cN||(x^2+1)^{-\frac{1}{2}}(\mathcal{I}_N v - \widehat{P}^1_N v)||\\
&\le |\widehat{P}^1_N v - v|_1 + cN^{2-r}||v||_{r,C}\\
&\le cN^{2-r}||v||_{r,C}.
\end{aligned}
\tag{4.9}
$$

We complete the proof by (4.8), (4.9) and space interpolation.

5. Modified Rational Spectral Method for Dirac Equation

This section is for some applications. We take the Dirac equation on the whole line as an example to show how to deal with nonlinear wave equations by using the modified Legendre rational approximation.

We first introduce some notations. Let $i=\sqrt{-1}$ and $v(x)$ be a complex valued function,

$$v(x)=v_R(x)+iv_I(x)$$

where $v_R(x)$ and $v_I(x)$ are the real part and the imaginary part of $v(x)$, respectively. Define

$$|v(x)|=(|v_R(x)|^2+|v_I(x)|^2)^{\frac{1}{2}}.$$

Let $\overline{v}$ be the complex conjugate of v, and

$$(u,v)=\int_\Lambda u(x)\overline{v(x)}dx,\quad ||v||=(v,v)^{\frac{1}{2}}.$$

If $v_R,\ v_I\in H^r(\Lambda)$, we say that $v\in H^r(\Lambda)$, with the following semi-norm and norm,

$$|v|_r=(|v_R|_r^2+|v_I|_r^2)^{\frac{1}{2}},\quad ||v||_r=(||v_R||_r^2+||v_I||_r^2)^{\frac{1}{2}}.$$

We define the space $H_A^r(\Lambda),\ H_B^r(\Lambda),\ H_C^r(\Lambda)$ and their norms $||v||_{r,A},\ ||v||_{r,B},\ ||v||_{r,C}$ similarly.

Now let $V=(v_1,v_2)^T$ be a complex-valued vector function,

$$|V|=(|v_1|^2+|v_2|^2)^{\frac{1}{2}},\quad ||V||=(||v_1||^2+||v_2||^2)^{\frac{1}{2}}.$$

If $v_1,\ v_2\in H^r(\Lambda)$, we say that $V\in H^r(\Lambda)$, with the following semi-norm and norm,

$$|V|_r=(|v_1|_r^2+|v_2|_r^2)^{\frac{1}{2}},\quad ||V||_r=(||v_1||_r^2+||v_2||_r^2)^{\frac{1}{2}}.$$

We define the space $H_A^r(\Lambda),\ H_B^r(\Lambda),\ H_C^r(\Lambda)$ and their norms $||v||_{r,A},\ ||v||_{r,B},\ ||v||_{r,C}$ similarly.

The Dirac equation plays an important role in quantum mechanics, see, e.g., Alvarez and Carreras [2], Alvarez, Kuo and Vazquez [3] and Makhankov [24]. Let m and λ be certain real numbers, $\Psi(x,t)=(\psi_1(x,t),\psi_2(x,t))^T$ and $f(x,t)=(f_1(x,t),f_2(x,t))^T$. For simplicity, set

$$Q_1(\Psi(x,t))=i(|\psi_2(x,t)|^2-|\psi_1(x,t)|^2)\psi_1(x,t),$$
$$Q_2(\Psi(x,t))=i(|\psi_1(x,t)|^2-|\psi_2(x,t)|^2)\psi_2(x,t).$$

Then the initial value problem of the Dirac equation is of the form

$$\begin{cases}\partial_t\psi_1(x,t)+\partial_x\psi_2(x,t)+im\psi_1(x,t)+2\lambda Q_1(\Psi(x,t))=f_1(x,t), & x\in\Lambda,\ 0<t\le T,\\ \partial_t\psi_2(x,t)+\partial_x\psi_1(x,t)-im\psi_2(x,t)+2\lambda Q_2(\Psi(x,t))=f_2(x,t), & x\in\Lambda,\ 0<t\le T,\\ \lim\limits_{|x|\to\infty}\Psi(x,t)=0, & 0<t\le T,\\ \Psi(x,0)=\Psi^{(0)}(x), & x\in\Lambda.\end{cases}\tag{5.1}$$

A weak formulation of (5.1) is as follows

$$\begin{cases}(\partial_t\psi_1(t)+\partial_x\psi_2(t)+im\psi_1(t)+2\lambda Q_1(\Psi(t)),v)=(f_1(t),v), & \forall v\in H^1(\Lambda),\ 0<t\le T,\\ (\partial_t\psi_2(t)+\partial_x\psi_1(t)-im\psi_2(t)+2\lambda Q_2(\Psi(t)),v)=(f_2(t),v), & \forall v\in H^1(\Lambda),\ 0<t\le T,\\ \Psi(0)=\Psi^{(0)}.\end{cases}\tag{5.2}$$

We next check the conservation. For simplicity, let $f(x,t)\equiv 0$. We note that for any complex-valued functions $u,\ v\in H^1(\Lambda)$,

$$(\partial_x u,v)+\overline{(\partial_x u,v)}+(u,\partial_x v)+\overline{(u,\partial_x v)}=0.\tag{5.3}$$

Next, for any complex-valued vector function $U=(u_1,u_2)^T\in L^4(\Lambda)$,

$$(Q_1(U),u_1)+\overline{(Q_1(U),u_1)}=0,\tag{5.4}$$

$$(Q_2(U),u_2)+\overline{(Q_2(U),u_2)}=0.\tag{5.5}$$

Furthermore, for any complex-valued function $v\in H^1(0,T;L^2(\Lambda))$,

$$(\partial_t v(t),v(t))+\overline{(\partial_t v(t),v(t))}=\partial_t||v(t)||^2.\tag{5.6}$$

Now, we take $v=\psi_1$ in the first formula of (5.2) with $f\equiv 0$ and $v=\psi_2$ in the second one, respectively. Then we take the complex conjugates of these two resulting equations. Putting the four results together, and using (5.3)-(5.6), we get that

$$\partial_t||\Psi(t)||^2=0,\quad 0<t\le T.$$

Thus the solution of (5.2) possesses the following conservation

$$||\Psi(t)||=||\Psi^{(0)}||.\tag{5.7}$$

We shall use the modified Legendre rational spectral method for (5.2). Let $\phi = \phi_R + i\phi_I$. If $\phi_R,\ \phi_I \in \mathcal{R}_N$, then we write $\phi \in \mathcal{R}_N$. For any vector function $\Phi = (\phi_1, \phi_2)^T$, if $\phi_1,\ \phi_2 \in \mathcal{R}_N$, then we write $\Phi \in \mathcal{R}_N$.

The modified Legendre rational spectral scheme for (5.2) is to find $\Psi_N(x,t) \in \mathcal{R}_N$ for all $0 \le t \le T$ such that

$$\begin{cases} (\partial_t\psi_{1,N}(t) + \partial_x\psi_{2,N}(t) + im\psi_{1,N}(t) + 2\lambda Q_1(\Psi_N(t)), \phi) = (f_1(t), \phi), & \forall\phi \in \mathcal{R}_N, 0 < t \le T, \\ (\partial_t\psi_{2,N}(t) + \partial_x\psi_{1,N}(t) - im\psi_{2,N}(t) + 2\lambda Q_2(\Psi_N(t)), \phi) = (f_2(t), \phi), & \forall\phi \in \mathcal{R}_N,\ 0 < t \le T. \\ \Psi_N(0) = P_N\Psi^{(0)} = (P_N\psi_1^{(0)}, P_N\psi_2^{(0)})^T. \end{cases} \tag{5.8}$$

Following the same line as in the derivation of (5.7), we get that for $f \equiv 0$,

$$||\Psi_N(t)|| = ||\Psi_N^{(0)}||. \tag{5.9}$$

So the numerical solution possesses exactly the same conservation as the genuine solution Ψ. Indeed, this is one of the main advantages of the modified Legendre rational spectral method.

We now deal with the convergence of (5.8). To do this, let

$$\Psi_N^*(t) = (\psi_{1,N}^*, \psi_{2,N}^*)^T = (P_N\psi_1, P_N\psi_2)^T.$$

By (5.2),

$$\begin{cases} (\partial_t\psi_{1,N}^*(t) + \partial_x\psi_{2,N}^*(t) + im\psi_{1,N}^*(t) + 2\lambda Q_1(\Psi_N^*(t)) + E_1(t) + 2\lambda F_1(t), \phi) = (f_1(t), \phi), \\ \qquad\qquad\qquad \forall\phi \in \mathcal{R}_N,\ 0 < t \le T, \\ (\partial_t\psi_{2,N}^*(t) + \partial_x\psi_{1,N}^*(t) - im\psi_{2,N}^*(t) + 2\lambda Q_2(\Psi_N^*(t)) + E_2(t) + 2\lambda F_2(t), \phi) = (f_2(t), \phi), \\ \qquad\qquad\qquad \forall\phi \in \mathcal{R}_N,\ 0 < t \le T, \\ \Psi_N^*(0) = P_N\Psi^{(0)}, \end{cases} \tag{5.10}$$

where

$$\begin{aligned} E_1(t) &= \partial_x\psi_2(t) - \partial_x\psi_{2,N}^*(t), \\ E_2(t) &= \partial_x\psi_1(t) - \partial_x\psi_{1,N}^*(t), \\ F_j(t) &= Q_j(\Psi(t)) - Q_j(\Psi_N^*(t)), \quad j = 1,\ 2. \end{aligned}$$

Next, let

$$\widetilde{\Psi}_N = (\widetilde{\psi}_{1,N}, \widetilde{\psi}_{2,N})^T = (\psi_{1,N} - \psi_{1,N}^*, \psi_{2,N} - \psi_{2,N}^*)^T.$$

Subtracting (5.10) from (5.8), we obtain

$$\begin{cases} (\partial_t\widetilde{\psi}_{1,N}(t) + \partial_x\widetilde{\psi}_{2,N}(t) + im\widetilde{\psi}_{1,N}(t) + 2\lambda Q_1(\widetilde{\Psi}_N(t)), \phi) = (E_1(t) + 2\lambda F_1(t) + 2\lambda G_1(t), \phi), \\ \qquad\qquad\qquad \forall\phi \in \mathcal{R}_N,\ 0 < t \le T, \\ (\partial_t\widetilde{\psi}_{2,N}(t) + \partial_x\widetilde{\psi}_{1,N}(t) - im\widetilde{\psi}_{2,N}(t) + 2\lambda Q_2(\widetilde{\Psi}_N(t)), \phi) = (E_2(t) + 2\lambda F_2(t) + 2\lambda G_2(t), \phi), \\ \qquad\qquad\qquad \forall\phi \in \mathcal{R}_N,\ 0 < t \le T, \\ \widetilde{\Psi}_N(0) = 0, \end{cases} \tag{5.11}$$

where

$$G_j(t) = -Q_j(\Psi_N^*(t) + \widetilde{\Psi}_N(t)) + Q_j(\Psi_N^*(t)) + Q_j(\widetilde{\Psi}_N(t)), \quad j = 1,\ 2.$$

We take $\phi = \widetilde{\psi}_{j,N}$ in the $j-$th formula of (5.11), and then take the complex conjugates of those two resulting equations. Putting the four results together, we use (5.3)-(5.6) to get that

$$\frac{d}{dt}||\widetilde{\Psi}_N(t)||^2 \le c||\widetilde{\Psi}_N(t)||^2 + c\sum_{j=1}^{2}(||E_j(t)||^2 + ||F_j(t)||^2 + ||G_j(t)||^2). \tag{5.12}$$

Then it remains to estimate the upper-bounds of the last term in (5.12). By Theorem 3.2,

$$||E_j(t)||^2 \le cN^{3-2r}||\Psi(t)||_{r,B}^2,\ j = 1, 2. \tag{5.13}$$

Next, thanks to Theorems 3.2 and 3.3,

$$\begin{aligned}||F_j(t)||^2 &\le c\sum_{k=0}^{2}\sum_{j=1}^{2}||\psi_j(t)||_\infty^{4-2k}||\psi_{j,N}^*(t)||_\infty^{2k}||\psi_j(t)-\psi_{j,N}^*(t)||^2 \\ &\le c^*(\Psi)N^{-2r}||\Psi(t)||_{r,B}^2, \quad j=1,\ 2,\end{aligned} \tag{5.14}$$

where $c^*(\Psi)$ is a positive constant depending only on $||\Psi||_{L^\infty(0,T;L^\infty(\Lambda)\cap H_B^1(\Lambda))}$. Finally Theorem 2.1 implies

$$\begin{aligned}||G_j(t)||^2 &\le c\sum_{k=1}^{2}\sum_{j=1}^{2}||\psi_{j,N}^*(t)||_\infty^{6-2k}||\widetilde{\psi}_{j,N}(t)||_{L^{2k}}^{2k} \\ &\le c^*(\Psi)\sum_{k=1}^{2}N^{2(k-1)}||\widetilde{\Psi}_N(t)||^{2k}.\end{aligned} \tag{5.15}$$

Substituting (5.13)-(5.15) into (5.12) and integrating the resulting inequality with respect to t, we get that

$$||\widetilde{\Psi}_N(t)||^2 \le c^*(\Psi)\int_0^t\sum_{k=1}^{2}N^{2(k-1)}||\widetilde{\Psi}_N(s)||^{2k}ds + c^*(\Psi)N^{3-2r}||\Psi||_{L^2(0,T;H_B^r(\Lambda))}^2. \tag{5.16}$$

Lemma 5.1. *(See Guo, Shen and Xu [20]) Assume that*

(i) *a and a_k are non-negative constants,*

(ii) *$E(t)$ is a non-negative function of t,*

(iii) *$\rho \ge 0$ and for all $0 \le t \le t_1$,*

$$E(t) \le \rho + a\int_0^t (E(s) + \sum_{k=2}^{n} N^{a_k}E^k(s))ds,$$

(iv) *for certain $t_1 > 0$, $\rho e^{nat_1} \le \min_{2\le k\le n} N^{-\frac{a_k}{k-1}}$.*

Then for all $0 \le t \le t_1$,

$$E(t) \le \rho e^{nat}.$$

Applying Lemma 5.1 to (5.16) and using Theorem 3.2, we obtain that

Theorem 5.1. *If for $r \ge \frac{5}{2}$, $\Psi \in L^\infty(0,T;L^\infty(\Lambda)\cap H_B^1(\Lambda))\cap L^2(0,T;H_B^r(\Lambda))$, then for all $0 \le t \le T$,*

$$||\Psi(t) - \Psi_N(t)|| \le c^*(\Psi)N^{\frac{3}{2}-r}||\Psi||_{L^2(0,T;H_B^r(\Lambda))}.$$

6. Numerical Results

In this section, we present some numerical results. Take the test function

$$\psi_1(x,t) = \psi_2(x,t) = \tanh(\frac{t}{x^2+1})(x^2+1)^{-5}.$$

We shall use (5.8) to solve (5.1). Let τ be the mesh size in time. The corresponding fully discrete rational spectral scheme is as follows,

$$\begin{cases} ((1+im\tau)\psi_{1,N}(t+\tau),R_k) = ((1-im\tau)\psi_{1,N}(t-\tau) - 2\tau\partial_x\psi_{2,N}(t) - 4\lambda\tau Q_1(\Psi_N(t)) \\ \qquad +\tau(f_1(t+\tau)+f_1(t-\tau)),R_k), \quad 0\le k\le N, \\ ((1-im\tau)\psi_{2,N}(t+\tau),R_k) = ((1+im\tau)\psi_{2,N}(t-\tau) - 2\tau\partial_x\psi_{1,N}(t) - 4\lambda\tau Q_2(\Psi_N(t)) \\ \qquad +\tau(f_2(t+\tau)+f_2(t-\tau)),R_k), \quad 0\le k\le N. \end{cases} \tag{6.1}$$

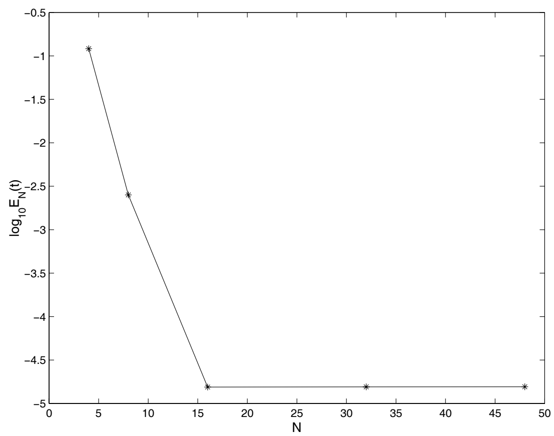

Figure 1: Convergence rates of scheme (6.1), $\tau = 10^{-2}, t = 1$.

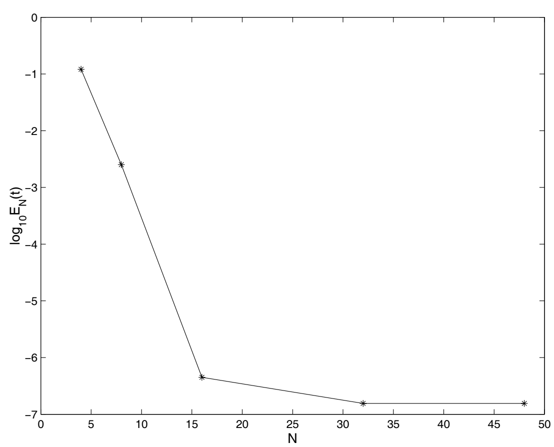

Figure 2: Convergence rates of scheme (6.1), $\tau = 10^{-3}, t = 1.$.

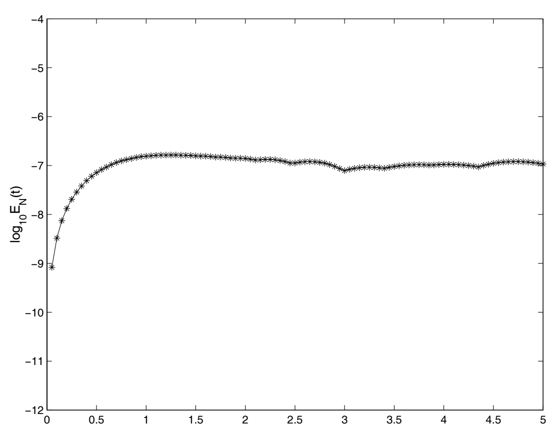

Figure 3: Stability of scheme (6.1), $\tau = 10^{-3}, N = 32$.

Set $m = \lambda = 1$. For any complex-valued function $v(x,t)$, we define the discrete maximum norm as

$$||v(t)||_{\infty,N} = \max_{0\le j\le N}\{|v_R(\xi_{N,j},t)|, |v_I(\xi_{N,j},t)|\}.$$

The numerical error $E_N(t)$ is given by

$$E_N(t) = \max_{j=1,2} ||\psi_{j,N}(t) - \psi_j(t)||_{\infty,N}.$$

We first take $\tau = 10^{-2}$ in scheme (6.1), In Figure 1, we plot the log_{10} of $E_N(t)$ at $t = 1$ vs. N. Figure 2 is for the corresponding cases with $\tau = 10^{-3}$. They indicate the high accuracy and the convergence of this method as N increases and τ decreases. In Figure 3, we plot the log_{10} of $E_N(t)$ of numerical solution of scheme (6.1) with $\tau = 10^{-3}$ and $N = 32$. It indicates the stability of scheme (6.1). They coincide with theoretical analysis very well.

For comparison, we also use the finite difference method as in [2] to solve (5.1). As usual, let A be a positive number and impose the artificial boundary condition $\Psi(-A,t) = \Psi(A,t) = 0$. Let $h = \frac{A}{N}$ and $x_l = lh, \ -N \le l \le N$. The numerical solution is denoted by Ψ_N. Then the corresponding fully discrete finite difference scheme for (5.1) is as follows,

$$\begin{cases} (1+im\tau)\ \psi_{1,N}(x_l,t+\tau) = (1-im\tau)\psi_{1,N}(x_l,t-\tau) - \frac{\tau}{h}(\psi_{2,N}(x_{l+1},t) \\ \qquad -\psi_{2,N}(x_{l-1},t)) - 4\lambda\tau Q_1(\Psi_N(x_l,t)) + \tau(f_1(x_l,t+\tau) + f_1(x_l,t-\tau), \\ (1-im\tau)\ \psi_{2,N}(x_l,t+\tau) = (1+im\tau)\psi_{2,N}(x_l,t-\tau) - \frac{\tau}{h}(\psi_{1,N}(x_{l+1},t) \\ \qquad -\psi_{1,N}(x_{l-1},t)) - 4\lambda\tau Q_2(\Psi_N(x_l,t)) + \tau(f_2(x_l,t+\tau) + f_2(x_l,t-\tau)). \end{cases} \tag{6.2}$$

For description of numerical errors, let

$$||v(t)||_{\infty,N} = \max_{-N\le l\le N}\{|v_R(x_l,t)|, |v_I(x_l,t)|\},$$

$$E_N(t) = \max_{j=1,2} ||\psi_{j,N}(t) - \psi_j(t)||_{\infty,N}.$$

In calculation, we take $A = 30$ and $\tau = 10^{-3}$. In Figure 4, we plot the $E_N(t)$ at $t = 1$ vs. N. It is clear that the new scheme (6.1) provides much better numerical results than the finite difference scheme (6.2).

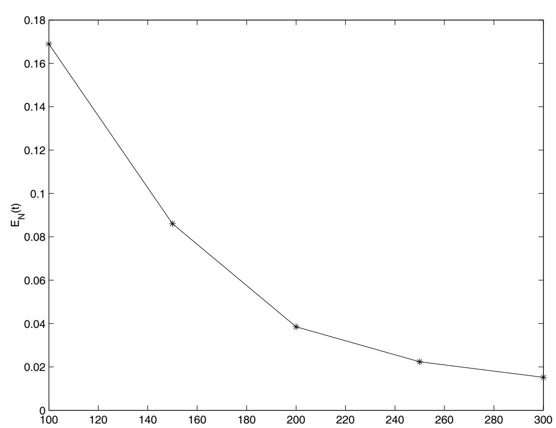

Figure 4: Convergence rates of scheme (6.2), $\tau = 10^{-3}, t = 1$.

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