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THE LOWER APPROXIMATION OF EIGENVALUE BY LUMPED MASS FINITE ELEMENT METHOD *[1)]

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Abstract

In the present paper, we investigate properties of lumped mass finite element method (LFEM hereinafter) eigenvalues of elliptic problems. We propose an equivalent formulation of LFEM and prove that LFEM eigenvalues are smaller than the standard finite element method (SFEM hereinafter) eigenvalues. It is shown, for model eigenvalue problems with uniform meshes, that LFEM eigenvalues are not greater than exact solutions and that they are increasing functions of the number of elements of the triangulation, and numerical examples show that this result equally holds for general problems.

Mathematics subject classification: 65N30.
Key words: Lumped mass, Eigenvalue, Min-max principle, Finite element.

1. Introduction

The finite element method has been widely and successfully applied to both boundary value and eigenvalue problems for a solid continuum. In the boundary value problem, it has been shown that if the interpolation functions satisfy certain criteria [7],the finite element solution will converge to the exact solution as the size of the element is diminished. The convergence likewise occurs for the eigenvalue problems (Ref. [7, 3, 5, 8, 1, 2] and references therein).

There exist two finite element methods for solving eigenvalue problems, one is SFEM (Ref.[1, 2]), the other is LFEM. LFEM has been extensively applied to science and engineering computations because of its simplicity. LFEM in particular can largely simplify the computation of generalized eigenvalue problems (Ref.[3, 5]). The convergence of LFEM for eigenvalue problems was established by Tong.et al [8] and Strang and Fix [7]. Strang and Fix in [7] gave an error expansion of LFEM eigenvalue for one dimensional Neumann problem, the error expansion of LFEM eigenvalue for one dimensional Dirichlet problem was presented in [1], where some comments on the asymptotic lower bound when h tends to zero for the problem therein were also given. Tong.et al in [8] proved that LFEM didn't lose the accuracy of approximation compared with SFEM as long as proper lumped mass method was chosen. The concept of lower approximations of eigenvalues was first introduced in [6]. Numerical experiments therein indicated that LFEM eigenvalues are lower approximations to the exact ones, however the analysis therein is not rigorous.

In the present paper, we investigate properties of LFEM eigenvalues. It is well known that SFEM eigenvalues approximate exact solutions from above [1, 2] and that they are in some sense decreasing functions of the number of elements of the partition of the domain considered, on the contrary, what we are interested in is to show that LFEM eigenvalues approximate exact solutions from below and that they are increasing functions of the number of elements. For model eigenvalue problems with uniform meshes, we provide a rigorous analysis for these properties. For general problems, we propose an equivalent form for LFEM and show that

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LFEM eigenvalues are smaller than SFEM eigenvalues, and the final numerical experiments demonstrate that LFEM eigenvalues are exactly increasing functions of the number of elements, then we can safely assert LFEM eigenvalues are not greater than exact ones. The paper is organized as following. In section 2, we recall the weak formulation of the elliptic eigenvalue problem. LFEM and its equivalent formulation will be described in section 3, and in section 4 we shall show that LFEM eigenvalues are not greater than SFEM eigenvalues, as applications, we shall also prove, for model problems with uniform meshes, that LFEM eigenvalues are lower approximations in the same section. Numerical results are illustrated in section 5. This paper ends with section 6, which brings our final remark.

2. Variational Formulation of Eigenvalue Problem

We shall consider the eigenvalue problem in the divergence form which is read as

$$\begin{cases} Lu = -\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial}{\partial x_i}u) + c(x)u = \lambda\rho u \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset R^d$ is a bounded open domain with smooth enough boundary $\partial\Omega$, $a_{ij}(x)$ have local integrable derivatives, $c(x) \in L^\infty(\Omega)$ and $c(x) \geq 0$. We assume L is a strict elliptic operator.

For the eigenvalue problem $Lu = \lambda\rho u$, there are two variational formulation forms: Rayleigh quotient and weak form, which are expressed as, respectively

$$R(v) = \frac{a(v,v)}{(\rho v, v)}$$

$$a(u,v) = \lambda(\rho u, v) \qquad \forall v \in H_0^1$$

where

$$a(u,v) = \int_\Omega [a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + c(x)uv]dx \text{ and } (\rho u, v) = \int_\Omega \rho uv dx$$

G.Strang and G.J.Fix in [7] show that the two forms are equivalent, in particular, one has

Lemma 2.1 (min-max principle). *Let λ_l be l-th eigenvalue of problem (2.1), it holds that*

$$\lambda_l = \min_{s_l} \max_{v \in s_l} R(v) \tag{2.2}$$

where s_l is any l-dimension subspace of $H_0^1(\Omega)$.

Let A and B be $n \times n$ real symmetric and positive definite matrixes, discrete counterpart of Rayleigh quotient with respect to A and B can be stated as

$$R(x) = \frac{x^T Ax}{x^T Bx} \quad (x \in R^n, x \neq 0)$$

where x^T denotes the transpose of n-dimensional vector x, then one has

Lemma 2.2. *Let λ_k be k-th generalized eigenvalue of A with respect to B, then*

$$\lambda_k = \min_{V_k} \max_{x \in V_k, x \neq 0} R(x) \tag{2.3}$$

where V_k is any k-dimension subspace of R^n.

3. LFEM And Its Equivalent Form

3.1. Example of LFEM

Lumped mass means that element mass is lumped at nodes of element in some average sense, and the mass matrix of the discrete generalized eigenvalue problem in right-hand side becomes a diagonal matrix and even becomes an unit matrix in the simplest case. To fix the idea, let us recall the example given by G.Strang and Fix as following,

$$\begin{cases} -p\frac{d^2u}{dx^2} + qu = \lambda u & 0 < x < \pi \\ u(0) = 0, & u^{'}(\pi) = 0 \end{cases} \tag{3.1}$$

where $p > 0$ and $q > 0$ are two constants.

Partitioning domain $[0, \pi]$ into N subintervals by introducing grid points $x_j = jh$, where $h = \pi/N$ is constant width of subintervals, and taking the piecewise linear function space with natural nodal basis $\varphi_i(x)$ as the finite element approximation space, denoting approximation eigenfunction by $u^h = \sum_{i=1}^{N} Q_i\varphi_i(x)$, where $Q_i, i = 1, \cdots, N$ are values of u^h at nodes, recalling the standard manipulation of finite element procedure, we get, in the end, the system of discrete eigenvalue equations as follows

$$K^hQ^h = \lambda^h M^h Q^h \tag{3.2}$$

where global stiff matrix $K^h = pK_1^h + qM^h$, and

$$M^h = h/6\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ \ddots & \ddots & \ddots & & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}, \quad K_1^h = 1/h\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

So-called lumped mass is to let a diagonal matrix, say M_1^h, take the place of the mass matrix M^h in right-hand side, whose diagonal component takes as the value the sum of components of M^h in the same row, in this case, the diagonal matrix M_1^h is

$$M_1^h = \frac{h}{6}\mathrm{diag}(5, 6, \cdots, 6, 3)$$

the system of eigenvalue equations becomes

$$K^hQ^h = \lambda_1^h M_1^h Q^h \tag{3.3}$$

In particular, (3.2) can be represented as the following difference formulation

$$\frac{p}{h}(2Q_i^h - Q_{i-1}^h - Q_{i+1}^h) + \frac{qh}{6}(4Q_i^h + Q_{i+1}^h + Q_{i-1}^h) = \lambda^h\frac{h}{6}(4Q_i^h + Q_{i+1}^h + Q_{i-1}^h) \tag{3.4}$$

where $Q_{N+1}^h = Q_{N-1}^h$ and $Q_0^h = 0$ because of $u(0) = 0$

Likewise, (3.3) can be rewritten as

$$\frac{p}{h}(2Q_i^h - Q_{i-1}^h - Q_{i+1}^h) + \frac{qh}{6}(4Q_i^h + Q_{i+1}^h + Q_{i-1}^h) = \lambda_1^h hQ_i^h \tag{3.5}$$

where $Q^h_{N+1} = Q^h_{N-1}$ and $Q^h_0 = 0$.

3.2. An Equivalent Form of LFEM

In this section, we establish an equivalent form of lumped mass finite element method here. Let J^h be a partition of domain Ω and S^h be the the usual conforming piecewise linear or bilinear function(trilinear when $d = 3$ respectively) space associated with J^h, and let $S^h_0 = S^h \cap H^1_0(\Omega)$. Then the standard finite element method for the eigenvalue problems is that: Find $u^h \in S^h_0$ and $0 \neq \lambda^h \in R$ such that

$$a(u^h, v^h) = \lambda^h(\rho u^h, v^h) \quad \forall v^h \in s^h_0 \tag{3.6}$$

Denote $\{p_i\}_{i=1}^N$ the interior nodes and $\varphi_1, \cdots, \varphi_N$ the natural nodal basis of S^h_0 associated, then $u^h \in s^h_0$ can be expressed as

$$u^h = \sum_{j=1}^{N} q_j \varphi_j \tag{3.7}$$

where $q_j, j = 1, \cdots, N$ are values of u^h at nodes. Substituting (3.7) into (3.6), and let $v^h = \varphi_k$ $(k = 1, \cdots N)$ successively, we obtain the system of eigenvalue equations for SFEM as follows

$$K^h Q = \lambda^h M^h Q \tag{3.8}$$

where K^h is the stiff matrix and M^h is the mass matrix, $Q = (q_1, q_2, \cdots, q_N)^T$.

As mentioned above, lumped mass is to let a diagonal matrix M^h_1, take the place of the mass matrix M^h in right-hand side, whose the diagonal component takes as the value the sum of components of M^h in the same row, thus the system of LFEM eigenvalue equations takes the form

$$K^h Q = \mu^h M^h_1 Q, \quad M^h = diag(m_1, m_2, \cdots, m_N) \tag{3.9}$$

where $m_i, i = 1, N$ denote the mass lumped in the sense described above at the nodes.

In the case of linear or bilinear(trilinear respectively) element, we can interpret lumped mass procedure from another point of view. We may think the above procedure of lumping mass is to change the density function $\rho(x)$ into the form $\rho(x) = \sum\limits_k m_k \delta(x - x_k)$, where x_k denote the coordinates of the nodes, and keep nodal shape functions φ_i invariant. Another interpreting of lumping mass is to introduce a set of special nodal shape functions, say $\{\varphi^1_j(x)\}_{j=1}^N$, for the right-hand side of weak form (3.6) and keep $\rho(x)$ invariant.

We now define this set of shape functions $\{\varphi^1_j(x)\}_{j=1}^N$. For element T , let $\{\varphi_j(x)\}_{j=1}^\nu$ be the linear or bilinear(trilinear respectively) basis defined on T, define

$$m^T_j = \sum_{i=1}^{\nu} \int_T \rho \varphi_j \varphi_i dx$$

then $\{\varphi_j(x)^1\}_{j=1}^\nu$ when restricted on T are defined such that:
1). $\varphi^1_j \equiv 1$ on T_j the neighbourhood of node P_j such that $\int_{T_j} \rho(x)dx = m^T_j$ and $\varphi^1_j \equiv 0$ on T/T_j
2). $\text{supp}\varphi^1_j \cap \text{supp}\varphi^1_k = \emptyset, if\ k \neq j$.
Note that $\sum\limits_{j=1}^{\nu} m^T_j = \int_T \rho dx$, it is easy to see there exists a partition $T_j, j = 1, \cdots, \nu$ of T such

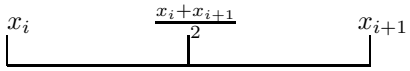
that the two condidtions 1) and 2) are satisfied.

To fix the idea and simplify the representation, we give some examples for the case where the density ρ is constant and linear element is used, for the general case, the argument is similar, in this case, any partition of T such that $\mid T_j \mid$ the volume of T_j equals to $\frac{|T|}{\nu}$ is desirable.

In the case of one dimension, let $T = [x_i, x_{i+1}]$, then φ_i^1, φ_{i+1}^1 can be defined as

$$\varphi_i^1(x) = \begin{cases} 1 & x_i \le x \le \frac{x_i+x_{i+1}}{2} \\ 0 \end{cases}$$

$$\varphi_{i+1}^1(x) = \begin{cases} 1 & \frac{x_i+x_{i+1}}{2} \le x \le x_{i+1} \\ 0 \end{cases}$$

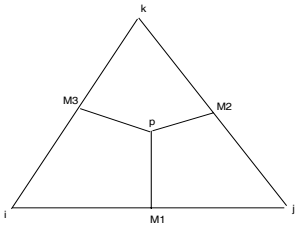


In the case of two dimension, let p be the center of gravity of the triangle T and M_1, M_2, M_3 are middle points, then φ_i^1, φ_j^1, φ_k^1 can be defined as.

$$\varphi_i^1(x) = \begin{cases} 1 & x \in T_i \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_j^1(x) = \begin{cases} 1 & x \in T_j \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_k^1(x) = \begin{cases} 1 & x \in T_k \\ 0 & \text{otherwise} \end{cases}$$



where T_i, T_j and T_k are quadrilaterals taking the node i, j, k as a vertex respectively as illustrated in the figure, for example, T_i is quadrilateral iM_1pM_3.

Set $v_1 = \sum_{i=1}^{N} q_i\varphi_i$ and $v_0 = \sum_{i=1}^{N} q_i\varphi_i^1$, namely, let v_1^h and v_0^h take the same values at nodes $p_i, i = 1, \cdots, N$, then the equivalent form of lumped mass finite method is that : find $u_1^h \in S_0^h$, $0 \ne \mu^h \in R$ such that

$$a(u_1^h, v_1^h) = \mu^h(\rho(x)u_0^h, v_0^h) \quad \forall v_1^h \in s_0^h \tag{3.10}$$

where u_0^h is similarly defined as v_0.

It is easy to show that (3.10) is an equivalent form of lumped mass finite element scheme, because we can get the same system of discrete lumped mass eigenequations.

4. Properties of LFEM Eigenvalues

In this section, we provide an analysis for properties of lumped finite element eigenvalues. Here we just consider the case where the density ρ is constant. We first show, for general problems, that LFEM eigenvalues are not greater than SFEM eigenvalues, then as applications of this result, we prove a stronger result for model problems with uniform meshes, that LFEM eigenvalues are lower approximations to exact ones. Our analysis is based on the equivalent form of LFEM proposed in the previous section. For the convenience of expression, we just consider the affine element.

Since K^h and M^h are symmetric and positive definite matrixes, the Rayleigh quotient follows

$$R(Q) = \frac{Q^T K^h Q}{Q^T M^h Q} \tag{4.1}$$

where $Q = [q_1, \cdots, q_N]^T$, q_i are the values of u^h at nodes p_i. From Lemma 2.2, we have

$$\lambda_l^h = \min_s \max_{Q \in s, Q \neq 0} R(Q) \tag{4.2}$$

where λ_l^h is the l-th eigenvalue of (3.8) and s is an arbitrary l-dimension subspace of R^N.

Obviously, M_1^h is a symmetric and positive definite matrix, Rayleigh quotient of K^h with respect to M_1^h is

$$R_1(Q) = \frac{Q^T K^h Q}{Q^T M_1^h Q} \tag{4.3}$$

Likewise

$$\mu_l^h = \min_s \max_{Q \in s, Q \neq 0} R_1(Q) \tag{4.4}$$

where μ_l^h is the l-th eigenvalue of (3.9) and s is an arbitrary l-dimension subspace of R^N.

We first establish some technical results by using the equivalent form of LFEM (3.10).

Lemma 4.1. *Let μ_l^h be the l-th eigenvalue of (3.9), it holds that*

$$\mu_l^h = \min_{s_2} \max_{v_1 \in s_2} \frac{a(v_1, v_1)}{(\rho v_0, v_0)} \tag{4.5}$$

where s_2 is any l-dimension subspace of s_0^h.

Proof. This is an immediate consequence of (4.4) and (3.10).

Lemma 4.2. *Let $\{\varphi_i\}_{i=1}^N$ and $\{\varphi_i^1\}_{i=1}^N$ be defined as in the previous section, $v_1 = \sum_{i=1}^N q_i \varphi_i$, $v_0 = \sum_{i=1}^N q_i \varphi_i^1$, $Q = \{q_1, \cdots, q_N\}$, it holds that*

$$\max_{Q \in R^N, Q \neq 0} \frac{(v_1, v_1)}{(v_0, v_0)} \leq 1 \tag{4.6}$$

Proof. Since $(v_1, v_1) = \sum_{T \in J^h} \int_\tau v_1^2 dx$, and $(v_0, v_0) = \sum_{T \in J^h} \int_\tau v_0^2 dx$ therefore, we just need to establish (4.6) on each element $T \in J^h$. Moreover, we just consider the affine element, T can be mapped into the reference element $\hat{T}$ by an affine mapping, and the affine transformation does not change the ratio $\frac{(v_1, v_1)}{(v_0, v_0)}$, with this in mind, we now establish (4.6) on $\hat{T}$.

1) In one dimension case, $\hat{T} = [0, 1]$,

$$v_1 = (q_2 - q_1)x + q_1$$

$$v_0 = \begin{cases} q_1 & 0 \leq x < 1/2 \\ q_2 & 1/2 \leq x \leq 1 \end{cases}$$

$$\frac{(v_1, v_1)_{0,T}}{(v_0, v_0)_{0,T}} = \frac{\frac{1}{3}(q_1^2 + q_2^2) + \frac{1}{3} q_1 q_2}{\frac{1}{2}(q_1^2 + q_2^2)} \leq 1$$

2) In two dimension case, the reference element $\hat{T}$ is an isosceles right angle triangle whose right angle zenith is placed at origin and two right angle sides with unit length lie along x axes and y axes, respectively,

$$v_1 = q_3(1-x-y) + q_1 x + q_2 y$$

$$v_0 = \begin{cases} q_1 & (x,y) \in \hat{T}_1 \\ q_2 & (x,y) \in \hat{T}_2 \\ q_3 & (x,y) \in \hat{T}_3 \end{cases}$$

where $\hat{T}_i, i = 1,2,3$ are chosen such that $\mid T_i \mid = \frac{1}{6}$.

Direct calculation yields

$$\frac{(v_1,v_1)_{0,T}}{(v_0,v_0)_{0,T}} = \frac{\frac{1}{12}(q_1^2+q_2^2+q_3^2) + \frac{1}{12}(q_1q_2+q_1q_3+q_2q_3)}{\frac{1}{6}(q_1^2+q_2^2+q_3^2)} \leq 1$$

Similar argument shows that (4.6) holds for the three dimension case.

Now we show our main result in this section

Theorem 4.1. *Let λ_l^h be the l-th eigenvalue of standard finite element method, and μ_l^h be l-th eigenvalue of lumped mass finite element method, assume ρ is constant, then we have*

$$\mu_l^h \leq \lambda_l^h \tag{4.7}$$

Proof. set $v_1 = \sum_{i=1}^{N} q_i\varphi_i$, $v_0 = \sum_{i=1}^{N} q_i\varphi_i^1$, $Q = (q_1, q_2, \cdots, q_N)^T$. From (4.2), and Lemma 4.1, we have

$$\lambda_l^h = \min_s \max_{v_1 \in s, v_1 \neq 0} \frac{a(v_1,v_1)}{(\rho v_1, v_1)}, \text{and} \quad \mu_l^h = \min_s \max_{v_1 \in s, v_1 \neq 0} \frac{a(v_1,v_1)}{(\rho v_0, v_0)}$$

where s is any l-dimension subspace of S_0^h.

For given s_1, by virtue of Lemma 4.2, we deduce

$$\begin{aligned} \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_0,v_0)} &= \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_1,v_1)} \frac{(v_1,v_1)}{(v_0,v_0)} \\ &\leq \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_1,v_1)} \max_{v_1 \in s_1} \frac{(v_1,v_1)}{(v_0,v_0)} \\ &\leq \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_1,v_1)} \end{aligned}$$

Then

$$\min_s \max_{v_1 \in s} \frac{a(v_1,v_1)}{\rho(v_0,v_0)} \leq \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_1,v_1)}$$

Thus, we get

$$\min_s \max_{v_1 \in s} \frac{a(v_1,v_1)}{\rho(v_0,v_0)} \leq \min_{s_1} \max_{v_1 \in s_1} \frac{a(v_1,v_1)}{\rho(v_1,v_1)}$$

$$\mu_l^h \leq \lambda_l^h$$

which ends the proof.

Remark 4.1. For the linear element, it is well-known that SFEM eigenvalue λ_l^h is in some sense a decreasing function of the number of elements of the triangulation of Ω and is not smaller than exact solution λ_l, therefore, if one show that LFEM eigenvalue μ_l^h is in some sense

an increasing function, then from Theorem 4.1, we can assert μ_l^h is a lower approximation to λ_l.

We shall in what follows examine LFEM eigenvalues of model problems with unform meshes.

Example 1. *One dimensional Neumann problem.* We recall the example given in section 3. By using the knowledge of constant coefficient difference equations and the special Toeplitz structure of eigenvalue equations (3.5), we get l-th eigenvector and eigenvalue as follow

$$(Q_l^h)_j = \sqrt{\frac{\pi}{2}}\sin[(l-\frac{1}{2})jh], \text{and } \mu_l^h = pk_h(l) + qm_h(l)$$

where $k_h(l) = 2h^{-2}[1-\cos(l-\frac{1}{2})h], m_h(l) = \frac{2+\cos(l-\frac{1}{2})h}{3}$. Assume that $(l-\frac{1}{2})h \le \frac{\pi}{2}$, it is easy to see μ_l^h is a decreasing function of h, which amouts to μ_l^h is a increasing function of the number of elements of the partition.

Remark 4.2. Without assumption $(l-\frac{1}{2})h \le \frac{\pi}{2}$, it equally holds that

$$\mu_l^h \le \lambda_l \tag{4.8}$$

Proof. In fact, carefully exploiting the Taylor expansion of $k_h(l), m_h(l)$, we obtain

$$k_h(l) = (l-\frac{1}{2})^2 - \frac{2}{h^2}(\frac{[(l-\frac{1}{2})h]^4}{4!} - \frac{[(l-\frac{1}{2})h]^6}{6!} + \frac{[(l-\frac{1}{2})h]^8}{8!}\cdots)$$

$$m_h(l) = 1 - \frac{1}{3}(\frac{[(l-\frac{1}{2})h]^2}{2!} - \frac{[(l-\frac{1}{2})h]^4}{4!} + \frac{[(l-\frac{1}{2})h]^6}{6!} - \cdots)$$

i) For $(l-\frac{1}{2})h \ge 2$

$$\begin{aligned}\mu_l^h &= \lambda_l - \frac{2p}{h^2}(\frac{[(l-\frac{1}{2})h]^2}{2!} - 1 + \cos(l-\frac{1}{2})h) - \frac{q}{3}(1-\cos(l-\frac{1}{2})h)\\ &\le \lambda_l\end{aligned}$$

ii) For $(l-\frac{1}{2})h < 2$
Obviously the series

$$\frac{[(l-\frac{1}{2})h]^4}{4!} - \frac{[(l-\frac{1}{2})h]^6}{6!} + \frac{[(l-\frac{1}{2})h]^8}{8!}\cdots$$

and the series

$$\frac{[(l-\frac{1}{2})h]^2}{2!} - \frac{[(l-\frac{1}{2})h]^4}{4!} + \frac{[(l-\frac{1}{2})h]^6}{6!} - \cdots$$

are Newton-Leibniz series, signs of sums are determined by the first term, then we come to

$$k_h(l) \le (l-\frac{1}{2})^2, m_h(l) \le 1$$

which ends the proof.

Remark 4.3. The one dimensional Dirichlet example in [1] can be put into this framework, and a similar result can be obtained. Note that, contrarily to [1], we needn't assume h is small enough and tends to zero.

Example 2. *Two dimensional Dirichlet Problem.* We now consider lumped mass finite element eigenvalues of Laplace operator in the square domain with Dirichlet boundary condition as following

$$\begin{cases} -\triangle u = \lambda u & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \partial\Omega \end{cases} \tag{4.9}$$

where Ω is the rectangle domain:$0 \le x \le 1$, $0 \le y \le 1$. Exact eigenvalues of this problem are $\lambda_{kl} = (k^2 + l^2)\pi^2$ $(k, l = 1, 2, \cdots)$. At this stage ,we confine ourselves to uniform triangle mesh.

Partitioning Ω into $2N^2$ triangles by introducing nodes $p_{i,j} = (ih, jh), i, j = 0, \cdots, N$ where $h = 1/N$ is constant width of subintervals, taking the piecewise linear function space with natural nodal basis as finite element approximation space, and applying lumped mass finite element method to discretize the continuous eigenvalue equation, the system of discrete eigenvalue equations reads as

$$-Q_{i-1,j} - Q_{i,j-1} + 4Q_{i,j} - Q_{i+1,j} - Q_{i,j-1} = h^2 \mu^h Q_{i,j} \tag{4.10}$$

$$Q_{i,0} = Q_{0,j} = Q_{N,j} = Q_{i,N} = 0 \tag{4.11}$$

$i, j = 1, 2, \cdots, N-1$, where $Q_{i,j}$ are values of u^h at nodes $p_{i,j}$.

Taking $q_{i,j} = A \sin ki\pi h \sin lj\pi h$ as trial eigenvector, then we obtain lumped mass finite element eigenvalues as follow

$$\begin{aligned} \mu^h_{kl} &= (4\sin^2 \frac{k\pi h}{2} + 4\sin^2 \frac{l\pi h}{2})/h^2 \\ &= [4 - 2(\cos k\pi h + \cos l\pi h)]/h^2 \end{aligned} \tag{4.12}$$

It is easy, from (4.12), to show that μ^h_{kl} is a decreasing function of h, provided that $\frac{k\pi h}{2} \le \frac{\pi}{2}$ and $\frac{l\pi h}{2} \le \frac{\pi}{2}$, thus μ^h_{kl} is a increasing function of the number of elements, by virtue of Remark 4.1, we come to

$$\mu^h_{kl} \le \lambda_{kl} \tag{4.13}$$

Similar argument shows that, (4.13) is also true for the rectangle element.

5. Numerical Examples

In the following figures, small box stands for SFEM, and plus LFEM, horizontal axis is refered to the number of elements of the partition and vertial axis is refered to errors or approximate eigenvalues.

5.1. Eigenvalue of Laplace operator with Dirichlet boundary condition

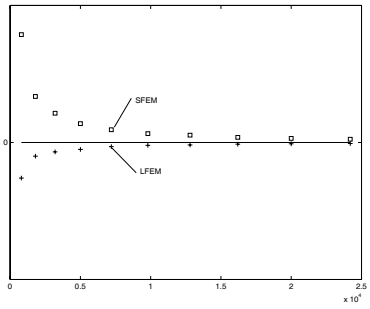


Fig.1 Errors of LFEM and SFEM eigenvalues with quasi-uniform triangular mesh (exact solution=19.7392088)

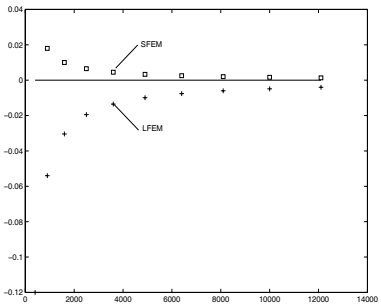


Fig.2 Errors of LFEM and SFEM eigenvalues with quasi-uniform rectangular mesh (exact solution=19.7392088)

5.2. Eigenvalue of Laplace operator in unit circle domain

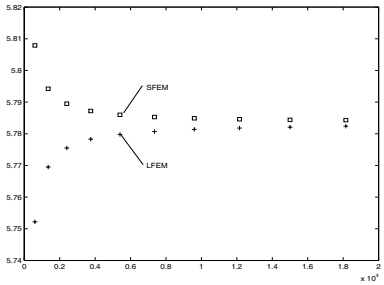


Fig.3 LFEM and SFEM eigenvalues with quasi-uniform triangular mesh

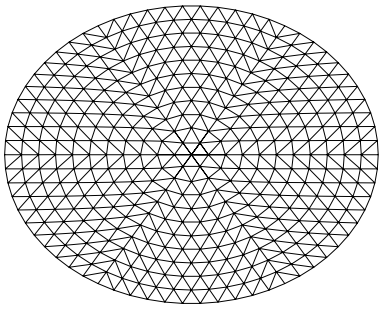


Fig.4 mesh of unit circle domain

5.3. Eigenvalue of Lu=-$\triangle u$+(x+y+1)u in unit square domain with ρ(x,y) =x+y+1

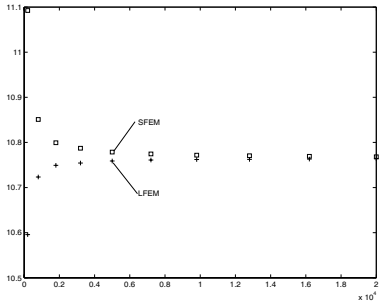


Fig.5 LFEM and SFEM eigenvalues with quasi-uniform triangular mesh

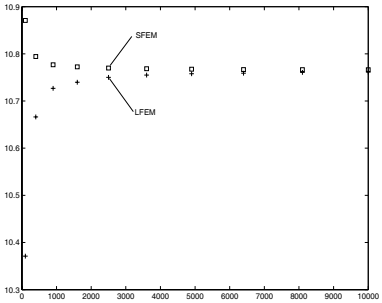


Fig.6 LFEM and SFEM eigenvalues with quasi-uniform rectangular mesh

5.4. Eigenvalue of Lu=-$\partial_y[(xy+1)\partial_x u]$+$(\sin(x)+\sin(y)+1)u$ in unit square domain with $\rho(x,y) = \cos(x)\cos(y)+1$

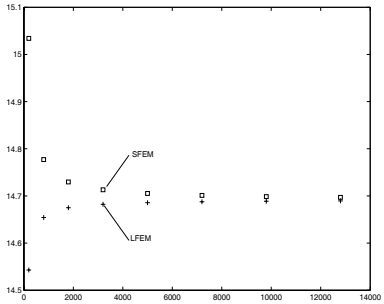


Fig.7 LFEM and SFEM eigenvalues with uniform triangular mesh

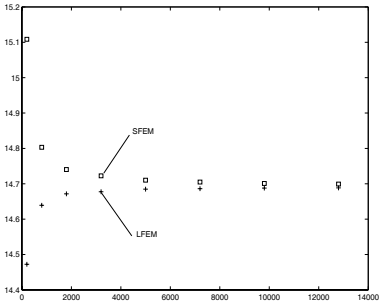


Fig.8 LFEM and SFEM eigenvalues with quasi-uniform triangular mesh

5.5. Eigenvalue of beam vibration problem

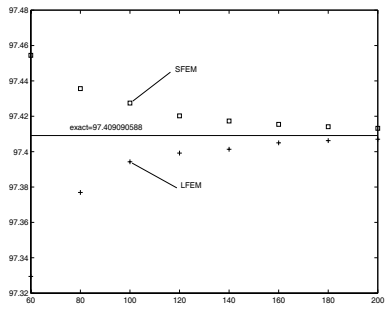


Fig.9 LFEM and SFEM eigenvalues with quasi-uniform mesh (exact solution=97.409090588)

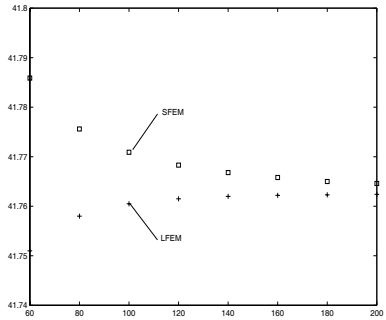


Fig.10 LFEM and SFEM eigenvalues with quasi-uniform mesh

We shall consider the transverse free vibration of elastic beams in this subsection. Let Y be the transverse displacement, which agrees the following fourth order differential equation

$$-\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 Y(x,t)}{\partial x^2}] = \rho(x)\frac{\partial^2 Y(x,t)}{\partial t^2}$$

The corresponding eigenvalue problems is

$$-\frac{d^2}{dx^2}[EI(x)\frac{d^2 Y(x)}{dx^2}] = \lambda\rho(x)Y(x)$$

Where E is elastic modulus, I(x) is moment of inertia.

In order to use piecewise linear function space as finite element approximation space, we have to reduce the fourth order differential equation into a system of second differential equations by introducing the bending moment $M(x) = EI(x)\frac{d^2Y(x)}{dx^2}$ as an independent variable, we obtain

$$\begin{cases} -\frac{d^2M(x)}{dx^2} = \lambda\rho(x)Y(x) \\ M(x) = EI(x)\frac{d^2Y(x)}{dx^2} \end{cases}$$

In our numerical examples, we set E=1,I=1,$\rho = 1$ and E=1,I=1, $\rho = 1.5 + \sin 4 * \pi x$, the results are illustrated in Fig.9 and Fig.10 respectively

6. Final Conclusion

In the present paper, we provide analysis for properties of LFEM eigenvalues. We show that SFEM eigenvalues are not smaller than LFEM eigenvalues for the elliptic eigenvalue problem where the density ρ is constant, however, numerical examples in the previous section denonstrate this result equally holds for the problem where ρ is no constant. As applications of this result, we examine the LFEM eigenvalues of model problems and prove they are lower approximations to exact solutions, for general problems, whether such a result is valid or not is still an open problem, but as Remark 4.1 points out, if the fact that has been demonstrated by our numerical examples, namely, LFEM eigenvalues are in some sense increasing functions of the number of elements, is provd, then one can safely assert LFEM eigenvalues are lower approximations. For one dimensional problem with constant coefficients, let J_2^h be the refinement of J_1^h by introducing a more node, by comparing the Rayleigh quotients of them, for the first LFEM

eigenvalue, we can show it is an increasing function in this sense, unfortunately, such an idea can not be extended to general problems, so we skill the proof here. Whether LFEM eigenvalues are increasing functions and therefore lower approximations or not for the general case needs and is worth further studying.

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References

[1] I. Babuska and J.E. Osborn, Finite element-Galerkin Approximation of the Eigenvalues and Eigenvectors of Selfadjoint Problem, *Math. Comp.*, **52** (1989), 275-297.

[2] I. Babuska and J.E. Osborn, Eigenvalue Problems, in Handbook of Numerical Analysis,V.II: Finite Element Methods(Part I), Edited by P.G.Ciarlet and J.L.Lions, 1991, Elsevier.

[3] K.J. Bathe and E.L. Wilson, Numerical methods in finite element analysis, Prentice-Hall, 1976.

[4] Chuanmao Chen and Yunqing Huang, High Accuracy Theory of Finite Element Methods, Hunan Science Press, 1995.

[5] L.S. Jiang and Z.Y. Pang, Finite Element and its Mathematical Foundation, People Education Press, 1979.

[6] Hongmei Shen, The lower approximation of eigenvalue and it's relative problem, Master thesis , Xiangtan unvi., 1999.

[7] G.Srang and G.J.Fix, Analysis of the finite element method, Prentice-Hall, 1973.

[8] P. Tong, T.H. Pian and L.L. Bucciarelli, Mode shapes and frequencies by the finite element method using consistent and lumped mass finite element, *J. Comp. Struct*, **1** (1971), 623-638.