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COMPUTING A NEAREST P-SYMMETRIC NONNEGATIVE DEFINITE MATRIX UNDER LINEAR RESTRICTION *1)

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Abstract

Let P be an $n \times n$ symmetric orthogonal matrix. A real $n \times n$ matrix A is called P-symmetric nonnegative definite if A is symmetric nonnegative definite and $(PA)^T = PA$. This paper is concerned with a kind of inverse problem for P-symmetric nonnegative definite matrices: Given a real $n \times n$ matrix $\widetilde{A}$, real $n \times m$ matrices X and B, find an $n \times n$ P-symmetric nonnegative definite matrix A minimizing $\|A - \widetilde{A}\|_F$ subject to $AX = B$. Necessary and sufficient conditions are presented for the solvability of the problem. The expression of the solution to the problem is given. These results are applied to solve an inverse eigenvalue problem for P-symmetric nonnegative definite matrices.

Mathematics subject classification: 15A24, 65F20.
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1. Introduction

Throughout this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m\times n}$, the set of all orthogonal matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{OR}^{n\times n}$, the transpose of a real matrix A by A^T, the Moore-Penrose pseudoinverse of a matrix A by A^+, the $n \times n$ identity matrix by I_n, the set of all symmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{SR}_0^{n\times n}$. $A > 0 (A \geq 0)$ means that A is a real symmetric positive(nonnegative) definite matrix. For $A, B \in \mathbf{R}^{m\times n}$, we define an inner product in $\mathbf{R}^{m\times n}$ $:< A, B >= tr(B^T A)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|_F$ induced by the inner product is the Frobenius norm.

Definition 1.1 (c.f.[5]). *A real $n \times n$ matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is called doubly symmetric or bisymmetric if*

$$a_{ij} = a_{ji} = a_{n+1-j,n+1-i}, \quad i = 1, 2, \cdots, n.$$

The set of all bisymmetric matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{BSR}^{n\times n}$. A real $n \times n$ matrix A is said to be bisymmetric nonnegative definite if A is bisymmetric and nonnegative definite. The set of all bisymmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{BSR}_0^{n\times n}$.

Definition 1.2. *Let $P \in \mathbf{R}^{n\times n}$ be a symmetric orthogonal matrix. $A \in \mathbf{R}^{n\times n}$ is called P-symmetric nonnegative definite matrix if A is symmetric nonnegative definite and $(PA)^T = PA$. The set of all P-symmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{SR}_P^{n\times n}$.*

If $P = I_n$, then $\mathbf{SR}_{I_n}^{n\times n} = \mathbf{SR}_0^{n\times n}$. e_i denotes the ith column of I_n. Let $S_n = [e_n, e_{n-1}, \cdots, e_1]$. If $P = S_n$, then $\mathbf{SR}_{S_n}^{n\times n} = \mathbf{BSR}_0^{n\times n}$.

In this paper, we consider the following problem.

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Problem IP. *Given a matrix* $\widetilde{A} \in \mathbf{R}^{n\times n}$, *two matrices* $X, B \in \mathbf{R}^{n\times m}$, *let*

$$S_A = \{A \in \mathbf{SR}_P^{n\times n} | AX = B\}, \tag{1.1}$$

find $\widehat{A} \in S_A$ *such that*

$$\|\widetilde{A} - \widehat{A}\|_F = \inf_{\forall A \in S_A} \|\widetilde{A} - A\|_F. \tag{1.2}$$

Problem IP is essentially computing the nearest P-symmetric nonnegative definite matrix in the Frobenius norm to an arbitrary real matrix $\widetilde{A}$ under the linear restriction $AX = B$. The problem arises in a remarkable variety of applications such as structural modification and system identification[4,11]. If $P = I_n$, Problem IP reduces to an inverse problem for real symmetric nonnegative definite matrices[12]. If $P = S_n$, Problem IP is an inverse problem for bisymmetric nonnegative definite matrices[10]. If $B = X\Lambda, \Lambda = diag(\lambda_1, \cdots, \lambda_m) \in \mathbf{R}^{m\times m}$, then the set S_A, further the solution $\widehat{A}$, is determined by the eigenvalues $\lambda_1, \cdots, \lambda_m$ and corresponding eigenvectors, Problem IP becomes an inverse eigenvalue problem[2].

The symmetric nonnegative definite solutions to the matrix inverse problem $AX = B$ were studied by Zhang[12]. The analysis of Re-positive and Re-nonnegative definite solutions can be found in [8] and [9], respectively. The bisymmetric nonnegative definite solutions to the matrix inverse problem $AX = B$ were treated in [10]. In this paper, the results from [10] are generalized and extended to P-symmetric nonnegative definite matrices.

In section 2, we give necessary and sufficient conditions for the set S_A to be nonempty and construct the set S_A explicitly when it is nonempty. In section 3, we show that there exists a unique solution in Problem IP if the set S_A is nonempty and present the expression of the solution to Problem IP. In section 4, we consider an inverse eigenvalue problem for P-symmetric nonnegative definite matrices.

2. The Set $\mathbf{S_A}$

To begin with, we introduce a lemma.

Lemma 2.1 (c.f.[7]). *Suppose that* $P \in \mathbf{OR}^{n\times n}$ *is symmetric. Then there exists an orthogonal matrix* $U \in \mathbf{OR}^{n\times n}$ *such that*

$$P = U \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} U^T. \tag{2.1}$$

The representation (2.1) is referred to as a spectral decomposition of the matrix P. For convenience, let us introduce the notations

$$k_1 = k, \quad k_2 = n - k.$$

It is easy to obtain the following lemma from Definition 1.2.

Lemma 2.2. $A \in \mathbf{SR}_P^{n\times n}$ *if and only if*

$$A^T = A \geq 0, \quad AP - PA = 0. \tag{2.2}$$

About the structure of $\mathbf{SR}_P^{n\times n}$, we have the following result.

Theorem 2.1. *Let the spectral decomposition of the matrix* $P \in \mathbf{OR}^{n\times n}$ *be (2.1),* $A \in \mathbf{SR}_P^{n\times n}$ *if and only if*

$$A = U \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} U^T, \tag{2.3}$$

where $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i} (i = 1, 2)$.

Proof. If $A \in \mathbf{SR}_P^{n\times n}$, it follows from (2.1) that

$$U^TAU\begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2}\end{pmatrix} - \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2}\end{pmatrix} U^TAU = 0. \tag{2.4}$$

Since $A \in \mathbf{SR}_0^{n\times n}$, so $U^TAU \in \mathbf{SR}_0^{n\times n}$. Let

$$U^TAU = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22}\end{pmatrix}, \tag{2.5}$$

where $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i}(i = 1, 2)$.

Substituting the expression (2.5) in (2.4) yields $A_{12} = 0$. Hence

$$A = U\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}\end{pmatrix}U^T.$$

On the other hand, it can be directly verified that the matrix A in form (2.3) belongs to $\mathbf{SR}_P^{n\times n}$ from Lemma 2.2.

Lemma 2.3 (c.f.[12]). *Let $Y, Z \in \mathbf{R}^{n\times m}$, and the singular value decomposition(SVD) of the matrix Y be*

$$Y = Q\begin{pmatrix} \Sigma & 0 \\ 0 & 0\end{pmatrix}V^T,$$

where $Q = [Q_1, Q_2] \in \mathbf{OR}^{n\times n}$, $V = [V_1, V_2] \in \mathbf{OR}^{m\times m}$, $\Sigma = diag(\sigma_1, \cdots, \sigma_r)$, $\sigma_i > 0(i = 1, \cdots, r)$, $r = rank(Y)$, $Q_1 \in \mathbf{R}^{n\times r}$, $V_1 \in \mathbf{R}^{m\times r}$. Then there exists a matrix $A \in \mathbf{SR}_0^{n\times n}$ such that

$$AY = Z \tag{2.6}$$

if and only if

$$Y^TZ = Z^TY \geq 0, \quad rank(Y^TZ) = rank(Z), \tag{2.7}$$

in which case the general symmetric nonnegative definite solution of (2.6) can be expressed as

$$A = ZY^+ + (ZY^+)^T(I_n - YY^+) + (I_n - YY^+)Z(Y^TZ)^+Z^T(I_n - YY^+) + Q_2GQ_2^T, \tag{2.8}$$

where $G \in \mathbf{SR}_0^{(n-r)\times(n-r)}$ is arbitrary.

Theorem 2.2. *For given matrices $X, B \in \mathbf{R}^{n\times m}$, let the spectral decomposition of the matrix $P \in \mathbf{OR}^{n\times n}$ be (2.1), and*

$$U^TX = \begin{pmatrix} X_1 \\ X_2\end{pmatrix}, \quad U^TB = \begin{pmatrix} B_1 \\ B_2\end{pmatrix}, \tag{2.9}$$

where $X_i, B_i \in \mathbf{R}^{k_i\times m}(i = 1, 2)$, and the SVD of the matrix X_i be

$$X_i = Q^{(i)}\begin{pmatrix} \Sigma^{(i)} & 0 \\ 0 & 0\end{pmatrix}V^{(i)T}, \quad i = 1, 2, \tag{2.10}$$

where $Q^{(i)} = [Q_1^{(i)}, Q_2^{(i)}] \in \mathbf{OR}^{k_i\times k_i}$, $V^{(i)} = [V_1^{(i)}, V_2^{(i)}] \in \mathbf{OR}^{m\times m}$, $\Sigma^{(i)} = diag(\sigma_1^{(i)}, \cdots, \sigma_{r_i}^{(i)})$, $\sigma_j^{(i)} > 0(j = 1, \cdots, r_i)$, $r_i = rank(X_i)$, $Q_1^{(i)} \in \mathbf{R}^{k_i\times r_i}$, $V_1^{(i)} \in \mathbf{R}^{m\times r_i}$. Then S_A is nonempty if and only if

$$X_i^TB_i = B_i^TX_i \geq 0, \quad rank(X_i^TB_i) = rank(B_i), \quad i = 1, 2 \tag{2.11}$$

in which case S_A can be expressed as

$$S_A = \{A_0 + U\begin{pmatrix} Q_2^{(1)}G_1Q_2^{(1)T} & 0 \\ 0 & Q_2^{(2)}G_2Q_2^{(2)T}\end{pmatrix}U^T | G_i \in \mathbf{SR}_0^{(k_i-r_i)\times(k_i-r_i)}\} \tag{2.12}$$

where

$$A_0 = U \begin{pmatrix} A_0^{(1)} & 0 \\ 0 & A_0^{(2)} \end{pmatrix} U^T, \tag{2.13}$$

$$\begin{aligned} A_0^{(i)} &= B_iX_i^+ + (B_iX_i^+)^T(I_{k_i} - X_iX_i^+) \\ &+(I_{k_i} - X_iX_i^+)B_i(X_i^TB_i)^+B_i^T(I_{k_i} - X_iX_i^+), \end{aligned} \quad i = 1,2 \tag{2.14}$$

Proof. Since S_A is nonempty if and only if $AX = B$ has a solution $A \in \mathbf{SR}_P^{n\times n}$, then it follows from Theorem 2.1 that $AX = B$ and $A \in \mathbf{SR}_P^{n\times n}$ are equivalent to

$$U \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} U^TX = B, \tag{2.15}$$

where $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i} (i = 1,2)$. Thus $AX = B$ has a solution in $\mathbf{SR}_P^{n\times n}$ if and only if $A_{ii}X_i = B_i$ $(i = 1,2)$ has a solution in $\mathbf{SR}_0^{k_i\times k_i}$. It follows from Lemma 2.3 that $A_{ii}X_i = B_i$ has a solution in $\mathbf{SR}_0^{k_i\times k_i}$ if and only if

$$X_i^TB_i = B_i^TX_i \geq 0, \quad rank(X_i^TB_i) = rank(B_i), \quad i = 1,2$$

in which case the general symmetric nonnegative definite solution of $A_{ii}X_i = B_i$ can be represented as

$$\begin{aligned} A_{ii} &= B_iX_i^+ + (B_iX_i^+)^T(I_{k_i} - X_iX_i^+) \\ &+(I_{k_i} - X_iX_i^+)B_i(X_i^TB_i)^+B_i^T(I_{k_i} - X_iX_i^+) + Q_2^{(i)}G_iQ_2^{(i)^T}, \end{aligned} \quad i = 1,2 \tag{2.16}$$

where $G_i \in \mathbf{SR}_0^{(k_i-r_i)\times(k_i-r_i)}$ $(i = 1,2)$. Using (2.3) and (2.16) we obtain the expression (2.12) of the set S_A.

From Theorem 2.2 we obtain the following conclusion.

Corollary 2.1. *If the SVD of the matrix $X_i (i = 1,2)$ is (2.10), X_i and B_i satisfy the following conditions*

$$X_i^TB_i = B_i^TX_i > 0, \quad i = 1,2$$

then S_A is nonempty, and S_A can be expressed as

$$S_A = \{A_0 + U \begin{pmatrix} Q_2^{(1)}G_1Q_2^{(1)^T} & 0 \\ 0 & Q_2^{(2)}G_2Q_2^{(2)^T} \end{pmatrix} U^T | G_i \in \mathbf{SR}_0^{(k_i-m)\times(k_i-m)}\}$$

where A_0 is expressed by (2.13), $Q_2^{(i)} \in \mathbf{R}^{k_i\times(k_i-m)}$ $(i = 1,2)$.

3. Expression of the Solution for Problem IP

Lemma 3.1 (c.f.[6]). *Let $\widetilde{C} \in \mathbf{R}^{n\times n}$, $C_1 = (\widetilde{C}+\widetilde{C}^T)/2$ and $C_1 = QH$ be a polar decomposition $(Q \in \mathbf{OR}^{n\times n}, H \geq 0)$. Then $\widehat{C} = (C_1 + H)/2$ is a unique positive approximation of $\widetilde{C}$ in the Frobenius norm, i.e.*

$$\|\widetilde{C} - \widehat{C}\|_F = \inf_{\forall C\in\mathbf{SR}_0^{n\times n}} \|\widetilde{C} - C\|_F. \tag{3.1}$$

We denote the unique positive approximation of $\widetilde{C}$ in the Frobenius norm by $[\widetilde{C}]_+$. Now we prove the following theorem.

Theorem 3.1. *Suppose $\widetilde{A} \in \mathbf{R}^{n\times n}$, $X, B \in \mathbf{R}^{n\times m}$, X_i and $B_i (i = 1,2)$ are defined by (2.9), and the SVD of the matrix $X_i (i = 1,2)$ is (2.10). If X_i and $B_i (i = 1,2)$ satisfy the condition (2.11), then Problem IP has a unique solution $\widehat{A} \in S_A$. Moreover, let*

$$U^T\widetilde{A}U = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{21} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \qquad \widetilde{A}_{22}^{(i)} = Q_2^{(i)^T}(\widetilde{A}_{ii} - B_i(X_i^TB_i)^+B_i^T)Q_2^{(i)}, \tag{3.2}$$

where U is defined by (2.1), $\widetilde{A}_{ii} \in \mathbf{R}^{k_i \times k_i} (i = 1, 2)$, then the unique solution $\widehat{A}$ of Problem IP can be expressed as

$$\widehat{A} = A_0 + U \begin{pmatrix} Q_2^{(1)} \widehat{G}_1 {Q_2^{(1)}}^T & 0 \\ 0 & Q_2^{(2)} \widehat{G}_2 {Q_2^{(2)}}^T \end{pmatrix} U^T, \tag{3.3}$$

where A_0 is defined by (2.13), $\widehat{G}_i = [\widetilde{A}_{22}^{(i)}]_+ (i = 1, 2)$.

Proof. Since X_i and $B_i (i = 1, 2)$ satisfy the condition (2.11), the set S_A is nonempty. It is easy to verify that S_A is a closed convex subset of the Hilbert space $\mathbf{R}^{n \times n}$. It follows from the best approximation theorem[1] that there exists a unique matrix $\widehat{A} \in S_A$ satisfying (1.2), i.e., Problem IP has the unique solution $\widehat{A} \in S_A$.

By Theorem 2.2, $A \in S_A$ can be expressed as

$$\begin{aligned} A &= A_0 + U \begin{pmatrix} Q_2^{(1)} G_1 {Q_2^{(1)}}^T & 0 \\ 0 & Q_2^{(2)} G_2 {Q_2^{(2)}}^T \end{pmatrix} U^T \\ &= A_0 + U \begin{pmatrix} Q^{(1)} \begin{pmatrix} 0 & 0 \\ 0 & G_1 \end{pmatrix} {Q^{(1)}}^T & 0 \\ 0 & Q^{(2)} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix} {Q^{(2)}}^T \end{pmatrix} U^T, \end{aligned} \tag{3.4}$$

where $G_i \in \mathbf{SR}_0^{(k_i - r_i) \times (k_i - r_i)} (i = 1, 2)$.

From (2.10) and (2.14), it is easy to verify that

$$\left\{ \begin{array}{l} X_i^+ Q_2^{(i)} = 0 \\ (I_{k_i} - X_i X_i^+) Q_1^{(i)} = 0 \\ {Q_2^{(i)}}^T A_0^{(i)} Q_2^{(i)} = {Q_2^{(i)}}^T B_i (X_i^T B_i)^+ B_i^T Q_2^{(i)} \end{array} \right. \qquad i = 1, 2. \tag{3.5}$$

From (2.13), (3.2), (3.4) and (3.5), we have

$$\begin{aligned} \|\widetilde{A} - A\|_F^2 &= \|\widetilde{A} - A_0 - U \begin{pmatrix} Q_2^{(1)} G_1 {Q_2^{(1)}}^T & 0 \\ 0 & Q_2^{(2)} G_2 {Q_2^{(2)}}^T \end{pmatrix} U^T \|_F^2 \\ &= \|U^T (\widetilde{A} - A_0) U - \begin{pmatrix} Q_2^{(1)} G_1 {Q_2^{(1)}}^T & 0 \\ 0 & Q_2^{(2)} G_2 {Q_2^{(2)}}^T \end{pmatrix} \|_F^2 \\ &= \| \begin{pmatrix} \widetilde{A}_{11} - A_0^{(1)} - Q_2^{(1)} G_1 {Q_2^{(1)}}^T & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} - A_0^{(2)} - Q_2^{(2)} G_2 {Q_2^{(2)}}^T \end{pmatrix} \|_F^2 \\ &= \sum_{i=1}^{2} \|\widetilde{A}_{ii} - A_0^{(i)} - Q_2^{(i)} G_i {Q_2^{(i)}}^T \|_F^2 + \|\widetilde{A}_{12}\|_F^2 + \|\widetilde{A}_{21}\|_F^2 \\ &= \sum_{i=1}^{2} \|\widetilde{A}_{ii} - A_0^{(i)} - Q^{(i)} \begin{pmatrix} 0 & 0 \\ 0 & G_i \end{pmatrix} {Q^{(i)}}^T \|_F^2 + \|\widetilde{A}_{12}\|_F^2 + \|\widetilde{A}_{21}\|_F^2 \\ &= \sum_{i=1}^{2} \|{Q^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q^{(i)} - \begin{pmatrix} 0 & 0 \\ 0 & G_i \end{pmatrix} \|_F^2 + \|\widetilde{A}_{12}\|_F^2 + \|\widetilde{A}_{21}\|_F^2 \\ &= \sum_{i=1}^{2} \| \begin{pmatrix} {Q_1^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_1^{(i)} & {Q_1^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_2^{(i)} \\ {Q_2^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_1^{(i)} & {Q_2^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_2^{(i)} - G_i \end{pmatrix} \|_F^2 \\ &\quad + \|\widetilde{A}_{12}\|_F^2 + \|\widetilde{A}_{21}\|_F^2 \\ &= \sum_{i=1}^{2} \|\widetilde{A}_{22}^{(i)} - G_i\|_F^2 + M, \end{aligned} \tag{3.6}$$

where $M = \|\widetilde{A}_{12}\|_F^2 + \|\widetilde{A}_{21}\|_F^2 + \sum_{i=1}^{2} \| \begin{pmatrix} {Q_1^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_1^{(i)} & {Q_1^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_2^{(i)} \\ {Q_2^{(i)}}^T (\widetilde{A}_{ii} - A_0^{(i)}) Q_1^{(i)} & 0 \end{pmatrix} \|_F^2.$

Hence, Problem IP is reduced to find $\widehat{G}_i \in \mathbf{SR}_0^{(k_i-r_i)\times(k_i-r_i)}$ such that

$$\|\widetilde{A}_{22}^{(i)} - \widehat{G}_i\|_F^2 = \inf_{\forall G_i \in \mathbf{SR}_0^{(k_i-r_i)\times(k_i-r_i)}} \|\widetilde{A}_{22}^{(i)} - G_i\|_F, \ i = 1, 2. \tag{3.7}$$

It then follows from Lemma 3.1 that

$$\widehat{G}_i = [\widetilde{A}_{22}^{(i)}]_+, \ i = 1, 2. \tag{3.8}$$

Substituting (3.8) in (3.4), we obtain (3.3).

If $n = 2k, P = S_n = [e_n, \cdots, e_1]$, then we have

$$P = U \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix} U^T, \tag{3.9}$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix} \in \mathbf{OR}^{n\times n}$. For the matrices $X, B \in \mathbf{R}^{n\times m}$, let

$$X_1 = \frac{1}{\sqrt{2}}[I_k, S_k]X, \quad B_1 = \frac{1}{\sqrt{2}}[I_k, S_k]B \tag{3.10}$$

$$X_2 = \frac{1}{\sqrt{2}}[I_k, -S_k]X, \quad B_2 = \frac{1}{\sqrt{2}}[I_k, -S_k]B \tag{3.11}$$

From (3.9) $\sim$ (3.11), Theorem 2.2 and Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Suppose $\widetilde{A} \in \mathbf{R}^{n\times n}$, $X, B \in \mathbf{R}^{n\times m}$. If $n = 2k$ and the SVD of the matrix $X_i (i = 1, 2)$ is (2.10), then $S_A = \{A \in \mathbf{BSR}_0^{n\times n} | AX = B\}$ is nonempty if and only if*

$$\begin{cases} X^T \begin{pmatrix} I_k & S_k \\ S_k & I_k \end{pmatrix} B = B^T \begin{pmatrix} I_k & S_k \\ S_k & I_k \end{pmatrix} X \geq 0 \\ rank[X^T \begin{pmatrix} I_k & S_k \\ S_k & I_k \end{pmatrix} B] = rank[(I_k, S_k)B], \end{cases} \tag{3.12}$$

and

$$\begin{cases} X^T \begin{pmatrix} I_k & -S_k \\ -S_k & I_k \end{pmatrix} B = B^T \begin{pmatrix} I_k & -S_k \\ -S_k & I_k \end{pmatrix} X \geq 0 \\ rank[X^T \begin{pmatrix} I_k & -S_k \\ -S_k & I_k \end{pmatrix} B] = rank[(I_k, -S_k)B]. \end{cases} \tag{3.13}$$

If X and B satisfy the conditions (3.12) and (3.13), then there exists a unique bisymmetric nonnegative definite matrix $\widehat{A} \in S_A$ satisfying (1.2). Moreover, let

$$\frac{1}{2} \begin{pmatrix} I_k & S_k \\ I_k & -S_k \end{pmatrix} \widetilde{A} \begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \widetilde{A}_{22} \in \mathbf{R}^{k\times k},$$

$$\widetilde{A}_{22}^{(1)} = Q_2^{(1)^T} [\widetilde{A}_{11} - (I_k, S_k)B(X^T \begin{pmatrix} I_k & S_k \\ S_k & I_k \end{pmatrix} B)^+ B^T \begin{pmatrix} I_k \\ S_k \end{pmatrix}] Q_2^{(1)},$$

$$\widetilde{A}_{22}^{(2)} = Q_2^{(2)^T} [\widetilde{A}_{22} - (I_k, -S_k)B(X^T \begin{pmatrix} I_k & -S_k \\ -S_k & I_k \end{pmatrix} B)^+ B^T \begin{pmatrix} I_k \\ -S_k \end{pmatrix}] Q_2^{(2)}.$$

$\widehat{A}$ can be expressed as

$$\widehat{A} = \frac{1}{2} \begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix} \begin{pmatrix} \widehat{A}_{11} & 0 \\ 0 & \widehat{A}_{22} \end{pmatrix} \begin{pmatrix} I_k & S_k \\ I_k & -S_k \end{pmatrix}, \tag{3.14}$$

where

$$\begin{aligned}\widehat{A}_{ii} = \quad & B_i X_i^+ + (B_i X_i^+)^T [I_k - X_i X_i^+] \\ & + [I_k - X_i X_i^+] B_i (X_i^T B_i)^+ B_i^T (I_k - X_i X_i^+) + Q_2^{(i)} [\widetilde{A}_{22}^{(i)}]_+ Q_2^{(i)T}, \quad i = 1, 2.\end{aligned} \tag{3.15}$$

If $n = 2k+1, P = S_n = [e_n, \cdots, e_1]$, *then*

$$P = U \begin{pmatrix} I_{k+1} & 0 \\ 0 & -I_k \end{pmatrix} U^T, \tag{3.16}$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix} \in \mathbf{OR}^{n\times n}$. *Let*

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & S_k \\ 0 & \sqrt{2} & 0 \end{pmatrix} X, \qquad B_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & S_k \\ 0 & \sqrt{2} & 0 \end{pmatrix} B, \tag{3.17}$$

$$X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & -S_k \end{pmatrix} X, \qquad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & -S_k \end{pmatrix} B. \tag{3.18}$$

It is easy to obtain a result similar to Corollary 3.1 from (3.16) ∼(3.18), Theorem 2.2 and Theorem 3.1.

4. An Inverse Eigenvalue Problem

Now we consider the following inverse eigenvalue problem for P-symmetric nonnegative definite matrices.

Problem IEP. *Given a matrix* $\widetilde{A} \in \mathbf{R}^{n\times n}, m (0 < m \le n)$ *eigenvalues* $\lambda_1, \cdots, \lambda_m \in \mathbf{R}$ *and corresponding eigenvectors* $x_1, \cdots, x_m \in \mathbf{R}^n$, *let*

$$S_A = \{A \in \mathbf{SR}_P^{n\times n} | A x_i = \lambda_i x_i, \quad i = 1, \cdots, m\}, \tag{4.1}$$

find $\widehat{A} \in S_A$ *such that*

$$\|\widetilde{A} - \widehat{A}\|_F = \inf_{\forall A \in S_A} \|\widetilde{A} - A\|_F. \tag{4.2}$$

Problem IEP is related to the frequently encountered engineering problem of a structural modification on the dynamic behavior of a structure[4,11]. Let

$$X = [x_1, \cdots, x_m], \quad \Lambda = diag(\lambda_1, \cdots, \lambda_m), \tag{4.3}$$

then

$$S_A = \{A \in \mathbf{SR}_P^{n\times n} | AX = X\Lambda\}. \tag{4.4}$$

Lemma 4.1 (c.f.[12]). *Suppose* $Y = [y_1, \cdots, y_m] \in \mathbf{R}^{n\times m} (y_i \in \mathbf{R}^n), \Lambda = diag(\lambda_1, \cdots, \lambda_m) \in \mathbf{R}^{m\times m}$. *Then there is a matrix* $A \in \mathbf{SR}_0^{n\times n}$ *such that* $AY = Y\Lambda$ *if and only if*

$$\lambda_i \ge 0, \quad (\lambda_i - \lambda_j) y_i^T y_j = 0, \quad i, j = 1, 2, \cdots, m. \tag{4.5}$$

The following result gives the solution to Problem IEP.

Theorem 4.1. *Suppose* $\widetilde{A} \in \mathbf{R}^{n\times n}, X = [x_1, \cdots, x_m] \in \mathbf{R}^{n\times m}, \Lambda = diag(\lambda_1, \cdots, \lambda_m) \in \mathbf{R}^{m\times m}$. *Let the spectral decomposition of the matrix* $P \in \mathbf{OR}^{n\times n}$ *be (2.1), and*

$$U^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{4.6}$$

where $X_1 = [x_1^{(1)}, \cdots, x_m^{(1)}] \in \mathbf{R}^{k_1\times m}, X_2 = [x_1^{(2)}, \cdots, x_m^{(2)}] \in \mathbf{R}^{k_2\times m}(x_j^{(i)} \in \mathbf{R}^{k_i}(i = 1, 2, j = 1, \cdots, m))$, *and the SVD of the matrix* X_i *be (2.10). Then* S_A *is nonempty if and only if*

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j) {x_i^{(l)}}^T x_j^{(l)} = 0, \quad l = 1, 2, \quad i, j = 1, \cdots, m. \tag{4.7}$$

If X *and* Λ *satisfy the condition (4.7), then there exists a unique P-symmetric nonnegative definite matrix* $\widehat{A} \in S_A$ *satisfying (4.2). Moreover, let*

$$U^T \widetilde{A} U = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \quad \widetilde{A}_{22}^{(i)} = {Q_2^{(i)}}^T [\widetilde{A}_{ii} - X_i \Lambda (X_i^T X_i \Lambda)^+ \Lambda X_i^T] Q_2^{(i)}, \ i = 1, 2$$

where $\widetilde{A}_{ii} \in \mathbf{R}^{k_i\times k_i}(i = 1, 2)$, *then* $\widehat{A}$ *can be expressed as*

$$\widehat{A} = U \begin{pmatrix} \widehat{A}_{11} & 0 \\ 0 & \widehat{A}_{22} \end{pmatrix} U^T, \tag{4.8}$$

where $\widehat{A}_{ii} = X_i \Lambda X_i^+ + Q_2^{(i)} [\widetilde{A}_{22}^{(i)}]_+ {Q_2^{(i)}}^T \ (i = 1, 2)$.

Proof. From Theorem 2.1, we know that $AX = X\Lambda$ and $A \in \mathbf{SR}_P^{n\times n}$ are equivalent to

$$U \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} U^T X = X\Lambda, \tag{4.9}$$

where $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i}(i = 1, 2)$. So S_A is nonempty if and only if $A_{ii}X_i = X_i\Lambda$ has a solution $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i}(i = 1, 2)$. It follows from Lemma 4.1 that $A_{ii}X_i = X_i\Lambda$ has a solution in $\mathbf{SR}_0^{k_i\times k_i}$ if and only if X_i and Λ satisfy the condition (4.7).

From Theorem 2.2, Theorem 3.1 and $X_i^T X_i X_i^+ = X_i^{T[3]}$, it follows that Problem IEP has a unique solution $\widehat{A}$ which can be expressed as (4.8).

Remarks. Our results here generalize those results in [10].

5. Numerical Algorithm and Example

Based on Theorem 2.2 and Theorem 3.1 we can describe an algorithm for solving Problem IP as follows.

Algorithm 5.1. 1) Compute the spectral decomposition (2.1) of the given matrix P;

2) Form the matrices X_i and $B_i(i = 1, 2)$ by (2.9);

3) If $X_i^T B_i = B_i^T X_i \geq 0$ and $rank(X_i^T B_i) = rank(B_i)(i = 1, 2)$, go to 4); otherwise Problem IP has no solution and stop;

4) Compute the SVD of X_i, and $X_i^+(i = 1, 2)$;

5) Compute A_0 by (2.13) and (2.14);

6) Compute $\widetilde{A}_{ii}$ and $\widetilde{A}_{22}^{(i)}(i = 1, 2)$ by (3.2);

7) Find the unique positive approximation $\widehat{G}_i$ of $\widetilde{A}_{22}^{(i)}(i = 1, 2)$ based on Lemma 3.1;

8) Compute the solution $\widehat{A}$ of Problem IP by (3.3).

Example 5.1. Let

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 2.7 & -1.2 & -0.1 & -1.2 \\ -1.1 & 2.1 & 1.2 & 1.1 \\ 0 & 1.1 & 2.7 & 1.1 \\ -1.1 & 1.2 & 1.2 & 4 \end{pmatrix},$$

$$X = \begin{pmatrix} \frac{\sqrt{3}-1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{3}+1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\sqrt{3}-1}{\sqrt{6}} & -\frac{2}{\sqrt{3}} \\ \frac{\sqrt{3}+1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ 0 & -2 \end{pmatrix}.$$

The spectral decomposition of the matrix P is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

We get

$$U^T X = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad U^T B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $X_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = B_1, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}$. It is easy to verify $X_i^T B_i = B_i^T X_i \geq 0$ and $rank(X_i^T B_i) = rank(B_i)(i = 1, 2)$. It follows from Theorem 2.2 that S_A is nonempty. Using MATLAB we obtain the unique solution of Problem IP as

$$\widehat{A} = \begin{pmatrix} 2.9527 & -0.7500 & -0.3098 & -1.1620 \\ -0.7500 & 2.0723 & 1.1903 & 1.1620 \\ -0.3098 & 1.1903 & 2.5125 & 1.1620 \\ -1.1620 & 1.1620 & 1.1620 & 4.0126 \end{pmatrix}.$$

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