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LONG-TIME BEHAVIOR OF FINITE DIFFERENCE SOLUTIONS OF THREE-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH WEAKLY DAMPED *1)

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Abstract

The three-dimensional nonlinear Schrödinger equation with weakly damped that possesses a global attractor are considered. The dynamical properties of the discrete dynamical system which generate by a class of finite difference scheme are analysed. The existence of global attractor is proved for the discrete dynamical system.

Mathematics subject classification: 65M06, 35Q55, 65P99.
Key words: Nonlinear Schrödinger equation, Finite difference method, Global attractor.

1. Introduction

The three-dimensional nonlinear schrödinger equation with weakly damped

$$i\frac{\partial u}{\partial t} + \triangle u + g(|u|^2)u + i\gamma u = f \quad x \in \Omega,\ t > 0 \tag{1.1}$$

where $i = \sqrt{-1}, \triangle u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, \gamma > 0, \Omega = (0, L_1)\times(0, L_2)\times(0, L_3)$, together with appropriate boundary and initial conditions, is arisen in many physical fields. The existence of an attractor is one of the most important characteristics for a dissipative system. The long-time dynamics is completely determined by the attractor of the system. J.M. Ghidaglia [2] studied the long-time behavior of the nonlinear Schrödinger equation (1.1) in dimension one and proved the existence of a compact global attractor $\mathcal{A}$ in H^1 which has the finite Hausdorff and fractal dimension under some conditions. The equation (1.1) in dimension three were studied by P. Laurencot[6], S. Y. Wu & Y. Zhao[9], and obtain also the existence of a compact global attractor $\mathcal{A}$ in H^1 under conditions (1.4)-(1.6). Guo Boling[3] construct the approximate inertial manifolds for the equation (1.1) and the order of approximation of these manifolds to the global attractor were derived. At the same time, a semidiscrete finite difference method of the equation was discussed by Yin Yan[10] and we studied also long-time behavior of completely discrete finite difference solutions of the equation in dimension one in [11]. In this paper, a completely discrete scheme is discussed by finite difference method for the equation (1.1) in dimension three with initial condition

$$u(x, 0) = u_0(x),\ x \in \Omega \tag{1.2}$$

and Dirichlet boundary condition:

$$u|_{\partial\Omega} = 0, t \in R^+, \tag{1.3}$$

where $f \in C(\overline{\Omega})$, $g(s)(0 \leq s < \infty)$ is a real valued smooth function that satisfies

$$\lim_{s\to+\infty} \frac{G_+(s)}{s^{\frac{5}{3}}} = 0, \tag{1.4}$$

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and for some $\omega > 0$

$$\lim_{s\to+\infty} sup\frac{h(s)-\omega G(s)}{s^{\frac{5}{3}}} \le 0. \tag{1.5}$$

$$|g'(s)| \le M, \quad s \in R^+. \tag{1.6}$$

where $h(s) = sg(s), G(s) = \int_0^s g(\sigma)d\sigma$ and $G_+(s) = max\{G(s), 0\}$.

In this paper we make the following assumptions on $g(s)$ besides the above conditions (1.4)–(1.6)

$$\lim_{s\to+\infty} \frac{G(s)}{s^{\frac{5}{3}}} = 0, \tag{1.4$'$}$$

$$g(s) \text{ and } g'(s) \text{ do not change sign in } R^+. \tag{1.7}$$

Finally, let us denote the first difference quotient of $G(s)$ by $G[s_2, s_1]$ on points s_1, s_2, i.e.

$$G[s_2, s_1] = \begin{cases} \frac{G(s_2)-G(s_1)}{s_2-s_1}, & if\ s_2 \ne s_1, \\ g(s_1), & if\ s_2 = s_1. \end{cases}$$

The paper is organized as follows. In §2 we prove embedding theorems and interpolation inequalities for discrete(grid) functions, which are the analogues of the embedding theorems and interpolation inequalities for the Sobolev space $W^{m,p}(\Omega)$, and a discrete system is established by finite difference method for the continuous system which is generated by the nonlinear Schrödinger equations (1.1) with Dirichlet boundary condition (1.3) and initial condition (1.2). In §3 we study the existence of absorbing sets and attractor for the discrete system.

2. Finite Difference Scheme

First, let us divide the domain $\overline{\Omega}$ into small grids by the parallel planes $x_1 = ih_1(0 \le i \le J_1), x_2 = jh_2(0 \le j \le J_2)$ and $x_3 = kh_3(0 \le k \le J_3)$, where h_1, h_2, h_3 are the spatial mesh lengths, J_1, J_2, J_3 are positive integers, and $J_1h_1 = L_1, J_2h_2 = L_2, J_3h_3 = L_3$. Denote the real or the complex value discrete functions on the grid point set $\overline{\Omega}_h = \{(ih_1, jh_2, kh_3); 0 \le i \le J_1, 0 \le j \le J_2, 0 \le k \le J_3\}$ by $\phi, \psi, \cdots$, and let $\Omega_h = \overline{\Omega}_h \cap \Omega, \partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$. We employ $\triangle_{+h_l}, \triangle_{-h_l}$ and δ_{h_l} to denote the forward difference, the backward difference and the forward difference quotient operators respectively in $x_l (1 \le l \le 3)$ direction, and $\triangle_h$ to denote the discrete Laplace operator, i.e.

$$\triangle_h \phi_{i,j,k} = \sum_{l=1}^{3} \frac{\triangle_{+h_l}\triangle_{-h_l}\phi_{i,j,k}}{h_l^2} \ .$$

We let $\triangle t$ denote the temporal mesh length, $J = (J_1 + 1)(J_2 + 1)(J_3 + 1)$, $h = \max_{1\le l\le 3} h_l$, and assume that there exists a positive constant $\delta \in (0, 1]$, such that $\delta h \le h_l (l = 1, 2, 3)$.

We introduce the discrete L^2 inner product

$$(\phi, \psi)_h = \sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3} \phi_{i,j,k}\overline{\psi}_{i,j,k} h_1h_2h_3$$

and the discrete H^1 inner product

$$(\phi, \psi)_{1,h} = \sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}\sum_{l=1}^{3} \delta_{h_l}\phi_{i,j,k}\overline{\delta_{h_l}\psi}_{i,j,k} h_1h_2h_3,$$

together with the associated norms

$$\|\phi\|_h = (\phi, \phi)_h^{\frac{1}{2}}, \quad \|\phi\|_{1,h} = (\phi, \phi)_{1,h}^{\frac{1}{2}}.$$

In addition, we can define the discrete L^p norm and the discrete L^∞ norm as follows

$$\|\phi\|_{L^p_h}=\Big(\sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3}|\phi_{i,j,k}|^p h_1h_2h_3\Big)^{\frac{1}{p}},\qquad \|\phi\|_\infty=\max_{i,j,k}|\phi_{i,j,k}|.$$

It is convenient to let L^p_h and H^1_h denote respectively normed vector spaces

$$\{C^J,\|\cdot\|_{L^p_h}\}\quad and \quad \{C^J_0,\|\cdot\|_{1,h}\}$$

where $C^J_0=\{\phi\in C^J;\phi|_{\partial\Omega_h}=0\}$.

We easily obtain by a simple calculation

Lemma 2.1. *For any discrete functions ϕ,ψ which are defined on $\overline{\Omega}_h$, and $\phi,\psi|_{\partial\Omega_h}=0$, there is the relation*

$$\sum_{i=1}^{J_1-1}\sum_{j=1}^{J_2-1}\sum_{k=1}^{J_3-1}\phi_{i,j,k}\triangle_h\psi_{i,j,k}h_1h_2h_3=-\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}\sum_{l=1}^{3}\delta_{h_l}\phi_{i,j,k}\,\delta_{h_l}\psi_{i,j,k}h_1h_2h_3.$$

Lemma 2.2. *Let discrete function ϕ be defined on $\overline{\Omega}_h$, and $\phi|_{\partial\Omega_h}=0$. Then we have*

$$\|\phi\|_{L^6_h}\le\frac{\sqrt[3]{36}}{2}\prod_{l=1}^{3}\|\delta_{h_l}\phi\|_h^{\frac{1}{3}}\le\frac{\sqrt[6]{48}}{2}\|\phi\|_{1,h}.$$

Proof. From $\phi|_{\partial\Omega_h}=0$ we can see easily the relations

$$\phi^3_{i,j,k}=\sum_{m=0}^{j-1}(\phi^2_{i,m+1,k}+\phi_{i,m+1,k}\phi_{i,m,k}+\phi^2_{i,m,k})\delta_{h_2}\phi_{i,m,k}h_2,$$
$$\phi^3_{i,j,k}=-\sum_{m=j}^{J_2-1}(\phi^2_{i,m+1,k}+\phi_{i,m+1,k}\phi_{i,m,k}+\phi^2_{i,m,k})\delta_{h_2}\phi_{i,m,k}h_2.$$

By Cauchy's inequality, we have

$$2|\phi_{i,j,k}|^3\le\frac{3}{2}\sum_{m=0}^{J_2-1}\big(|\phi_{i,m+1,k}|^2+|\phi_{i,m,k}|^2\big)|\delta_{h_2}\phi_{i,m,k}|h_2\le3\Big(\sum_{m=0}^{J_2-1}|\phi_{i,m,k}|^4h_2\Big)^{\frac{1}{2}}\Big(\sum_{m=0}^{J_2-1}|\delta_{h_2}\phi_{i,m,k}|^2h_2\Big)^{\frac{1}{2}},$$

or

$$\max_{0\le j\le J_2}|\phi_{i,j,k}|^3\le\frac{3}{2}\Big(\sum_{m=0}^{J_2-1}|\phi_{i,m,k}|^4h_2\Big)^{\frac{1}{2}}\Big(\sum_{m=0}^{J_2-1}|\delta_{h_2}\phi_{i,m,k}|^2h_2\Big)^{\frac{1}{2}}.\tag{2.1}$$

Multiplying both sides in (2.1) by h_3 and summing up for k from 0 to J_3-1, then using Cauchy's inequality, we have

$$\sum_{k=0}^{J_3-1}\max_{0\le j\le J_2}|\phi_{i,j,k}|^3h_3\le\frac{3}{2}\Big(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|\phi_{i,m,k}|^4h_2h_3\Big)^{\frac{1}{2}}\Big(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|\delta_{h_2}\phi_{i,m,k}|^2h_2h_3\Big)^{\frac{1}{2}}.$$

Similarly

$$\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|\phi_{i,j,k}|^3h_2\le\frac{3}{2}\Big(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|\phi_{i,j,n}|^4h_2h_3\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|\delta_{h_3}\phi_{i,j,n}|^2h_2h_3\Big)^{\frac{1}{2}}.$$

Hence

$$\Big(\sum_{k=0}^{J_3-1}\max_{0\le j\le J_2}|\phi_{i,j,k}|^3h_3\Big)\Big(\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|\phi_{i,j,k}|^3h_2\Big)$$

$$\leq \frac{9}{4}\Big(\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^4 h_2 h_3\Big)\Big(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|\delta_{h_2}\phi_{i,m,k}|^2 h_2 h_3\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|\delta_{h_3}\phi_{i,j,n}|^2 h_2 h_3\Big)^{\frac{1}{2}}.$$

Multiplying both sides the above inequality by h_1 and summing up for i from 0 to J_1-1, then using Cauchy's inequality, we have

$$\begin{aligned}
&\sum_{i=0}^{J_1-1}\Big(\sum_{k=0}^{J_3-1}\max_{0\leq j\leq J_2}|\phi_{i,j,k}|^3 h_3\Big)\Big(\sum_{j=0}^{J_2-1}\max_{0\leq k\leq J_3}|\phi_{i,j,k}|^3 h_2\Big)h_1\\
\leq&\frac{9}{4}\Big(\max_{0\leq i\leq J_1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^4 h_2 h_3\Big)\Big(\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\delta_{h_2}\phi_{i,j,k}|^2 h_1 h_2 h_3\Big)^{\frac{1}{2}}\cdot\\
&\Big(\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\delta_{h_3}\phi_{i,j,k}|^2 h_1 h_2 h_3\Big)^{\frac{1}{2}}.
\end{aligned}\tag{2.2}$$

However

$$\begin{aligned}
|\phi_{i,j,k}|^4 &\leq \frac{1}{2}\sum_{l=0}^{J_1-1}|(\phi_{l+1,j,k}^2+\phi_{l,j,k}^2)(\phi_{l+1,j,k}+\phi_{l,j,k})||\delta_{h_1}\phi_{l,j,k}|h_1\\
&\leq 2\Big(\sum_{l=0}^{J_1-1}|\phi_{l,j,k}|^6 h_1\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{J_1-1}|\delta_{h_1}\phi_{l,j,k}|^2 h_1\Big)^{\frac{1}{2}},
\end{aligned}$$

thus

$$\max_{0\leq i\leq J_1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^4 h_2 h_3\leq 2\Big(\sum_{l=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{l,j,k}|^6 h_1 h_2 h_3\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\delta_{h_1}\phi_{l,j,k}|^2 h_1 h_2 h_3\Big)^{\frac{1}{2}}.$$

Substituting the above inequality into (2.2), we have

$$\begin{aligned}
&\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^6 h_1 h_2 h_3=\sum_{i=0}^{J_1-1}\Big(\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^3|\phi_{i,j,k}|^3 h_2 h_3\Big)h_1\\
\leq&\sum_{i=0}^{J_1-1}\Big(\sum_{j=0}^{J_2-1}\max_{0\leq k\leq J_3}|\phi_{i,j,k}|^3 h_2\Big)\Big(\sum_{k=0}^{J_3-1}\max_{0\leq j\leq J_2}|\phi_{i,j,k}|^3 h_3\Big)h_1\\
\leq&\frac{9}{2}\Big(\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^6 h_1 h_2 h_3\Big)^{\frac{1}{2}}\prod_{l=1}^{3}\Big(\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\delta_{h_l}\phi_{i,j,k}|^2 h_1 h_2 h_3\Big)^{\frac{1}{2}},
\end{aligned}$$

dividing both sides by $\Big(\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|\phi_{i,j,k}|^6 h_1 h_2 h_3\Big)^{\frac{1}{2}}$, the lemma is proved.

By Lemma 2.2, Hölder's inequality and the define of the discrete L^p norm and the discrete L^∞ norm, we have

Lemma 2.3. *Let ϕ be defined on $\overline{\Omega}_h$, and $\phi|_{\partial\Omega_h}=0$, then for $q\in[2,6]$*

$$\|\phi\|_{L_h^q}\leq\frac{48^{\frac{\mu}{6}}}{2^\mu}\|\phi\|_{1,h}^{\mu}\|\phi\|_h^{1-\mu},$$

where $\mu=\frac{3}{2}-\frac{3}{q}\in[0,1]$, and

$$\|\phi\|_\infty\leq\frac{\sqrt[6]{48}}{2\delta^{\frac{1}{2}}}h^{-\frac{1}{2}}\|\phi\|_{1,h}.$$

Remark 2.1. The analogous results can be extended to the case of variable step size and the case of general convex domain $\Omega \subset R^3$ by almost the same procedures as the proof of Lemma 2.1–Lemma 2.3.

We construct the finite difference system

$$i\frac{e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k} - e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}}{\triangle t} + \frac{1}{2}\triangle_h(e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k})$$

$$+G[|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2]\frac{e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k}+e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}}{2}=f_{i,j,k}\ , \tag{2.3}$$

$$i=1,2,\cdots,J_1-1, j=1,2,\cdots,J_2-1, k=1,2,\cdots,J_3-1, n=0,1,\cdots$$

The finite difference boundary condition is as follows

$$\phi^n|_{\partial\Omega_h} = 0, n = 0, 1, \cdots . \tag{2.4}$$

The initial condition is as

$$\phi^0_{i,j,k} = u_0(ih_1, jh_2, kh_3), i = 0, 1, \cdots, J_1, j = 0, 1, \cdots, J_2, k = 0, 1, \cdots, J_3. \tag{2.5}$$

Now we are going to prove the existence of the solutions ϕ^{n+1} for the finite difference system (2.3) with the boundary conditions (2.4). For any discrete function ϕ that define on $\overline{\Omega}_h$, and $\phi|_{\partial\Omega_h} = 0$, let us define a discrete function Φ as follows

$$i(e^{\frac{\gamma}{2}\triangle t}\Phi_{i,j,k} - e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}) + \frac{1}{2}\triangle t\triangle_h(e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k})$$
$$+\frac{1}{2}\triangle tG[|e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2](e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}+e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k})=\triangle t f_{i,j,k},$$
$$i = 1, 2, \cdots, J_1 - 1, j = 1, 2, \cdots, J_2 - 1, k = 1, 2, \cdots, J_3 - 1.$$

where $\Phi|_{\partial\Omega_h} = 0$. It defines a mapping $\Phi = T(\phi)$ of H^1_h into itself. Obvious, the mapping $T(\phi)$ is continuous for any $\phi \in H^1_h$. In order to obtain the existence of the solutions for the finite difference system (2.3) with boundary conditions (2.4), it is sufficient to prove the uniform boundedness for all the possible fixed point Φ for the mapping λT with respect to the parameter $0 \le \lambda \le 1$ by Leray–Schauder fixed point theorem. Then the fixed point Φ of the mapping λT satisfy that

$$i(e^{\frac{\gamma}{2}\triangle t}\Phi_{i,j,,k} - \lambda e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}) + \frac{1}{2}\lambda\triangle t\triangle_h(e^{\frac{\gamma}{2}\triangle t}\Phi_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k})$$
$$+\frac{1}{2}\lambda\triangle tG[|e^{\frac{\gamma}{2}\triangle t}\Phi_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2](e^{\frac{\gamma}{2}\triangle t}\Phi_{i,j,k}+e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k})=\lambda\triangle t f_{i,j,k},$$
$$i = 1, 2, \cdots, J_1 - 1, j = 1, 2, \cdots, J_2 - 1, k = 1, 2, \cdots, J_3 - 1.$$

Multiplying the both side of the above equalities by $(e^{\frac{\gamma}{2}\triangle t}\overline{\Phi}_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\overline{\phi}^{n}_{i,j,k})h_1h_2h_3$, then summing them up for i from 1 to $J_1 - 1$, for j from 1 to $J_2 - 1$ and for k from 1 to $J_3 - 1$, then taking the imaginary part, we have

$$e^{\frac{\gamma}{2}\triangle t}\|\Phi\|^2_h = (\lambda - 1)e^{-\frac{\gamma}{2}\triangle t}Re(\Phi, \phi^n)_h + \lambda e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|^2_h + \lambda\triangle tIm(f, \Phi + e^{-\gamma\triangle t}\phi^n)_h.$$

By Cauchy's inequality and ε-inequality, we have

$$\begin{aligned} e^{\frac{\gamma}{2}\triangle t}\|\Phi\|^2_h \le& e^{-\frac{\gamma}{2}\triangle t}\|\Phi\|_h\|\phi^n\|_h + e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|^2_h + \triangle t(\|f\|_h\|\Phi\|_h + e^{-\gamma\triangle t}\|f\|_h\|\phi^n\|_h) \\ \le& \frac{1}{2}\|\Phi\|^2_h + \frac{1}{2}e^{-\gamma\triangle t}\|\phi\|^2_h + e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|^2_h + \frac{\gamma}{2}\triangle t\|\Phi\|^2_h + \frac{\triangle t}{2\gamma}\|f\|^2_h \end{aligned}$$

$$+\triangle te^{-\gamma\triangle t}\frac{\gamma}{2}e^{-\frac{\gamma}{2}\triangle t}\|\phi^n\|_h^2+\frac{\triangle t}{2\gamma}e^{-\frac{\gamma}{2}\triangle t}\|f\|_h^2,$$

it implies that

$$\|\Phi\|_h^2\le 3e^{-\gamma\triangle t}\|\phi^n\|_h^2+\frac{2\triangle t}{\gamma}\|f\|_h^2.$$

This means that $\sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3}|\Phi_{i,j,k}|^2h_1h_2h_3$ is uniformly bounded with respect to the parameter $0\le\lambda\le 1$. Thus the solution of the finite difference system (2.3) with boundary conditions (2.4) exists. The uniqueness of the solution of the finite difference system is proved by lemma 3.4 in section 3.

3. Long-time Behavior of Discrete System

In this section, let us put (2.3) with boundary conditions (2.4) in framework of dissipative dynamical systems. For fixed h_1,h_2,h_3 and $\triangle t$ let us define operator $S_{h,\triangle t}:H_h^1\to H_h^1$ by

$$\phi^1=S_{h,\triangle t}\phi^0,$$

hence, by (3.17) in the follow, for every $n\ge 0$, the family of solution operators $\{(S_{h,\triangle t})^n\}_{n\ge 0}$ defined by $\phi^n=(S_{h,\triangle t})^n\phi^0$, forms a continuous semigroup on H_h^1.

New we turn to prove the existence of absorbing sets in L_h^2 under the discrete system $(S_{h,\triangle t})^n$.

Lemma 3.1. *For the solutions ϕ^n of the discrete system, there is priori estimate as follows*

$$\|\phi^n\|_h^2\le e^{-\gamma n\triangle t}\|\phi^0\|_h^2+\gamma^{-2}e^{\gamma\triangle t}\|f\|_h^2(1-e^{-\gamma n\triangle t}),\quad n=0,1,2,\cdots$$

in particular

$$\sup_{n\ge 0}\|\phi^n\|_h\le\max\left\{\|\phi^0\|_h,\gamma^{-1}e^{\frac{\gamma}{2}\triangle t}\|f\|_h\right\}=C_0.$$

Furthermore, there exists a constant $\rho_0>\frac{1}{\gamma}e^{\frac{\gamma}{2}\triangle t}\|f\|_h$ such that the ball

$$B_0^h=\{\phi\in L_h^2;\|\phi\|_h\le\rho_0\}$$

is a absorbing set in L_h^2 under the semigroup $(S_{h,\triangle t})^n$.

Proof. Multiplying relation (2.3) by $\triangle t(e^{\frac{\gamma}{2}\triangle t}\overline{\phi}_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\overline{\phi}_{i,j,k}^{n})h_1h_2h_3$, and summing them up for i from 1 to J_1-1, for j from 1 to J_2-1 and for k from 1 to J_3-1 respectively, then taking the imaginary part, we have

$$\begin{aligned}&Re(e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}-e^{-\frac{\gamma}{2}\triangle t}\phi^n,e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n)_h\\&+\frac{1}{2}\triangle tIm(\triangle_h(e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n),e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n)_h\\=&\triangle tIm(f,e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n)_h.\end{aligned}\tag{3.1}$$

It is easy to see that

$$Re(e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}-e^{-\frac{\gamma}{2}\triangle t}\phi^n,e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n)_h=\|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}\|_h^2-\|e^{-\frac{\gamma}{2}\triangle t}\phi^n\|_h^2.$$

By Lemma 2.1

$$Im(\triangle_h(e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n),e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi^n)_h=0.$$

Therefore, (3.1) can be rewritten as follows

$$e^{\frac{\gamma}{2}\triangle t}\|\phi^{n+1}\|_h^2=e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|_h^2+\triangle tIm(f,\phi^{n+1}+e^{-\gamma\triangle t}\phi^n)_h.$$

It follows that

$$\begin{aligned} e^{\frac{\gamma}{2}\triangle t}\|\phi^{n+1}\|_h^2 \le & e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|_h^2 + \triangle t|(f,\phi^{n+1})_h| + \triangle t e^{-\gamma\triangle t}|(f,\phi^n)_h| \\ \le & e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|_h^2 + \triangle t\|f\|_h\|\phi^{n+1}\|_h + \triangle t e^{-\gamma\triangle t}\|f\|_h\|\phi^n\|_h \\ \le & e^{-\frac{3\gamma}{2}\triangle t}\|\phi^n\|_h^2 + \frac{\gamma}{2}\triangle t\|\phi^{n+1}\|_h^2 + \frac{\triangle t}{2\gamma}\|f\|_h^2 \\ & + \triangle t e^{-\gamma\triangle t}\frac{\gamma}{2}e^{-\frac{\gamma}{2}\triangle t}\|\phi^n\|_h^2 + \frac{\triangle t}{2\gamma}e^{-\frac{\gamma}{2}\triangle t}\|f\|_h^2, \end{aligned}$$

it implies that

$$\begin{aligned} \|\phi^{n+1}\|_h^2 & \le e^{-\gamma\triangle t}\|\phi^n\|_h^2 + \frac{\triangle t}{\gamma}\|f\|_h^2 \\ & \le \cdots\ \cdots \\ & \le e^{-\gamma(n+1)\triangle t}\|\phi^0\|_h^2 + \frac{1}{\gamma^2}e^{\gamma\triangle t}\|f\|_h^2(1-e^{-\gamma(n+1)\triangle t}), n=0,1,\cdots. \end{aligned}$$

The proof is complete.

Lemma 3.2. *Under the condition* $(1.4)'$, *for every* $\varepsilon > 0$, *we have*

$$|(G(|e^{\beta\triangle t}\varphi|^2),1)_h| \le \varepsilon e^{\frac{10}{3}\beta\triangle t}\|\varphi\|_{1,h}^2\|\varphi\|_h^{\frac{4}{3}} + C'_\varepsilon|\Omega|.$$

Where β *is a arbitrary real number,* C'_ε *is a constant only dependent on* ε *and* $g(s)$, *and* $|\Omega|$ *denotes the volume of the domain* Ω.

Proof. From $(1.4)'$ one know that for every $\varepsilon > 0$, there exists a constant $C'_\varepsilon > 0$, such that

$$|G(s)| \le \varepsilon s^{\frac{5}{3}} + C'_\varepsilon,\ s \ge 0. \tag{3.2}$$

By Lemma 2.3 , we have

$$\begin{aligned} |(G(|e^{\beta\triangle t}\varphi|^2),1)_h| & \le (\varepsilon|e^{\beta\triangle t}\varphi|^{\frac{10}{3}} + C'_\varepsilon, 1)_h \\ & \le \varepsilon e^{\frac{10}{3}\beta\triangle t}\|\varphi\|_{L_h^{\frac{10}{3}}}^{\frac{10}{3}} + C'_\varepsilon|\Omega| \\ & \le \varepsilon e^{\frac{10}{3}\beta\triangle t}\|\varphi\|_{1,h}^2\|\varphi\|_h^{\frac{4}{3}} + C'_\varepsilon|\Omega|. \end{aligned}$$

From this we can obtain the conclusions of the lemma.

Now we turn to prove the existence of an absorbing set in the space H_h^1 under the discrete system $(S_{h,\triangle t})^n$.

Lemma 3.3. *Under the conditions* $(1.4)'$, *(1.5) and (1.7), there are priori estimates for the solutions* ϕ^n *of the discrete system as follows*

$$\begin{aligned} \|\phi^n\|_{1,h}^2 \le & 2e^{-\gamma n\triangle t}(E^0 + \|\phi^0\|_h^2) + 4e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0} + C''_{\varepsilon_0}\big)|\Omega|(1-e^{-\gamma n\triangle t}) \\ & + 2C'_{\varepsilon_0}|\Omega| + 2\|f\|_h^2 + \frac{2}{\gamma^2}e^{\gamma\triangle t}\|f\|_h^2(1-e^{-\gamma n\triangle t}). \end{aligned} \tag{3.3}$$

In particular

$$\begin{aligned} \sup_{n\ge 0}\|\phi^n\|_{1,h}^2 \le & max\big\{2E^0 + 2\|\phi^0\|_h^2, 4e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0} + C''_{\varepsilon_0}\big)|\Omega| + \frac{2}{\gamma^2}e^{\gamma\triangle t}\|f\|_h^2\big\} \\ & + 2C'_{\varepsilon_0}|\Omega| + 2\|f\|_h^2 = C_1. \end{aligned} \tag{3.4}$$

Furthermore, there exists a constant $\rho_1 > \Big(4e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1} + C''_{\varepsilon_1}\big)|\Omega| + 2C'_{\varepsilon_1}|\Omega| + 4\rho_0\|f\|_h\Big)^{\frac{1}{2}}$ *such that the ball*

$$B_1^h = \{\phi \in H_h^1; \|\phi\|_{1,h} \le \rho_1\}$$

is a absorbing set in H_h^1 under the semigroup $(S_{h,\triangle t})^n$. Where E^0 is defined by (3.6), $\varepsilon_0 = min\Big\{\frac{1}{4(\omega+1)}C_0^{-\frac{4}{3}}e^{-\frac{7\gamma}{6}\triangle t},\frac{1}{2}C_0^{-\frac{4}{3}}e^{\frac{2\gamma}{3}\triangle t}\Big\}$, $\varepsilon_1 = min\Big\{\frac{1}{4(\omega+1)}\rho_0^{-\frac{4}{3}}e^{-\frac{7\gamma}{6}\triangle t},\frac{1}{2}\rho_0^{-\frac{4}{3}}e^{\frac{2\gamma}{3}\triangle t}\Big\}$, and C'_{ε_0}, C''_{ε_1} are decided respectively in (3.2) and (3.7).

Proof. Multiplying the relation (2.3) by $(e^{\frac{\gamma}{2}\triangle t}\overline{\phi}_{i,j,k}^{n+1} - e^{-\frac{\gamma}{2}\triangle t}\overline{\phi}_{i,j,k}^{n})h_1h_2h_3$, and summing them up for i from 1 to J_1-1, for j from 1 to J_2-1 and for k from 1 to J_3-1, then taking the real part, we have

$$\begin{aligned}&e^{\gamma\triangle t}\|\phi^{n+1}\|_{1,h}^2 - (G(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1)_h + 2e^{\frac{\gamma}{2}\triangle t}Re(f,\phi^{n+1})_h\\ &= e^{-\gamma\triangle t}\|\phi^{n}\|_{1,h}^2 - (G(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}|^2),1)_h + 2e^{-\frac{\gamma}{2}\triangle t}Re(f,\phi^{n})_h,\end{aligned}$$

or

$$\begin{aligned}&e^{\frac{\gamma}{2}\triangle t}\|\phi^{n+1}\|_{1,h}^2 - e^{-\frac{\gamma}{2}\triangle t}(G(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1)_h + 2Re(f,\phi^{n+1})_h\\ &=e^{-\gamma\triangle t}\big(e^{-\frac{\gamma}{2}\triangle t}\|\phi^{n}\|_{1,h}^2 - e^{\frac{\gamma}{2}\triangle t}(G(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}|^2),1)_h + 2Re(f,\phi^{n})_h\big).\end{aligned}\tag{3.5}$$

We take that

$$E^n = \|\phi^n\|_{1,h}^2 - e^{-\frac{\gamma}{2}\triangle t}(G(|e^{-\frac{\gamma}{2}\triangle t}\phi^n|^2),1)_h + 2Re(f,\phi^n)_h.\tag{3.6}$$

In the case of $g(s)>0$, It follows from (3.5) that

$$E^{n+1}+(e^{\frac{\gamma}{2}\triangle t}-1)\|\phi^{n+1}\|_{1,h}^2 \le e^{-\gamma\triangle t}E^n + e^{-\frac{\gamma}{2}\triangle t}\big(G(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2)-G(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1\big)_h.$$

We infer from (1.5), (3.2) that, for every $\varepsilon>0$, there exists a constant $C''_\varepsilon>0$, such that

$$\begin{aligned}h(s) &\le \omega G(s) + \varepsilon s^{\frac{5}{3}} + C''_\varepsilon\\ &\le (\omega+1)\varepsilon s^{\frac{5}{3}} + \omega C'_\varepsilon + C''_\varepsilon. \quad s\ge 0\end{aligned}\tag{3.7}$$

According to (1.7), $g(s)$ is a monotone function in R^+. By Lemma 2.3, (3.7) and the inequality $1+x\le e^x$, $\forall x\in R$, when $g(s)$ is a monotone increasing function in R^+, we deduce that

$$\begin{aligned}\big(G(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2) - G(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1\big)_h &\le (g(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2)(e^{\gamma\triangle t}-e^{-\gamma\triangle t})|\phi^{n+1}|^2,1)_h\\ &\le 2\gamma\triangle t(g(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2)|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2,1)_h\\ &\le 2\gamma\triangle t((\omega+1)\varepsilon e^{\frac{5\gamma}{3}\triangle t}|\phi^{n+1}|^{\frac{10}{3}} + \omega C'_\varepsilon + C''_\varepsilon,1)_h\\ &\le 2\gamma\triangle t(\omega+1)e^{\frac{5\gamma}{3}\triangle t}\varepsilon\|\phi^{n+1}\|_{1,h}^2\|\phi^{n+1}\|_h^{\frac{4}{3}}\\ &\quad + 2\gamma\triangle t(\omega C'_\varepsilon + C''_\varepsilon)|\Omega|,\end{aligned}$$

when $g(s)$ is a monotone decreasing function in R^+, we have

$$\begin{aligned}(G(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2)-G(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1)_h &\le (g(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2)(e^{\gamma\triangle t}-e^{-\gamma\triangle t})|\phi^{n+1}|^2,1)_h\\ &\le 2\gamma\triangle t e^{3\gamma\triangle t}(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2 g(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2),1)_h\\ &\le 2\gamma\triangle t e^{3\gamma\triangle t}((\omega+1)\varepsilon e^{-\frac{5\gamma}{3}\triangle t}|\phi^{n+1}|^{\frac{10}{3}}+\omega C'_\varepsilon + C''_\varepsilon,1)_h\\ &\le 2\gamma\triangle t(\omega+1)e^{\frac{4\gamma}{3}\triangle t}\varepsilon\|\phi^{n+1}\|_{1,h}^2\|\phi^{n+1}\|_h^{\frac{4}{3}}\\ &\quad + 2\gamma\triangle t e^{3\gamma\triangle t}(\omega C'_\varepsilon + C''_\varepsilon)|\Omega|.\end{aligned}$$

Therefore, in the case of $g(s)>0$, we derive that

$$\begin{aligned}E^{n+1} + (e^{\frac{\gamma}{2}\triangle t}-1)\|\phi^{n+1}\|_{1,h}^2 \le& e^{-\gamma\triangle t}E^n + 2\gamma\triangle t(\omega+1)e^{\frac{7\gamma}{6}\triangle t}\varepsilon\|\phi^{n+1}\|_{1,h}^2\|\phi^{n+1}\|_h^{\frac{4}{3}}\\ &+ 2\gamma\triangle t e^{\frac{5\gamma}{2}\triangle t}(\omega C'_\varepsilon + C''_\varepsilon)|\Omega|.\end{aligned}\tag{3.8}$$

In the case of $g(s)\le 0$, by Lemma 3.2 and the inequality $1+x\le e^x\le 1+xe^x,\ \forall x\in R$, It follows from (3.5) that

$$\begin{aligned}E^{n+1}\le & e^{-\gamma\triangle t}E^n-e^{-\frac{3\gamma}{2}\triangle t}(e^{\frac{\gamma}{2}\triangle t}-1)\|\phi^n\|_{1,h}^2+e^{-\gamma\triangle t}(e^{-\frac{\gamma}{2}\triangle t}-e^{\frac{\gamma}{2}\triangle t})(G(|e^{-\frac{\gamma}{2}\triangle t}\phi^n|^2),1)_h\\ \le & e^{-\gamma\triangle t}E^n-e^{-\frac{3\gamma}{2}\triangle t}(e^{\frac{\gamma}{2}\triangle t}-1)\|\phi^n\|_{1,h}^2+\gamma\triangle te^{-\frac{\gamma}{2}\triangle t}(-G(|e^{-\frac{\gamma}{2}\triangle t}\phi^n|^2),1)_h\\ \le & e^{-\gamma\triangle t}E^n-e^{-\frac{3\gamma}{2}\triangle t}(e^{\frac{\gamma}{2}\triangle t}-1)\|\phi^n\|_{1,h}^2+\gamma\triangle te^{-\frac{13\gamma}{6}\triangle t}\varepsilon\|\phi^n\|_{1,h}^2\|\phi^n\|_h^{\frac{4}{3}}+\gamma\triangle tC'_\varepsilon|\Omega|.\end{aligned}\tag{3.9}$$

Choosing $\varepsilon=\varepsilon_0=min\Big\{\frac{1}{4(\omega+1)}C_0^{-\frac{4}{3}}e^{-\frac{7\gamma}{6}\triangle t},\frac{1}{2}C_0^{-\frac{4}{3}}e^{\frac{2\gamma}{3}\triangle t}\Big\}$ in (3.8) and (3.9), $g(s)$ is either greater than zero or less than or equal to zero, we always derive that

$$E^{n+1}\le e^{-\gamma\triangle t}E^n+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|,\ n\ge 0,$$

or

$$E^n\le e^{-\gamma\triangle t}E^{n-1}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|,\ n>0.\tag{3.10}$$

Using frequently the inequality (3.10), we derive that

$$\begin{aligned}E^n&\le e^{-\gamma\triangle t}E^{n-1}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|\\ &\le e^{-\gamma 2\triangle t}E^{n-2}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|e^{-\gamma\triangle t}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}(\omega C'_{\varepsilon_0}+C''_{\varepsilon_0})|\Omega|\\ &\le\cdots\cdots\\ &\le e^{-\gamma n\triangle t}E^0+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|(e^{-\gamma(n-1)\triangle t}+\cdots+e^{-\gamma\triangle t}+1)\\ &\le e^{-\gamma n\triangle t}E^0+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|(1-e^{-\gamma n\triangle t}),\quad n>0.\end{aligned}\tag{3.11}$$

By Lemma 3.1, Lemma 3.2 and the definition of E^n, it follows from (3.11)

$$\begin{aligned}\|\phi^n\|_{1,h}^2=&E^n+e^{-\frac{\gamma}{2}\triangle t}(G(|e^{-\frac{\gamma}{2}\triangle t}\phi^n|^2),1)_h-2Re(f,\phi^n)_h\\ \le& e^{-\gamma n\triangle t}E^0+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|(1-e^{-\gamma n\triangle t})\\ &+\varepsilon_0e^{-\frac{13\gamma}{6}\triangle t}\|\phi^n\|_{1,h}^2\|\phi^n\|_h^{\frac{4}{3}}+\omega C'_{\varepsilon_0}|\Omega|+2\|f\|_h\|\phi^n\|_h\\ \le& e^{-\gamma n\triangle t}E^0+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_0}+C''_{\varepsilon_0}\big)|\Omega|(1-e^{-\gamma n\triangle t})+\frac{1}{2}\|\phi^n\|_{1,h}^2+\omega C'_{\varepsilon_0}|\Omega|\\ &+\|f\|_h^2+e^{-\gamma n\triangle t}\|\phi^0\|_h^2+\frac{1}{\gamma^2}e^{\gamma\triangle t}\|f\|_h^2(1-e^{-\gamma n\triangle t}),\quad n>0,\end{aligned}$$

it implies (3.3).

If initial value ϕ^0 satisfied that $\|\phi^0\|_h\le R$. Then by Lemma 3.1, there exists a positive integer $N=N(R)$, such that

$$\|\phi^n\|_h\le\rho_0,\quad n>N(R).\tag{3.12}$$

Hence, choosing $\varepsilon=\varepsilon_1=min\Big\{\frac{1}{4(\omega+1)}\rho_0^{-\frac{4}{3}}e^{-\frac{7\gamma}{6}\triangle t},\frac{1}{2}\rho_0^{-\frac{4}{3}}e^{\frac{2\gamma}{3}\triangle t}\Big\}$ in (3.8) and (3.9), $g(s)$ is either greater than zero or less than or equal to zero , we always derive that

$$E^{n+1}\le e^{-\gamma\triangle t}E^n+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|,\ n>N(R),$$

or

$$E^n\le e^{-\gamma\triangle t}E^{n-1}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|,\ n>N(R)+1,\tag{3.13}$$

Using frequently the inequality (3.13), we derive that

$$\begin{aligned} E^n &\le e^{-\gamma\triangle t}E^{n-1}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega| \\ &\le e^{-\gamma 2\triangle t}E^{n-2}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|e^{-\gamma\triangle t}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega| \\ &\le \cdots\cdots \\ &\le e^{-\gamma(n-N-1)\triangle t}E^{N+1}+2\gamma\triangle te^{\frac{5\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|(e^{-\gamma(n-N-2)\triangle t}+\cdots+e^{-\gamma\triangle t}+1) \\ &\le e^{-\gamma(n-N-1)\triangle t}E^{N+1}+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|,\quad n>N+1. \end{aligned} \tag{3.14}$$

By Lemma 3.2 and the definition of E^n, it follows from (3.14)

$$\begin{aligned} \|\phi^n\|_{1,h}^2 =&E^n+e^{-\frac{\gamma}{2}\triangle t}(G(|e^{-\frac{\gamma}{2}\triangle t}\phi^n|^2),1)_h-2Re(f,\phi^n)_h \\ \le& e^{-\gamma(n-N-1)\triangle t}E^{N+1}+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|+\varepsilon_1 e^{-\frac{13\gamma}{6}\triangle t}\|\phi^n\|_{1,h}^2\|\phi^n\|_h^{\frac{4}{3}} \\ &+C'_{\varepsilon_1}|\Omega|+2\|f\|_h\|\phi^n\|_h \\ \le& e^{-\gamma(n-N-1)\triangle t}E^{N+1}+2e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|+\frac{1}{2}\|\phi^n\|_{1,h}^2 \\ &+C'_{\varepsilon_1}|\Omega|+2\rho_0\|f\|_h,\quad n>N+1, \end{aligned}$$

thus

$$\overline{\lim_{n\to\infty}}\|\phi^n\|_{1,h}^2\le 4e^{\frac{7\gamma}{2}\triangle t}\big((\omega+1)C'_{\varepsilon_1}+C''_{\varepsilon_1}\big)|\Omega|+2C'_{\varepsilon_1}|\Omega|+4\rho_0\|f\|_h.$$

The proof of the lemma is new complete.

Now we prove the uniqueness of the solution of the finite difference system (2.3) with boundary condition (2.4). Let $\epsilon^n=\phi^n-\psi^n$, where ϕ^n and ψ^n be two solutions of the difference scheme with initial value ϕ^0 and ψ^0 respectively, and ϕ^0 and ψ^0 satisfy

$$\|\phi^0\|_{1,h}\le R,\quad \|\psi^0\|_{1,h}\le R.$$

By Lemma 3.3, there exists a constant $C_1(R)$, such that

$$\sup_{n\ge 0}\|\phi^n\|_{1,h}^2\le C_1(R),\quad \sup_{n\ge 0}\|\psi^n\|_{1,h}^2\le C_1(R).$$

Then ϵ^n satisfies that

$$\begin{aligned} &i\frac{e^{\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^{n+1}-e^{-\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^n}{\triangle t}+\frac{1}{2}\triangle_h(e^{\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^n) \\ &+\frac{1}{2}G[|e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n|^2](e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n) \\ &-\frac{1}{2}G[|e^{\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^n|^2](e^{\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^n)=0, \\ &i=1,2,\cdots,J_1-1,j=1,2,\cdots,J_2-1,k=1,2,\cdots,J_3-1,n=0,1,\cdots. \end{aligned} \tag{3.15}$$

Because of

$$\begin{aligned} &G[|e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n|^2](e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n) \\ &\quad-G[|e^{\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^n|^2](e^{\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\psi_{i,j,k}^n) \\ =&G[|e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n|^2](e^{\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^{n+1}+e^{-\frac{\gamma}{2}\triangle t}\epsilon_{i,j,k}^n) \\ &\quad+(G[|e^{\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^{n+1}|^2,|e^{-\frac{\gamma}{2}\triangle t}\phi_{i,j,k}^n|^2] \end{aligned}$$

$$- G[|e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\psi^{n}_{i,j,k}|^2])(e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\psi^{n}_{i,j,k}),$$

and by the property of difference quotient, there exist constants $\varsigma^{n+1}_{i,j,k}, \xi^{n}_{i,j,k} \geq 0$, such that

$$\begin{aligned} &G[|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2] - G[|e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k}|^2, |e^{-\frac{\gamma}{2}\triangle t}\psi^{n}_{i,j,k}|^2] \\ =&\frac{1}{2}g'(\varsigma^{n+1}_{i,j,k})(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k}|^2 - |e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k}|^2) + \frac{1}{2}g'(\xi^{n}_{i,j,k})(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2 - |e^{-\frac{\gamma}{2}\triangle t}\psi^{n}_{i,j,k}|^2). \end{aligned}$$

where $\varsigma^{n+1}_{i,j,k}$ lies between the largest and smallest of $|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2$, $|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}_{i,j,k}|^2$ and $|e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k}|^2$, and $\xi^{n}_{i,j,k}$ lies between the largest and smallest of $|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}_{i,j,k}|^2$, $|e^{-\frac{\gamma}{2}\triangle t}\psi^{n}_{i,j,k}|^2$ and $|e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}_{i,j,k}|^2$. Multiplying the relation (3.15) by $\triangle t(e^{\frac{\gamma}{2}\triangle t}\bar{\epsilon}^{n+1}_{i,j,k} + e^{-\frac{\gamma}{2}\triangle t}\bar{\epsilon}^{n}_{i,j,k})h_1h_2h_3$, and summing them up for i from 1 to $J_1 - 1$, for j from 1 to $J_2 - 1$, for k from 1 to $J_3 - 1$, then taking the imaginary part, we have

$$\begin{aligned} &\|e^{\frac{\gamma}{2}\triangle t}\epsilon^{n+1}\|^2_h - \|e^{-\frac{\gamma}{2}\triangle t}\epsilon^{n}\|^2_h + \frac{1}{4}\triangle t Im\big(\{g'(\varsigma^{n+1})(|e^{\frac{\gamma}{2}\triangle t}\phi^{n+1}|^2 - |e^{\frac{\gamma}{2}\triangle t}\psi^{n+1}|^2) \\ &+g'(\xi^n)(|e^{-\frac{\gamma}{2}\triangle t}\phi^{n}|^2 - |e^{-\frac{\gamma}{2}\triangle t}\psi^{n}|^2)\}(e^{\frac{\gamma}{2}\triangle t}\psi^{n+1} + e^{-\frac{\gamma}{2}\triangle t}\psi^{n}), e^{\frac{\gamma}{2}\triangle t}\epsilon^{n+1} + e^{-\frac{\gamma}{2}\triangle t}\epsilon^{n}\big)_h = 0. \end{aligned} \tag{3.16}$$

Under the hypothesis (1.6) and Lemma 2.3, Lemma 3.3, we have

$$\begin{aligned} &|\text{the third term of the left-hand side of (3.16)}| \\ \leq&\frac{M}{4}\triangle t e^{2\gamma\triangle t}\big((|\phi^{n+1}| + |\psi^{n+1}|)|\epsilon^{n+1}| + (|\phi^{n}| + |\psi^{n}|)|\epsilon^{n}|, (|\psi^{n+1}| + |\psi^{n}|)(|\epsilon^{n+1}| + |\epsilon^{n}|)\big)_h \\ \leq&\frac{M}{8}\triangle t e^{2\gamma\triangle t}\big(\|\psi^{n+1}\|_\infty + \|\psi^{n}\|_\infty\big)\Big\{\big(3\|\phi^{n+1}\|_\infty + 3\|\psi^{n+1}\|_\infty + \|\phi^{n}\|_\infty + \|\psi^{n}\|_\infty\big)\|\epsilon^{n+1}\|^2_h \\ &+ \big(3\|\phi^{n}\|_\infty + 3\|\psi^{n}\|_\infty + \|\phi^{n+1}\|_\infty + \|\psi^{n+1}\|_\infty\big)\|\epsilon^{n}\|^2_h\Big\} \\ \leq&\frac{M}{8\delta}h^{-1}\triangle t e^{2\gamma\triangle t}\big(\|\psi^{n+1}\|_{1,h} + \|\psi^{n}\|_{1,h}\big)\Big\{\big(3\|\phi^{n+1}\|_{1,h} + 3\|\psi^{n+1}\|_{1,h} + \|\phi^{n}\|_{1,h} + \|\psi^{n}\|_{1,h}\big)\|\epsilon^{n+1}\|^2_h \\ &+ \big(3\|\phi^{n}\|_{1,h} + 3\|\psi^{n}\|_{1,h} + \|\phi^{n+1}\|_{1,h} + \|\psi^{n+1}\|_{1,h}\big)\|\epsilon^{n}\|^2_h\Big\} \\ \leq&2MC_1(R)e^{2\gamma\triangle t}\delta^{-1}h^{-1}\triangle t\big(\|\epsilon^{n}\|^2_h + \|\epsilon^{n+1}\|^2_h\big). \end{aligned}$$

Substituting the above estimate in (3.16), we have

$$\|\epsilon^{n+1}\|^2_h \leq e^{-2\gamma\triangle t}\|\epsilon^{n}\|^2_h + 2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t\big(\|\epsilon^{n}\|^2_h + \|\epsilon^{n+1}\|^2_h\big).$$

If $2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t < 1$, that is that $\triangle t < \frac{\delta h}{2MC_1(R)e^{\gamma\triangle t}}$, we derive that

$$\begin{aligned} \|\epsilon^{n+1}\|^2_h &\leq \frac{1 + 2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t}{1 - 2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t}\|\epsilon^{n}\|^2_h \\ &\leq \cdots\cdots \\ &\leq \Big(\frac{1 + 2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t}{1 - 2MC_1(R)e^{\gamma\triangle t}\delta^{-1}h^{-1}\triangle t}\Big)^{n+1}\|\epsilon^{0}\|^2_h, n = 0, 1, \cdots. \end{aligned} \tag{3.17}$$

Hence we obtain

Lemma 3.4. *Under the conditions* $(1.4)'$, *(1.5)–(1.7), and* $\triangle t < \frac{\delta}{2MC_1(R)e^{\gamma\triangle t}}h$. *Then the solution of the difference scheme (2,3) with boundary conditions (2.4) and initial condition (2.5) is unique, where* $C_1(R)$ *is defined by (3.4).*

Now we prove the existence of attractor $\mathcal{A}_{h,\triangle t}$ for the discrete system on H^1_h. Obvious, a family operators $(S_{h,\triangle t})^n$ satisfy the semigroup properties

$$(S_{h,\triangle t})^m(S_{h,\triangle t})^n = (S_{h,\triangle t})^{m+n}, \forall m, n \geq 0, \quad (S_{h,\triangle t})^0 = I.$$

According to (3.17), for every $n \geq 0$, $(S_{h,\triangle t})^n$ is a continuous operator from the finite dimensional space H_h^1 into itself. By Lemma 3.3, there exists a bounded set B_1^h which is absorbing in H_h^1 under $(S_{h,\triangle t})^n$. Using theorem 1.1 in [8], we obtain the main result as follows

Theorem 3.1. *We assume that* $(1.4)'$, *(1.5)–(1.7) are hold, and* $\triangle t < \frac{\delta h}{2MC_1(\rho_1)e^{\gamma \triangle t}}$. *Then the discrete dynamical system associated with the finite difference system (2.3) with boundary condition (2.4) possesses a global attractor* $\mathcal{A}_{h,\triangle t}$ *on* H_h^1, *and*

$$\mathcal{A}_{h,\triangle t} = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} (S_{h,\triangle t})^m B_1^h}.$$

Remark 3.1. As the parameter $\gamma > 0$, the discrete system can remain well dissipative properties of the original system, and as the parameter $\gamma = 0$, the discrete system can also remain well conservation properties of the original system.

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