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THE UPWIND FINITE ELEMENT SCHEME AND MAXIMUM PRINCIPLE FOR NONLINEAR CONVECTION-DIFFUSION PROBLEM *1)

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Abstract

In this paper, a kind of partial upwind finite element scheme is studied for two-dimensional nonlinear convection-diffusion problem. Nonlinear convection term approximated by partial upwind finite element method considered over a mesh dual to the triangular grid, whereas the nonlinear diffusion term approximated by Galerkin method. A linearized partial upwind finite element scheme and a higher order accuracy scheme are constructed respectively. It is shown that the numerical solutions of these schemes preserve discrete maximum principle. The convergence and error estimate are also given for both schemes under some assumptions. The numerical results show that these partial upwind finite element scheme are feasible and accurate.

Mathematics subject classification: 65F10, 65N30.
Key words: Convection-diffusion problem, Partial upwind finite element, Maximum principle.

1. Introduction

Convection-diffusion processes appear in many areas of science and technology. For example, fluid dynamics, heat and mass transfer, hydrology and so on. This is the reason that the numerical solution of convection-diffusion problem attracts a number of speciality. From an extensive literature devoted to linear problems, let us mentioned some papers [2], [3], monographs[1], and the references therein, few approaches to the solution of nonlinear problems mentioned in the papers [4], [10] and [11].

It is a well-known fact that the use of a classical Galerkin method with continuous piecewise linear finite elements leads to spurious oscillations when the local Péclet number is large. To obtain an effective scheme in the case of that convection term is dominate or the Peclét number is large, it is required to consider a suitable approximate for the convection term $\nabla \cdot \vec{b}(u)$. The partial upwind finite element scheme is known as the method solve convection-diffusion problem when the convection term is dominated [3]. In [10], the partial upwind finite element scheme for two-dimensional nonlinear Burgers equation is studied. In [4] and [11], M. Feistauer and his fellows investigates a combined finite volume-finite element methods for two-dimensional nonlinear convection-diffusion problem which the convection term only is nonlinear, and the convection term is explicit scheme. The purpose of this paper is to present an partial upwind finite element scheme for a more general type of two-dimensional nonlinear convection-diffusion problem which approximate the diffusion term by standard Galerkin method and approximate the convection term by partial upwind finite element method on the mesh dual to the triangular grid of weakly acute type.

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The method is easy to be carried out and it is applicable in multi-dimensional problem. Especially, it preserve maximum principle of original problem. Under some assumptions on the regularity of the exact solution of the continuous problem, we prove error estimates of the scheme. The numerical computations for the system of compressible viscous flow have demonstrated that the partial upwind finite element scheme is feasible and produces numerical results which are very promising.

This paper consists of seven sections. In Section 2, the notation and the nonlinear problem is given. In Section 3, the finite element space is defined, and the partial upwind finite element scheme. The discrete maximum principle and the convergence of the scheme is shown in Section 4 and Section 5 respectively. On the base of above work, a higher order accuracy scheme is studied in Section 6. In Section 7, we give another partial upwind finite element scheme, and prove its discrete maximum principle and convergence. To test above schemes, we give some numerical examples in Section 8, these numerical results show that these partial upwind finite element schemes are feasible and accurate.

2. Formulation of the Problem and Some Notations

Throughout this paper, we will use C (with or without subscript or superscript) to denote generic constant independent of discrete parameter. $W^{m,p}(\Omega)$ denotes usual Sobolev spaces, where $\Omega \subset R^2$ is a convex polygon domain, m, p are nonnegative integer. The corresponding norm and semi-norm are $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ [6]. Particular, for $p=2$, $H^m(\Omega)=W^{m,2}(\Omega)$, the corresponding norm and semi-norm are $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$, then

$$(u,v)=\int_\Omega uv\mathrm{d}x, \quad \|u\|_{0,\Omega}=(u,u).$$

As usual $H_0^1(\Omega)=\{v\in H^1(\Omega); v|_{\partial\Omega}=0\}$ denotes the subspace of $H^1(\Omega)$.

We consider the following two-dimensional nonlinear convection-diffusion initial boundary problem (P)

$$\begin{cases} \dfrac{\partial u}{\partial t}-\nabla\cdot(a(u)\nabla u)+\nabla\cdot\vec{b}(u)=f(u) & (x,t)\in\Omega\times[0,T]=D; & (2.1)\\ u(x,t)=0 & (x,t)\in\Gamma\times[0,T]; & (2.2)\\ u(x,0)=u^0(x) & x\in\bar{\Omega}. & (2.3)\end{cases}$$

where Γ is the boundary of Ω, $x=(x_1,x_2)$.

We define the bound set on $\mathbf{R}$:

$$G=\{u:|u|\le K_0\}$$

where K_0 is a positive constant which will be fixed later.

We assume the coefficient of problem (P) satisfied the following condition:

(**A1**) There exist constants m, M_a, C_1 and C_2 which depend on K_0 such that

$$0<m<a(u)<M_a, \quad \forall (x,t)\in\Omega\times(0,T], u\in G.$$

$$|f(u)|\le C_1|u|+C_2, \quad \forall (x,t)\in\Omega\times(0,T], u\in\mathbf{R}.$$

$\vec{b}(u)=\big(b^{(1)}(u),b^{(2)}(u)\big)\in W^1_\infty(G)\times W^1_\infty(G)$, $f(u)\in W^1_\infty(G\times\Omega\times(0,T])$, $u^0(x)\in C(\bar{\Omega})\bigcap H_0^1(\Omega)$.

(**A2**) $a(u)$, $\vec{b}(u)$ and $f(u)$ are locally Lipschitz continuous

$$|a(u)-a(v)|\le L|u-v|, \quad \forall u,v\in G.$$

$$\|\vec{b}(u)-\vec{b}(v)\| \le L|u-v|, \quad \forall u,v\in G.$$
$$|f(u)-f(v)| \le L|u-v|, \quad \forall u,v\in G.$$

where L is constant related to K_0 and $\|\vec{b}(u)-\vec{b}(v)\|$ is defined as

$$\|\vec{b}(u)-\vec{b}(v)\| = \{[b_1(u)-b_1(v)]^2+[b_2(u)-b_2(v)]^2\}^{1/2}.$$

The weak form of problem (P) is, find $u:[0,T]\to H_0^1(\Omega)$ such that

$$\begin{cases} (u_t,v)+(a(u)\nabla u,\nabla v)+(\nabla\cdot\vec{b}(u),v)=(f(u),v), \quad \forall v\in H_0^1(\Omega) & (2.4)\\ u(0)=u^0 & (2.5)\end{cases}$$

and we assume that the weak solution u of problem (P) satisfies the following regularity:

(**A3**) $u\in L^\infty(0,T;H^2(\Omega))\cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$, $|u(x,t)|\le K_0, \forall(x,t)\in\Omega\times[0,T]$, u_t, $u_{tt}\in L^2(0,T;H^2(\Omega))$.

3. The Finite Element Space and Partial Upwind Finite Element Scheme

Let us consider a family of regular triangulation $\{T_h\}$ in $\bar{\Omega}$ (see [9]). For a fixed triangulation T_h, we define a closed triangulation set $\{e_i\}_{i=1}^{N_e}$ and node set $\{p_i\}_{i=1}^{K}$, where $p_i(1\le i\le N)$ are inner nodes of Ω, $p_j(N+1\le j\le K)$ are boundary nodes on Γ. Let h_e denotes the diameter of element e, κ_e denotes the minimum altitude length of e. We denote the mesh parameter:

$$h=\max_{e\in T_h}\{h_e\}, \qquad \kappa=\min_{e\in T_h}\{\kappa_e\}$$

We assume that the triangulation family $\{T_h\}$ is regular and weakly acute type, i.e.

(**A4**) There exists $\alpha_0\in(0,\frac{\pi}{2})$ independent of h, such that all interior angles α of the triangles are bounded as follows:

$$\alpha\in\left[\alpha_0,\frac{\pi}{2}\right]$$

For a given triangulation T_h with nodes $\{p_i\}\in\bar{\Omega}$ we construct a secondary partition. Namely, we introduce regions

$$\Omega_i^e=\{p:p\in e,|p-p_i|\le|p-p_j|,\forall p_j\in e\},$$

where $|p-p_i|$ is the distance of node p and node p_i. We consider the dual decomposition $\tilde{T_h}=\{\Omega_i\}$, where Ω_i is circumcentric domain associated with nodal point P_i: $\Omega_i=\bigcup_{e\in T_h}\Omega_i^e$.

We define the area of Ω_i and Ω_i^e by $|\Omega_i|=meas_2(\Omega_i)$ and $|\Omega_i^e|=meas_2(\Omega_i^e)$ respectively, and associate the index set $\Lambda_i=\{j:j\ne i,p_j \text{ is adjacent to } p_i\}$. We say that nodes p_i, p_j are adjacent iff $\Gamma_{ij}=\partial\Omega_i\cap\partial\Omega_j\ne\emptyset$. The set of indices of all interior nodes $p_i\in\Omega$ is denoted by Λ, $T_h(i)=\{e:p_i\in e,e\in T_h\}$.

For $\tilde{T_h}$, we define the mass lumping operator $\hat{}: C(\bar{\Omega})\to L^\infty(\Omega)$.

$$\hat{\omega}(p)=\sum_{i=1}^{K}\omega(p_i)\mu_i(p)$$

where μ_i is the characteristic function defined on the circumcentric domain Ω_i associated with nodal point p_i. For $e\in T_h$, we define the adjacent node set index of node p_i: $\Lambda_i^e=\{j|j\ne i,p_j\in e\}$, Γ_i denotes the boundary of Ω_i, and $|\Gamma_i^e|$ is the length of Γ_i^e. $\Gamma_i^e=\Gamma\cap e$. Let

$$V_h=\{v|v\in C^0(\bar{\Omega});v \text{ is linear function on } e,\forall e\in T_h\}=span\{\varphi_1,\cdots,\varphi_{N+K}\}$$

be the subspace of $H^1(\Omega)$, where $\{\varphi_i\}_{i=1}^{N+K}$ is the nodal basis of two dimensional linear finite element space V_h, and the subspace of $H_0^1(\Omega)$:

$$V_{0h} = \{v | v \in V_h, v|_{\partial\Omega} = 0\} = span\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$$

It is clear that $\dim V_h = K$, $\dim V_{0h} = N$.

Then we have some important lemmas:

Lemma 1[1]. *For all node p_i $(1 \le i \le K)$,*

$$|\Gamma_i^e| \le 4|\Omega_i^e|/\kappa_e$$

Lemma 2[1]. *For all $w \in V_h$*

$$(\nabla w, \nabla \varphi_i) = -\sum_{j\in\Lambda_i} (w(p_j) - w(p_i))|\Gamma_{ij}|/|p_i - p_j|, \qquad 1 \le i \le K$$

where φ_i is the basis of the finite element space of V_h, $|\Gamma_{ij}|$ is the length of Γ_{ij}.

Lemma 3[8]. *There are constants C_*, $C^*, C^{**} > 0$ such that*

$$C_* \|w\|_{0,\Omega} \le \|\hat{w}\|_{0,\Omega} \le C^* \|w\|_{0,\Omega}, \qquad \forall w \in V_h.$$

and

$$|(\hat{\phi}, \hat{\chi}) - (\phi, \chi)| \le C^{**} h^2 \|\phi\|_{1,\Omega} \|\chi\|_{1,\Omega}, \quad \forall \phi, \chi \in V_h$$

Lemma 4[9]. *There is a constant C such that*

$$\|w - \hat{w}\|_{0,\Omega} \le Ch|w|_{1,\Omega}, \qquad \forall w \in C(\bar{\Omega}) \cap H^1(\Omega).$$

Let $\tau > 0$ is time step and $N_\tau = T/\tau$, then the partial upwind finite element scheme (P_h) of problem (P) is that for $n = 0, 1, \cdots, N_\tau - 1$, find $U^{n+1} \in V_{0h}$ such that

$$\begin{cases} (D_\tau \hat{U}^n, \hat{v}) + (a(U^n)\nabla U^{n+\frac{1}{2}}, \nabla v) + R(\vec{B}^n, U^n, v) = (\hat{f}_h(U^n), \hat{v}), \forall v \in V_{0h} \\ U^0 = I_h u^0 \end{cases}$$

where

$$D_\tau \hat{U}^n = (\hat{U}^{n+1} - \hat{U}^n)/\tau, \qquad U^{n+\frac{1}{2}} = (U^{n+1} + U^n)/2, \qquad f_h(U^n) = I_h f(U^n)$$

$$R(\vec{B}^n, U^n, v) = \sum_{i=1}^N v(p_i) \sum_{j\in\Lambda_i} (\sigma_{ij}^n U_i^n + \sigma_{ji}^n U_j^n - U_i^n)\beta_{ij}^n.$$

$$U_i^n = U^n(p_i), \qquad \beta_{ij}^n = \int_{\Gamma_{ij}} \vec{B}_{ij}^n \cdot \vec{n}_{ij} \mathrm{d}s, \qquad \vec{B}_{ij}^n = \frac{\vec{b}(U_j^n) - \vec{b}(U_i^n)}{U_j^n - U_i^n}.$$

In numerical computation, we define $\vec{B}_{ij}^n = \dfrac{\partial \vec{b}}{\partial u}(U_i^n)$ if $U_j^n = U_i^n$.

In scheme (P_h), the partial upwind parameter σ_{ij}^n and σ_{ji}^n in convection term $R(\vec{B}^n, U^n, v)$ could be chosen as

$$\sigma_{ij}^n = \begin{cases} 1 - \frac{1}{\rho_{ij}^n} + \frac{1}{e^{\rho_{ij}^n} - 1} \;, & \text{if } \beta_{ij}^n > 0 \\ \frac{1}{2}, & \text{if } \beta_{ij}^n = 0 \\ -\frac{1}{\rho_{ij}^n} - \frac{1}{e^{-\rho_{ij}^n} - 1}, & \text{if } \beta_{ij}^n < 0 \end{cases}$$

where ρ_{ij}^n is local Péclet number defined as:

$$\rho_{ij}^n = \frac{2\beta_{ij}^n}{|a_{ij}^n|}, \quad a_{ij}^n = (a(U^n)\nabla\varphi_j, \nabla\varphi_i)\,(i=1,2,\cdots,K; j \in \Lambda_i).$$

It is easy to see that

(i) $0 \le \sigma_{ij}^n, \sigma_{ji}^n \le 1;\ \sigma_{ij}^n + \sigma_{ji}^n = 1.$

(ii) $\begin{cases} \max\left\{\frac{1}{2}, 1-\frac{1}{\rho_{ij}^n}\right\} \le \sigma_{ij}^n \le 1, \text{if } \beta_{ij}^n \ge 0 \\ \max\left\{\frac{1}{2}, 1-\frac{1}{\rho_{ji}^n}\right\} \le \sigma_{ji}^n \le 1, \text{if } \beta_{ji}^n \ge 0 \end{cases}$

4. The Discrete Maximum Principle

At first, we introduce an important lemma:

Lemma 5. *Let $m \times k$ $(m \le k)$ matrices $A = (a_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ satisfy the conditions:*

(i) $\sum\limits_{j=1}^k a_{ij} \ge \sum\limits_{j=1}^k c_{ij} \ge 0$, $\sum\limits_{j=1}^k a_{ij} \ge \sum\limits_{j=1}^k d_{ij} \ge 0$ *and* $\sum\limits_{j=1}^k a_{ij} > 0$ $(i = 1, \cdots, m)$;

(ii) $c_{ij} \ge 0 (1 \le i \le m, 1 \le j \le k)$;

(iii) $d_{ij} \ge 0 (1 \le i \le m, 1 \le j \le k)$;

(iv) $a_{ij} \le 0 (1 \le i \le m, 1 \le j \le k, j \ne i)$.

if the vector $\vec{u} = (u_1, u_2, \cdots, u_k)^T$ satisfy

$$A\vec{u} = C\vec{w} + \tau D\vec{g} \tag{4.1}$$

where $\vec{w}$,$\vec{g}$ is k dimensional vectors, $\tau > 0$, then each component u_i $(1 \le i \le m)$ can be estimated by:

$$\max_{1\le i\le m} |u_i| \le \max\left\{\max_{1\le j\le k} |w_j| + \tau \max_{1\le j\le k} |g_j|, \max_{m+1\le j\le k} |u_j|\right\}. \tag{4.2}$$

Proof. Take i such that $|u_i| = \max\limits_{1\le j\le m}\{|u_j|\}$, if $|u_i| \le \max\limits_{m+1\le j\le k}\{|u_j|\}$, equation (4.2) holds; Then we only need consider the cases $|u_i| \ge \max\limits_{m+1\le j\le k}\{|u_j|\}$. The i-th component of equation (4.1) is written as:

$$a_{ii}u_i = -\sum_{j\ne i} a_{ij}u_j + \sum_{j=1}^k c_{ij}w_j + \tau\sum_{j=1}^k d_{ij}g_j$$

It is clear that $a_{ii} > 0$ from conditions (i) and (iv), from triangle-inequality and conditions (ii), (iii) and (iv) we have

$$a_{ii}|u_i| \le -\sum_{j\ne i} a_{ij}|u_i| + \sum_{j=1}^k c_{ij} \max_{1\le j\le k} |w_j| + \tau\sum_{j=1}^k d_{ij} \max_{1\le j\le k} |g_j|$$

then

$$|u_i| \le \frac{\sum\limits_{j=1}^k c_{ij}}{\sum\limits_{j=1}^k a_{ij}} \max_{1\le j\le k} |w_j| + \tau\frac{\sum\limits_{j=1}^k d_{ij}}{\sum\limits_{j=1}^k a_{ij}} \max_{1\le j\le k} |g_j|$$

From (i), we get

$$|u_i| \le \max_{1\le j\le k} |w_j| + \tau \max_{1\le j\le k} |g_j|.$$

then equation (4.2) hold.

For the Banach space X and a function $\phi(t)$ defined on discrete set: $\{0, \tau, 2\tau, \cdots, N_\tau \tau\} \to X$, let $\phi^n = \phi(n\tau)$,we define new norm:

$$\|\phi\|_{\tilde{L}^\infty(X)} = \max_{0\le n\le N_\tau} \|\phi^n\|_X.$$

Lemma 6. *Let $\tau > 0$ is time step and $N_\tau = T/\tau$, $\phi(t)$ is a discrete function defined as above, if $\phi(t)$ satisfy:*

$$\|\phi^{n+1}\|_X \le (1 + C_1\tau)\|\phi^n\|_X + C_2\tau, \quad n = 0, 1, \cdots, N_\tau - 1. \tag{4.3}$$

where $C_1, C_2 > 0$ is constant independent of τ. Then for $n = 1, 2, \cdots, N_\tau - 1$

$$\|\phi\|_{\tilde{L}^\infty(X)} \le e^{C_1T}\left(\|\phi^0\|_X + C_2T\right).$$

proof. From (4.3), we get for $n = 1, 2, \cdots, N_\tau$

$$\|\phi^n\|_X \le (1 + C_1\tau)^n\|\phi^0\|_X + \sum_{i=0}^{n-1} C_2(1 + C_1\tau)^i\tau.$$

For $x > 0$, by $(1 + x)^{\frac{1}{x}} \le e$, we get

$$(1 + C_1\tau)^n \le (1 + C_1\tau)^{N_\tau} \le e^{C_1T}.$$

Hence

$$\|\phi^n\|_X \le e^{C_1T}\|\phi^0\|_X + N_\tau C_2 e^{C_1T}\tau = e^{C_1T}\left(\|\phi^0\|_X + C_2T\right).$$

Then, we can get the following discrete maximum principle:

Theorem 1. *Suppose the conditions (A1),(A2) and (A4) be satisfied and time step τ satisfies the condition:*

$$0 < \tau \le \frac{\kappa^2}{2M_a + 4L\kappa}, \tag{4.4}$$

then the solution U of scheme (P_h) is bounded and is estimated by

$$\|U\|_{\tilde{L}^\infty(L^\infty(\Omega))} \le e^{C_1T}\left(\|u^0\|_{0,\infty,\Omega} + C_2T\right). \tag{4.5}$$

where C_1 and C_2 is defined in assumption (A1).

Remark. With the Theorem 1 the constant K_0 related the set G can be fixed by

$$K_0 = \max\left\{e^{C_1T}\left(\|u^0\|_{0,\infty,\Omega} + C_2T\right); \|u\|_{L^\infty(0,T,L^\infty(\Omega))}\right\} \tag{4.6}$$

proof of theorem. We prove the theorem by mathematics induction. It is clear that at $n = 0$, $\|U^0\|_{0,\infty,\Omega} = \|I_hu^0\|_{0,\infty,\Omega} \le \|u^0\|_{0,\infty,\Omega} \le K_0$.

Let $M = (m_{ij})$, $A = (a^n_{ij})$, $B = (b^n_{ij})$, where

$$m_{ij} = (\hat{\varphi}_j, \hat{\varphi}_i). \;\; a^n_{ij} = (a(U^n)\nabla\varphi_j, \nabla\varphi_i). \;\; b^n_{ij} = R(\vec{B}^n, \varphi_j, \varphi_i).$$

then we can write the equation (P_h) in the matrix form:

$$\left(M + \frac{\tau}{2}A\right)U^{n+1} = \left[M - \frac{\tau}{2}(A + 2B)\right]U^n + \tau M f(U^n).$$

where vectors $U^n = (U_1^n, U_2^n \cdots, U_K^n)^T$ and $f(U^n) = (f_1^n, f_2^n, \cdots, f_K^n)^T$ with $f_i^n = (\hat{f}_h(U^n), \hat{\varphi}_i)$.

It is clear that

$$m_{ii} > 0, \quad m_{ij} = 0 (j \neq i), \quad a_{ij}^n \leq 0 (i \neq j).$$

$$\sum_{j=1}^K m_{ij} = m_{ii} > 0, \quad \sum_{j=1}^K a_{ij}^n = 0.$$

$$\sum_{j=1}^K b_{ij}^n = b_{ii}^n + \sum_{j\in\Lambda_i} b_{ij}^n = \sum_{j\in\Lambda_i} (\sigma_{ij}^n - 1)\beta_{ij}^n + \sum_{j\in\Lambda_i} \sigma_{ji}^n \beta_{ij}^n = 0$$

(i) $\sum_{j=1}^K \left(m_{ij} + \frac{\tau}{2} a_{ij}^n\right) = \sum_{j=1}^K \left[m_{ij} - \frac{\tau}{2}(a_{ij}^n + 2b_{ij}^n)\right] = \sum_{j=1}^K m_{ij} = m_{ii} > 0, \ (1 \leq i \leq N);$

(ii) If $\beta_{ij}^n \geq 0$, because

$$\sigma_{ji}^n = 1 - \sigma_{ij}^n \leq 1/\rho_{ij}^n = |a_{ij}^n|/(2\beta_{ij}^n),$$

$$a_{ij}^n + 2b_{ij}^n = a_{ij}^n + 2\sigma_{ji}^n \beta_{ij}^n \leq 0 \ \ (j \neq i).$$

then

$$m_{ij} - \frac{\tau}{2}(a_{ij}^n + 2b_{ij}^n) \geq 0, (1 \leq i \leq N, 1 \leq j \leq K, j \neq i).$$

We can get the same result if $\beta_{ij}^n \leq 0$.

From lemma 1, lemma 2 and the assumption of theorem, we can get

$$\begin{aligned}
& m_{ii} - \tfrac{\tau}{2}(a_{ii}^n + 2b_{ii}^n) \\
= \ & |\Omega_i| - \tfrac{\tau}{2}\left[(a(U^n)\nabla\varphi_i, \nabla\varphi_i) + 2\sum_{j\in\Lambda_i}(\sigma_{ij}^n - 1)\beta_{ij}^n\right] \\
= \ & |\Omega_i| - \tfrac{\tau}{2}(a(U^n)\nabla\varphi_i, \nabla\varphi_i) - \tau \sum_{j\in\Lambda_i}(\sigma_{ij}^n - 1)\int_{\Gamma_{ij}} \vec{B}_{ij}^n \cdot \vec{n}_{ij} \mathrm{d}s \\
\geq \ & |\Omega_i| - \tfrac{M_a\tau}{2}\sum_{j\in\Lambda_i}\tfrac{|\Gamma_{ij}|}{|p_i - p_j|} - L\tau\sum_{j\in\Lambda_i}|\Gamma_{ij}| \\
\geq \ & \sum_{e\in T_h(i)}\left\{|\Omega_i^e| - \tfrac{M_a\tau}{2}4\tfrac{|\Omega_i^e|}{\kappa_e^2} - 4L\tau\tfrac{|\Omega_i^e|}{\kappa_e}\right\} \\
= \ & \sum_{e\in T_h(i)}\left[\left(1 - \tfrac{2M_a\tau}{\kappa_e^2} - \tfrac{4L\tau}{\kappa_e}\right)|\Omega_i^e|\right]
\end{aligned}$$

Hence if τ satisfy the condition (4.4), then

$$m_{ii} - \frac{\tau}{2}(a_{ii}^n + 2b_{ii}^n) \geq 0, \ (1 \leq i \leq N).$$

(iii) $m_{ij} \geq 0, \ (1 \leq i \leq N, 1 \leq j \leq K)$.

(iv) if $i \neq j$, $m_{ij} + \frac{\tau}{2}a_{ij}^n = \frac{\tau}{2}a_{ij}^n \leq 0, \ (1 \leq i \leq N, 1 \leq j \leq K)$.

Then by lemma 5 we get

$$\max_{1\leq i\leq N}|U_i^{n+1}| \leq \max_{1\leq j\leq K}|U_j^n| + \tau\max_{1\leq j\leq K}|f_j^n|, \quad n = 0, 1, \cdots, N_\tau - 1.$$

With $U_{N+1}^{n+1} = \cdots = U_K^{n+1} = 0$ that is

$$\|U^{n+1}\|_{L^\infty(\Omega)} \leq \|U^n\|_{L^\infty(\Omega)}(1 + C_1\tau) + C_2\tau.$$

From lemma 6, we get

$$\|U\|_{\tilde{L}^\infty(0,T;L^\infty(\Omega))} \leq e^{C_1T}\left(\|u^0\|_{0,\infty,\Omega} + C_2T\right)$$

5. The Error Analysis

For error analysis we need the following assumption:
(**A5**) There is a constant C_3 and C_4 independent of h and τ, such that

$$C_3h^2 \le \tau \le C_4h^2.$$

Lemma 7. *Let U be the solution of scheme (P_h), if $\|U^n - u^n\|_{0,\Omega} \le C_5(h+\tau)$, where C_5 is a constant independent of h and τ. then*

$$\|\nabla U^n\|_{0,\Omega} \le C_6,$$

where C_6 is a constant independent of h and τ.

Proof. Choose the function $\phi \in V_{0,h}$ such that $\|\phi - u^n\|_{1,\Omega} \le Ch$, then

$$\begin{aligned}\|\nabla U^n\|_{0,\Omega} &\le \|\nabla(U^n - \phi + \phi - u^n)\|_{0,\Omega} + \|\nabla u^n\|_{0,\Omega}\\ &\le \|\nabla(U^n - \phi)\|_{0,\Omega} + \|\nabla(\phi - u^n)\|_{0,\Omega} + \|\nabla u^n\|_{0,\Omega}\end{aligned}$$

From (A3), we know $\|\nabla u^n\|_{0,\Omega}$is bounded. Notice the choice of ϕ, $\|\nabla(\phi - u^n)\|_{0,\Omega}$ also is bounded. To estimate the first term in the last inequality, we use the inverse estimate

$$\|\nabla(U^n - \phi)\|_{0,\Omega} \le Ch^{-1}\|U^n - \phi\|_{0,\Omega} \le Ch^{-1}\left(\|U^n - u^n\|_{0,\Omega} + \|u^n - \phi\|_{0,\Omega}\right)$$

Hence

$$\|\nabla U^n\|_{0,\Omega} \le C_6.$$

Lemma 8[4]**.** *Under assumptions (A3), for $t_k \in [0,T)$ we have*

$$|(u^{k+1} - u^k, v) - \tau(u_t^{k+\frac{1}{2}}, v)| \le C\tau^2\|v\|_{0,\Omega}, \quad \forall v \in V_{0h},$$

where C is a constant independent of τ and h.

Theorem 2. *Let u be the exact solution of problem (P) which is sufficient smooth, conditions (A1), (A2), (A3), (A4), (A5) hold, and τ be the time step which is sufficient small, U be the solution of problem (P_h), then we have*

$$\|U - u\|_{\tilde{L}^\infty(0,T;L^2(\Omega))} \le C(h+\tau)$$

where C is a constant independent of h and τ.

*Proof.*Let $w = I_hu$, we set

$$U - u = (U - w) + (w - u) = \theta + \rho$$

Use well-known estimate[9], we get

$$\|\rho\|_{0,\Omega} = \|I_hu - u\|_{0,\Omega} \le Ch^2|u|_{2,\Omega}$$

where C is a constant independent of u, h and τ.

We proof the theorem by mathematics induction. It is clear that at $n = 0$, $\|U^0 - u^0\|_{0,\Omega} \le Ch$. We assume $\|U^m - u^m\|_{0,\Omega} \le C(h+\tau)$,$0 \le m \le n$. Then we must proof $\|U^{n+1} - u^{n+1}\|_{0,\Omega} \le C(h+\tau)$. From equation (P_h) and equation (2.4)

$$\begin{aligned}&(D_\tau\hat{\theta}^n, \hat{v}) + (a(U^n)\nabla\theta^{n+\frac{1}{2}}, \nabla v)\\ &= (\hat{f}_h(U^n), \hat{v}) - (D_\tau\hat{w}^n, \hat{v}) - (a(U^n)\nabla w^{n+\frac{1}{2}}, \nabla v) - R(\vec{B}, U^n, v)\\ &= [(\hat{f}_h(U^n), \hat{v}) - (f(u^{n+\frac{1}{2}}), v)] - [(D_\tau\hat{w}^n, \hat{v}) - (u_t^{n+\frac{1}{2}}, v)] - [(a(U^n)\nabla w^{n+\frac{1}{2}}, \nabla v)\end{aligned}$$

$$-(a(u^{n+\frac{1}{2}})\nabla u^{n+\frac{1}{2}},\nabla v)]-[R(\vec{B}^n,U^n,v)-(\nabla\cdot\vec{b}(u^{n+\frac{1}{2}}),v)],\qquad \forall v\in V_{0h}$$

put $v=\theta^{n+\frac{1}{2}}$, since

$$(D_\tau\hat{\theta}^n,\hat{\theta}^{n+\frac{1}{2}})=\left(\frac{\hat{\theta}^{n+1}-\hat{\theta}^n}{\tau},\frac{\hat{\theta}^{n+1}+\hat{\theta}^n}{2}\right)$$
$$=\frac{1}{2}D_\tau\|\hat{\theta}^n\|^2_{0,\Omega},\ \left(a(U^n)\nabla\theta^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}}\right)\geq m|\theta^{n+\frac{1}{2}}|^2_{1,\Omega}.$$

then

$$\frac{1}{2}D_\tau\|\hat{\theta}^n\|^2_{0,\Omega}+m|\theta^{n+\frac{1}{2}}|^2_{1,\Omega}$$
$$\leq\left[(\hat{f}_h(U^n),\hat{\theta}^{n+\frac{1}{2}})-(f(u^{n+\frac{1}{2}}),\theta^{n+\frac{1}{2}})\right]-\left[(D_\tau\hat{w}^n,\hat{\theta}^{n+\frac{1}{2}})-(u_t^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right]-\Big(a(U^n)\nabla w^{n+\frac{1}{2}}$$
$$-a(u^{n+\frac{1}{2}})\nabla u^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}}\Big)-\left[R(\vec{B}^n,U^n,\theta^{n+\frac{1}{2}})-(\nabla\cdot\vec{b}(u^{n+\frac{1}{2}}),\theta^{n+\frac{1}{2}})\right]=\sum_{i=1}^{4}A^{(i)}\qquad(5.1)$$

We start with the estimation of $A^{(1)}$:

$$A^{(1)}=(\hat{f}_h(U^n)-f_h(U^n),\hat{\theta}^{n+\frac{1}{2}})+(f_h(U^n)-f(U^n),\hat{\theta}^{n+\frac{1}{2}})+(f(U^n),\hat{\theta}^{n+\frac{1}{2}}-\theta^{n+\frac{1}{2}})$$
$$+(f(U^n)-f(u^{n+\frac{1}{2}}),\theta^{n+\frac{1}{2}})=\sum_{i=1}^{4}A^{(1i)}$$

From Lemma 4 and assumption (A1), and notice that $\|\cdot\|^2_{0,\Omega}$ is equivalent to $\|\hat{\cdot}\|^2_{0,\Omega}$,

$$A^{(11)}\leq\|\hat{f}_h(U^n)-f_h(U^n)\|_{0,\Omega}\|\hat{\theta}^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch\|\hat{\theta}^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch^2+C\|\theta^n\|^2_{0,\Omega}+C\|\theta^{n+1}\|^2_{0,\Omega}$$

$$A^{(12)}\leq\|f_h(U^n)-f(U^n)\|_{0,\Omega}\|\hat{\theta}^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch\|\hat{\theta}^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch^2+C\|\theta^n\|^2_{0,\Omega}+C\|\theta^{n+1}\|^2_{0,\Omega}$$

By Young's inequality,

$$A^{(13)}\leq\|f(U^n)\|_{0,\Omega}\|\hat{\theta}^{n+\frac{1}{2}}-\theta^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch|\theta^{n+\frac{1}{2}}|_{1,\Omega}\leq Ch^2+\varepsilon|\theta^{n+\frac{1}{2}}|^2_{1,\Omega}$$

From condition (A2),

$$\begin{aligned}A^{(14)}&\leq\|f(U^n)-f(u^{n+\frac{1}{2}})\|_{0,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}\\&\leq L\|U^n-u^{n+\frac{1}{2}}\|_{0,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}\\&\leq L(\|U^n-u^n\|_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega})\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}\\&\leq L(\|\rho^n\|_{0,\Omega}+\|\theta^n\|_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega})\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}\\&\leq C(h^2+\|\theta^n\|^2_{0,\Omega}+\|\theta^{n+1}\|^2_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|^2_{0,\Omega})\end{aligned}$$

Hence

$$A^{(1)}\leq C(h^2+\|\theta^n\|^2_{0,\Omega}+\|\theta^{n+1}\|^2_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|^2_{0,\Omega})+\varepsilon|\theta^{n+\frac{1}{2}}|^2_{1,\Omega}\qquad(5.2)$$

To estimate $A^{(2)}$,

$$A^{(2)}=|(D_\tau\hat{w}^n,\hat{\theta}^{n+\frac{1}{2}})-(D_\tau w^n,\theta^{n+\frac{1}{2}})|+|(D_\tau w^n-D_\tau u^n,\theta^{n+\frac{1}{2}})|$$
$$+|(D_\tau u^n-u_t^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})|=\sum_{i=1}^{3}A^{(2i)}$$

By Lemma 3 , the Cauchy inequality and conditions (A3), (A5), we get

$$\begin{aligned} A^{(21)} &\leq Ch^2\|\nabla D_\tau w^n\|_{0,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\|D_\tau \nabla u^n\|_{0,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\tau^{-1}\int_{t_n}^{t_{n+1}}\|\nabla u_t\|_{0,\Omega}dt\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\tau^{-\frac{1}{2}}\left(\int_{t_n}^{t_{n+1}}\|u_t\|_{1,\Omega}^2dt\right)^{\frac{1}{2}}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch\|u_t\|_{L^2(0,T;H^1(\Omega))}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \end{aligned}$$

$$\begin{aligned} A^{(22)} &\leq Ch^2\|D_\tau u^n\|_{2,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\|\tau^{-1}\int_{t_n}^{t_{n+1}}u_tdt\|_{2,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\tau^{-1}\int_{t_n}^{t_{n+1}}\|u_t\|_{2,\Omega}dt\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2\tau^{-\frac{1}{2}}\left(\int_{t_n}^{t_{n+1}}\|u_t\|_{2,\Omega}^2dt\right)^{\frac{1}{2}}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch\|u_t\|_{L^2(0,T;H^2(\Omega))}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq Ch^2+C\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2 \end{aligned}$$

From Lemma 8,

$$A^{(23)}\leq C\tau\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}\leq C\tau^2+C\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2$$

then

$$A^{(2)}\leq C(h^2+\tau^2+\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2)+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \tag{5.3}$$

To estimate $A^{(2)}$,

$$\begin{aligned} A^{(3)} \leq &\left|\left(a(U^n)\left(\nabla w^{n+\frac{1}{2}}-\frac{\nabla u^{n+1}+\nabla u^n}{2}\right),\nabla\theta^{n+\frac{1}{2}}\right)\right| \\ +&\left|\left(a(U^n)\left(\frac{\nabla u^{n+1}+\nabla u^n}{2}-\nabla u^{n+\frac{1}{2}}\right),\nabla\theta^{n+\frac{1}{2}}\right)\right| \\ +&\left|\left(\left(a(U^n)-a(u^{n+\frac{1}{2}})\right)\nabla u^{n+\frac{1}{2}},\nabla\theta^{n+\frac{1}{2}}\right)\right| = \sum_{i=1}^{3}A^{(3i)} \end{aligned}$$

where

$$A^{(31)}\leq\frac{1}{2}M_a\left(\|\nabla\rho^{n+1}\|_{0,\Omega}+\|\nabla\rho^n\|_{0,\Omega}\right)\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega}\leq Ch|\theta^{n+\frac{1}{2}}|_{1,\Omega}\leq Ch^2+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2$$

From conditions (A3)

$$A^{(32)}\leq M_a\tau^2\|u_{tt}\|_{L^2(0,T;H^2(\Omega))}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega}\leq C\tau^4+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2$$

From condition (A2), (A3)

$$\begin{aligned} A^{(33)} &\leq \|\nabla u^{n+1}\|_{0,\infty,\Omega}\|a(U^n)-a(u^{n+\frac{1}{2}})\|_{0,\Omega}\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\ &\leq C\|U^n-u^{n+\frac{1}{2}}\|_{0,\Omega}|\theta^{n+\frac{1}{2}}|_{1,\Omega} \\ &\leq C(\|U^n-u^n\|_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega})|\theta^{n+\frac{1}{2}}|_{1,\Omega} \\ &\leq C(\|\rho^n\|_{0,\Omega}+\|\theta^n\|_{0,\Omega}+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega})|\theta^{n+\frac{1}{2}}|_{1,\Omega} \\ &\leq C(h^4+\|\theta^n\|_{0,\Omega}^2+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega}^2)+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \end{aligned}$$

then

$$A^{(3)}\leq C(h^2+\tau^4+\|\theta^n\|_{0,\Omega}^2+\|u^n-u^{n+\frac{1}{2}}\|_{0,\Omega}^2)+3\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \tag{5.4}$$

To estimate $A^{(4)}$,

$$A^{(4)} \le |(\nabla\cdot\vec{b}(u^{n+\frac12}) - \nabla\cdot\vec{b}(U^n), \theta^{n+\frac12})| + |(\nabla\cdot\vec{b}(U^n), \theta^{n+\frac12} - \hat{\theta}^{n+\frac12})|$$

$$+|(\nabla\cdot\vec{b}(U^n), \hat{\theta}^{n+\frac12}) - R(\vec{B}^n, U^n, \theta^{n+\frac12})| = \sum_{i=1}^{3} A^{(4i)}$$

By Green theorem and boundary condition, we get

$$A^{(41)} \le \left|\left(\vec{b}(u^{n+\frac12}) - \vec{b}(U^n), \nabla\theta^{n+\frac12}\right)\right| \le C(h^2 + \|\theta^{n+\frac12}\|^2_{0,\Omega} + \|u^n - u^{n+\frac12}\|^2_{0,\Omega}) + \varepsilon|\theta^{n+\frac12}|^2_{1,\Omega}$$

By (A1), Lemma 4 and Lemma 7

$$A^{(42)} \le Ch^2 + \varepsilon|\theta^{n+\frac12}|^2_{1,\Omega}$$

Next we consider $A^{(43)}$. Notice that

$$\begin{aligned}
(\nabla\cdot\vec{b}(U^n), \hat{\theta}^{n+\frac12}) &= \sum_{i=1}^{N} \int_{\Omega_i} \hat{\theta}^{n+\frac12}\nabla\cdot\vec{b}(U^n)\mathrm{d}x \\
&= \sum_{i=1}^{N} \theta_i^{n+\frac12} \int_{\Omega_i} \nabla\cdot\vec{b}(U^n)\mathrm{d}x \\
&= \sum_{i=1}^{N} \theta_i^{n+\frac12} \int_{\Omega_i} \nabla\cdot[\vec{b}(U^n) - \vec{b}(U_i^n)]\mathrm{d}x \\
&= \sum_{i=1}^{N} \theta_i^{n+\frac12} \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} \vec{n}_{ij}\cdot[\vec{b}(U^n) - \vec{b}(U_i^n)]\mathrm{d}s \\
&= \sum_{i=1}^{N} \theta_i^{n+\frac12} \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} \vec{n}_{ij}\cdot\left[\vec{B}_{ij}^n(U_j^n - U_i^n) + \vec{b}(U^n) - \vec{b}(U_j^n)\right]\mathrm{d}s
\end{aligned}$$

Since $\nabla U|_e = const$ for each $e \in T_h$, we easily find that

$$|U(x) - U_j| \le h_e\,|\nabla U|_e| \le C\left(\int_e |\nabla U|_e|^2\,\mathrm{d}x\right)^{\frac12} = C\|\nabla U\|_{0,e}, \qquad \text{if } p_i, p_j \in e, x \in \Gamma_{ij}^e$$

$$|U_i - U_j| \le h_e\,|\nabla U|_e| \le C\left(\int_e |\nabla U|_e|^2\,\mathrm{d}x\right)^{\frac12} = C\|\nabla U\|_{0,e}, \qquad \text{if } p_i, p_j \in e$$

It follows from [1] and (A2),

$$\begin{aligned}
A^{(43)} &= |(\nabla\cdot\vec{b}(U^n), \hat{\theta}^{n+\frac12}) - R(\vec{B}^n, U^n, \theta^{n+\frac12})| \\
&= |\sum_{i=1}^{N} \theta_i^{n+\frac12} \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} \vec{n}_{ij}\cdot\left[\vec{B}_{ij}^n\sigma_{ij}^n(U_j^n - U_i^n) + \vec{b}(U^n) - \vec{b}(U_j^n)\right]\mathrm{d}s| \\
&= \left|\sum_{e\in T_h}\sum_{p_i,p_j\in e, i<j} (\theta_i^{n+\frac12} - \theta_j^{n+\frac12}) \int_{\Gamma_{ij}^e} \vec{n}_{ij}\cdot\left[\vec{B}_{ij}^n\sigma_{ij}^n(U_j^n - U_i^n) + \vec{b}(U^n) - \vec{b}(U_j^n)\right]\mathrm{d}s\right| \\
&\le \sum_{e\in T_h}\sum_{p_i,p_j\in e, i<j} |\theta_i^{n+\frac12} - \theta_j^{n+\frac12}||\Gamma_{ij}^e|\left(|\vec{B}_{ij}^n||\sigma_{ij}^n||U_j^n - U_i^n| + L|U^n - U_j^n|\right) \\
&\le \sum_{e\in T_h} 3C\left\|\nabla\theta^{n+\frac12}\right\|_{0,e} h_e 2CL\,\|\nabla U^n\|_{0,e}
\end{aligned}$$

Furthermore, by the Cauchy inequality and Lemma 7, we have

$$A^{(43)} \le Ch\|\nabla\theta^{n+\frac12}\|_{0,\Omega}\|\nabla U^n\|_{0,\Omega} \le Ch|\theta^{n+\frac12}|_{1,\Omega} \le Ch^2 + \varepsilon|\theta^{n+\frac12}|^2_{1,\Omega}$$

Hence

$$A^{(4)} \le C\left(h^2 + \|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2 + \|u^n - u^{n+\frac{1}{2}}\|_{0,\Omega}^2\right) + 3\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \tag{5.5}$$

By substituting of equations (5.2), (5.3), (5.4) and (5.5) in equation (5.1), taking ε small enough such that $\varepsilon < \frac{m}{8}$, we get

$$\frac{1}{2}D_\tau\|\theta^n\|_{0,\Omega}^2 \le C\left(h^2 + \tau^2 + \|\theta^n\|_{0,\Omega}^2 + \|\theta^{n+1}\|_{0,\Omega}^2 + \|u^n - u^{n+\frac{1}{2}}\|_{0,\Omega}^2\right)$$

Notice that

$$\|u^n - u^{n+\frac{1}{2}}\|_{0,\Omega} \le C\tau\|u_t\|_{0,\Omega}.$$

then

$$\|\theta^{n+1}\|_{0,\Omega}^2 \le \frac{1+2C\tau}{1-2C\tau}\|\theta^n\|_{0,\Omega}^2 + \frac{2C\tau}{1-2C\tau}(h^2+\tau^2)$$

This implies that if τ is small enough such that $1 - 2C\tau > 0$, then

$$\|\theta^{n+1}\|_{0,\Omega}^2 \le (1+C\tau)\|\theta^n\|_{0,\Omega}^2 + C\tau(h^2+\tau^2)$$

By induction over $n = 0, 1, \cdots, N_\tau - 1$, this easily deduce that

$$\|\theta^{n+1}\|_{0,\Omega}^2 \le (1+C\tau)^{n+1}\|\theta^0\|_{0,\Omega}^2 + \left[(1+C\tau)^{n+1} - 1\right](h^2+\tau^2)$$

Since $(1+C\tau)^{n+1} \le (1+C\tau)^{N_\tau} \le e^{CT}$, it follows that

$$\|\theta^{n+1}\|_{0,\Omega}^2 \le e^{CT}\|\theta^0\|_{0,\Omega}^2 + \left[e^{CT} - 1\right](h^2+\tau^2)$$

That is

$$\|\theta^{n+1}\|_{0,\Omega} \le C(h+\tau).$$

and also

$$\|\rho^{n+1}\|_{0,\Omega} \le Ch^2.$$

Hence the theorem is complete.

6. A Higher Order Accuracy Scheme

In this section, we will introduce a higher order of τ scheme (6.2). To simplify the theoretical analysis, we consider the predictor-corrector scheme at first.

For convenience, we denote $\tilde{U}^{n,k} = \frac{U^{n+1,k}+U^n}{2}$, then the predictor-corrector scheme is: for $n = 0, 1, \cdots, N_\tau - 1$, find $U^{n+1,0}, U^{n+1,1}, U^{n+1} \in V_{0h}$ such that

$$\left(\frac{\hat{U}^{n+1,0} - \hat{U}^n}{\tau}, \hat{v}\right) + \left(a(U^n)\nabla\left(\tilde{U}^{n,0}\right), \nabla v\right) + R\left(\vec{B}^n, U^n, v\right) = \left(\hat{f}_h(U^n), \hat{v}\right), \quad \forall v \in V_{0h} \tag{6.1a}$$

$$\left(\frac{\hat{U}^{n+1,1} - \hat{U}^n}{\tau}, \hat{v}\right) + \left(a\left(\tilde{U}^{n,0}\right)\nabla\left(\tilde{U}^{n,1}\right), \nabla v\right) + R\left(\vec{B}^{n,0}, \tilde{U}^{n,0}, v\right) = \left(\hat{f}_h\left(\tilde{U}^{n,0}\right), \hat{v}\right), \quad \forall v \in V_{0h} \tag{6.1b}$$

$$\left(\frac{\hat{U}^{n+1} - \hat{U}^n}{\tau}, \hat{v}\right) + \left(a\left(\tilde{U}^{n,1}\right)\nabla\left(\frac{U^{n+1}+U^n}{2}\right), \nabla v\right) + R\left(\vec{B}^{n,1}, \tilde{U}^{n,1}, v\right)$$
$$= \left(\hat{f}_h\left(\tilde{U}^{n,1}\right), \hat{v}\right), \quad \forall v \in V_{0h} \tag{6.1c}$$

$$U^0 = I_h u^0 \tag{6.1d}$$

where

$$R\left(\vec{B}^{n,k}, \tilde{U}^{n,k}, v\right) = \sum_{i=1}^{N} v(p_i) \sum_{j\in\Lambda_i} (\sigma_{ij}^n \tilde{U}_i^{n,k} + \sigma_{ji}^n \tilde{U}_j^{n,k} - \tilde{U}_i^{n,k})\beta_{ij}^{n,k}, \quad k = 0, 1. \qquad f_h = I_h f$$

$$\tilde{U}_i^{n,k} = \frac{U^{n+1,k}(p_i) + U^n(p_i)}{2}, \beta_{ij}^{n,k} = \int_{\Gamma_{ij}} \vec{B}_{ij}^{n,k} \cdot \vec{n}_{ij} \mathrm{d}s, \vec{B}_{ij}^{n,k} = \frac{\vec{b}(\tilde{U}_j^{n,k}) - \vec{b}(\tilde{U}_i^{n,k})}{\tilde{U}_j^{n,k} - \tilde{U}_i^{n,k}}.$$

In Numerical Computing we define $\vec{B}_{ij}^{n,k} = \frac{\partial \vec{b}}{\partial u}(\tilde{U}_i^{n,k})$if $\tilde{U}_j^{n,k} = \tilde{U}_i^{n,k}$.

Theorem 3. *Suppose the conditions (A1), (A2) and (A4) be satisfied and time step τ satisfies the condition (4.4), then the solution U of problem* (6.1) *is bounded and is estimated by*

$$\|U\|_{\tilde{L}^\infty(0,T;L^\infty(\Omega))} \le e^{C_1 T}\left(\|u^0\|_{0,\infty,\Omega} + C_2 T\right).$$

where C_1 and C_2 is defined in assumption (A1).

Proof. It is clear that $\|U^0\|_{L^\infty(\Omega)} \le K_0$. We suppose U^m $(0 \le m \le n)$ is bounded. From (6.1a), as the proof of Theorem 1, we get $\|U^{n+1,0}\|_{L^\infty(\Omega)} \le K_0$, then $\tilde{U}^{n,0}$ is bounded by K_0,

$$\|\tilde{U}^{n,0}\|_{L^\infty(\Omega)} = \left\| \frac{U^{n+1,0} + U^n}{2} \right\|_{L^\infty(\Omega)} \le K_0$$

Also, from (6.1b) we have $\tilde{U}^{n,1}$ is bounded by K_0.

Finally, as the analysis of Theorem 1, we get U^{n+1} is bounded.

To get the error estimation, we need the following assumption:

(A6) $u_{ttt} \in L^\infty(0, T; H^2(\Omega))$.

Theorem 4. *Let u be the exact solution of problem (P) which is sufficient smooth, conditions(A1), (A2), (A3), (A4), (A5), (A6) hold, and τ be the time step which is sufficient small, U be the solution of problem* (6.1)*, then we have*

$$\|U - u\|_{\tilde{L}^\infty(0,T;L^2(\Omega))} \le C(h + \tau^2)$$

where C is a constant independent of h and τ.

Proof. We proof the theorem by mathematics induction. It is easy to proof $\|U^n - u^n\|_{0,\Omega} \le Ch$ if $n = 0$. We assume that $\|U^m - u^m\|_{0,\Omega} \le C(h + \tau^2)$, $0 \le m \le n$ holds. Then we go on to proof that $\|U^{n+1} - u^{n+1}\|_{0,\Omega} \le C(h + \tau^2)$ is still true. First, we discuss the estimation of (6.1a). (6.1a) is equivalent to (P_h). From the proof of Theorem 2, we get

$$\|\theta^{n+1,0}\|_{0,\Omega}^2 \le \frac{1 + 2C\tau}{1 - 2C\tau}\|\theta^n\|_{0,\Omega}^2 + \frac{2C\tau}{1 - 2C\tau}(h^2 + \tau^2)$$

For (6.1b), Notice that in $a(u)$, $b(u)$ and $f(u)$, we use $\tilde{U}^{n,k} = \frac{U^{n+1,k} + U^n}{2}$ in (6.1b) instead of U^n in (P_h), which is different in (6.1b) and (P_h). Then we need to estimate $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ and $A^{(4)}$ again. To estimate $A^{(14)}$,

$$\begin{aligned}
A^{(14)} &\le \|f(\tilde{U}^{n,k}) - f(u^{n+\frac{1}{2}})\|_{0,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\
&\le L\|\tilde{U}^{n,k} - u^{n+\frac{1}{2}}\|_{0,\Omega}\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\
&\le \tfrac{L}{2}(\|U^{n+1,0} - u^{n+1}\|_{0,\Omega} + \|U^n - u^n\|_{0,\Omega} + 2\|\tfrac{u^{n+1}+u^n}{2} - u^{n+\frac{1}{2}}\|_{0,\Omega})\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \\
&\le \tfrac{L}{2}(\|\theta^{n+1,0}\|_{0,\Omega} + \|\rho^{n+1,0}\|_{0,\Omega} + \|\theta^n\|_{0,\Omega} + \|\rho^n\|_{0,\Omega} + C\tau^2)\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2 \\
&\le C(h^2 + \tau^4 + \|\theta^n\|_{0,\Omega}^2 + \|\theta^{n+1}\|_{0,\Omega}^2 + \|\theta^{n+1,0}\|_{0,\Omega}^2)
\end{aligned}$$

The estimations of $A^{(11)}$, $A^{(12)}$ and $A^{(13)}$ are similar to that in Theorem 2, then

$$A^{(1)} \leq C(h^2+\tau^4+\|\theta^n\|_{0,\Omega}^2+\|\theta^{n+1}\|_{0,\Omega}^2+\|\theta^{n+1,0}\|_{0,\Omega}^2)+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2$$

As the estimation of $A^{(14)}$, we can get the estimations of $A^{(3)}$ and $A^{(4)}$,

$$A^{(3)} \leq C(h^2+\tau^4+\|\theta^n\|_{0,\Omega}^2+\|\theta^{n+1,0}\|_{0,\Omega}^2)+3\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2$$

$$A^{(4)} \leq C\left(h^2+\tau^4+\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\theta^{n+1,0}\|_{0,\Omega}^2\right)+3\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2.$$

To estimate $A^{(2)}$,from condition (A6), we get

$$A^{(23)} \leq C\tau^2\|\theta^{n+\frac{1}{2}}\|_{0,\Omega} \leq C\tau^4+C\|\theta^{n+\frac{1}{2}}\|_{0,\Omega}^2$$

The estimations of $A^{(21)}$ and $A^{(22)}$ are similar to that in Theorem 2, then

$$A^{(2)} \leq C(h^2+\tau^4+\|\hat{\theta}^{n+\frac{1}{2}}\|_{0,\Omega}^2)+\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2$$

Finally, we get

$$\|\theta^{n+1,1}\|_{0,\Omega}^2 \leq \frac{1+2C\tau}{1-2C\tau}\|\theta^n\|_{0,\Omega}^2+\frac{2C\tau}{1-2C\tau}(h^2+\tau^3)$$

As the estimation of $\|\theta^{n+1,1}\|_{0,\Omega}$, we get

$$\|\theta^{n+1}\|_{0,\Omega}^2 \leq \frac{1+2C\tau}{1-2C\tau}\|\theta^n\|_{0,\Omega}^2+\frac{2C\tau}{1-2C\tau}(h^2+\tau^4)$$

As the proof of Theorem 2, we get

$$\|\theta^{n+1}\|_{0,\Omega} \leq C(h+\tau^2).$$

and

$$\|\rho^{n+1}\|_{0,\Omega} \leq Ch^2.$$

Hence the theorem is complete.

Then we give a three time level scheme : for $n=1,2,\cdots,N_\tau-1$, find $U^{n+1}\in V_{0h}$ such that

$$\left(\frac{\hat{U}^{n+1}-\hat{U}^n}{\tau},\hat{v}\right)+\left(a\left(\tilde{U}\right)\nabla\left(\frac{U^{n+1}+U^n}{2}\right),\nabla v\right)+R\left(\tilde{\vec{B}}^n,\tilde{U},v\right)=\left(\hat{f}_h\left(\tilde{U}\right),\hat{v}\right),\forall v\in V_{0h} \tag{6.2}$$

Where U^0, U^1 is given by (6.1),

$$\tilde{U}=\frac{3U^n-U^{n-1}}{2},\qquad R\left(\tilde{\vec{B}}^n,\tilde{U},v\right)=\sum_{i=1}^N v(p_i)\sum_{j\in\Lambda_i}(\sigma_{ij}^n\tilde{U}_i^n+\sigma_{ji}^n\tilde{U}_j^n-\tilde{U}_i^n)\tilde{\beta}_{ij}^n,\qquad f_h(\tilde{U})=I_hf(\tilde{U})$$

$$\tilde{U}_i^n=\frac{3U^n(p_i)-U^{n-1}(p_i)}{2},\tilde{\beta}_{ij}^n=\int_{\Gamma_{ij}}\tilde{\vec{B}}_{ij}^n\cdot\vec{n}_{ij}\mathrm{d}s,\tilde{\vec{B}}_{ij}^n=\frac{\vec{b}(\tilde{U}_j^n)-\vec{b}(\tilde{U}_j^n)}{\tilde{U}_j^n-\tilde{U}_i^n}.$$

In Numerical Computing we define $\tilde{\vec{B}}_{ij}^n=\frac{\partial\vec{b}}{\partial u}(\tilde{U}_i^n)$ if $\tilde{U}_j^n=\tilde{U}_i^n$.

To get the discrete maximum principle and the error estimate of this scheme, we need define the set on $\mathbf{R}$:

$$G'=\{u:|u|\leq 2K_0'\}$$

and modify conditions (A1) and (A2):

(A1$'$) There exist constants m', $\tilde{M}_a$, C_1', C_2' which depend on K_0' such that

$$0 < m' < a(u) < \tilde{M}_a, \quad |f(u)| \le C_1'|u| + C_2', \quad \forall (x,t) \in \Omega \times (0,T], u \in G'.$$

$\vec{b}(u) = \left(b^{(1)}(u), b^{(2)}(u)\right) \in W^1_\infty(G') \times W^1_\infty(G')$, $f(u) \in W^1_\infty(G' \times \Omega \times (0,T])$,$u^0(x) \in C(\bar{\Omega})$.

(A2$'$)$a(u)$,$\vec{b}(u)$,$f(u)$ are locally Lipschitz continuous

$$|a(u) - a(v)| \le \tilde{L}|u - v|.\|\vec{b}(u) - \vec{b}(v)\| \le \tilde{L}|u - v|.|f(u) - f(v)| \le \tilde{L}|u - v|.\forall u, v \in G'.$$

where $\tilde{L}$ is constant related to K_0'.

Then we have following theorems:

Theorem 5. *Suppose the conditions (A1$'$), (A2$'$) and (A4) be satisfied and time step τ satisfies the condition*

$$0 < \tau \le \frac{\kappa^2}{2\tilde{M}_a + 4\tilde{L}\kappa}$$

then the solution U of problem (6.2) is bounded and it is estimated by

$$\|U\|_{\tilde{L}^\infty(0,T;L^\infty(\Omega))} \le e^{C_1'T}\left(\|u^0\|_{0,\infty,\Omega} + C_2'T\right).$$

where C_1' and C_2' is defined in assumption (A1$'$).

Then, K_0' is fixed by

$$K_0' = \max\left\{e^{C_1'T}\left(\|u^0\|_{0,\infty,\Omega} + C_2'T\right); \|u\|_{L^\infty(0,T,L^\infty(\Omega))}\right\}$$

Theorem 6. *Let u be the exact solution of problem (P) which is sufficient smooth, conditions (A1$'$), (A2$'$), (A3), (A4), (A5), (A6) hold, and τ be the time step which is sufficient small, U be the solution of problem* (6.2), *then there exits a constant C independent of h and τ, such that*

$$\|U - u\|_{\tilde{L}^\infty(0,T;L^2(\Omega))} \le C(h + \tau^2)$$

The proof of discrete maximum principle and the error estimate of this scheme are similar to that of predictor-corrector scheme. It is clear that the theoretical analysis of this scheme is more complicated than that of predictor-corrector scheme. But it is more practicably than predictor-corrector scheme because it is one step scheme.

7. Another Partial Upwind Finite Element Scheme

In the above schemes, the convection term is explicit. In this section we will give a scheme which the convection term is implicit.

Let $\tau > 0$ is time step and $N_\tau = T/\tau$, then the partial upwind finite element scheme of problem (P) is ($P_h 1$): for $n = 0, 1, \cdots, N_\tau - 1$, find $U^{n+1} \in V_{0h}$ such that

$$\begin{cases} (D_\tau \hat{U}^n, \hat{v}) + (a(U^n)\nabla U^{n+\frac{1}{2}}, \nabla v) + R(\vec{B}^n, U^{n+\frac{1}{2}}, v) = (\hat{f}_h(U^n), \hat{v}), \forall v \in V_{0h} \\ U^0 = I_h u^0 \end{cases}$$

where

$$D_\tau \hat{U}^n = (\hat{U}^{n+1} - \hat{U}^n)/\tau, \qquad U^{n+\frac{1}{2}} = (U^{n+1} + U^n)/2, \qquad f_h(U^n) = I_h f(U^n)$$

$$R(\vec{B}^n, U^{n+\frac{1}{2}}, v) = \sum_{i=1}^{N} v(p_i) \sum_{j \in \Lambda_i} (\sigma_{ij}^n U_i^{n+\frac{1}{2}} + \sigma_{ji}^n U_j^{n+\frac{1}{2}} - U_i^{n+\frac{1}{2}})\beta_{ij}^n.$$

$$U_i^n = U^n(p_i), \quad \beta_{ij}^n = \int_{\Gamma_{ij}} \vec{B}_{ij}^n \cdot \vec{n}_{ij} \mathrm{d}s, \quad \vec{B}_{ij}^n = \frac{\vec{b}(U_j^n) - \vec{b}(U_i^n)}{U_j^n - U_i^n}.$$

In numerical computation, we define $\vec{B}_{ij}^n = \frac{\partial \vec{b}}{\partial u}(U_i^n)$ if $U_j^n = U_i^n$. And here ρ_{ij}^n is mesh Péclet number defined as:

$$\rho_{ij}^n = \frac{\beta_{ij}^n}{|a_{ij}^n|}, \ \ a_{ij}^n = (a(U^n)\nabla\varphi_j, \nabla\varphi_i)\,(i = 1, 2, \cdots, K; j \in \Lambda_i).$$

Then, we can get the following discrete maximum principle:

Theorem 7. *Suppose the conditions (A1),(A2) and (A4) be satisfied and time step τ satisfies the condition:*

$$0 < \tau \le \frac{\kappa^2}{2M_a + 4L\kappa}, \tag{7.1}$$

then the solution U of problem $(P_h 1)$ is bounded and is estimated by

$$\|U\|_{\tilde{L}^\infty(0,T;L^\infty(\Omega))} \le e^{C_1 T}\left(\|u^0\|_{0,\infty,\Omega} + C_2 T\right). \tag{7.2}$$

where C_1 and C_2 is defined in assumption (A1). Then, K_0 is fixed by

$$K_0 = \max\left\{e^{C_1 T}\left(\|u^0\|_{0,\infty,\Omega} + C_2 T\right); \|u\|_{L^\infty(0,T,L^\infty(\Omega))}\right\}$$

Proof. We only need check four conditions in Lemma 5.

(i) $\sum\limits_{j=1}^{K}\left(m_{ij} + \frac{\tau}{2}(a_{ij}^n + b_{ij}^n)\right) = \sum\limits_{j=1}^{K}\left[m_{ij} - \frac{\tau}{2}(a_{ij}^n + b_{ij}^n)\right] = \sum\limits_{j=1}^{K} m_{ij} = m_{ii} > 0$

(ii)If $\beta_{ij}^n \ge 0$, because

$$\sigma_{ji}^n = 1 - \sigma_{ij}^n \le 1/\rho_{ij}^n = |a_{ij}^n|/\beta_{ij}^n,$$
$$a_{ij}^n + b_{ij}^n = a_{ij}^n + \sigma_{ji}^n\beta_{ij}^n \le 0 \ \ (j \ne i).$$

then

$$m_{ij} - \frac{\tau}{2}(a_{ij}^n + b_{ij}^n) \ge 0.(1 \le i, j \le N, j \ne i)$$

We can get the same result if $\beta_{ij}^n \le 0$.

From lemma 1, lemma 2 and the assumption of Theorem, we can get

$$\begin{aligned}
& m_{ii} - \tfrac{\tau}{2}(a_{ii}^n + b_{ii}^n) \\
= \ & |\Omega_i| - \tfrac{\tau}{2}\left[(a(U^n)\nabla\varphi_i, \nabla\varphi_i) + \sum_{j\in\Lambda_i}(\sigma_{ij}^n - 1)\beta_{ij}^n\right] \\
= \ & |\Omega_i| - \tfrac{\tau}{2}(a(U^n)\nabla\varphi_i, \nabla\varphi_i) - \tfrac{\tau}{2}\sum_{j\in\Lambda_i}(\sigma_{ij}^n - 1)\int_{\Gamma_{ij}} \vec{B}_{ij}^n \cdot \vec{n}_{ij}\mathrm{d}s \\
\ge \ & |\Omega_i| - \tfrac{M_a\tau}{2}\sum_{j\in\Lambda_i}\tfrac{|\Gamma_{ij}|}{|p_i - p_j|} - \tfrac{L\tau}{2}\sum_{j\in\Lambda_i}|\Gamma_{ij}| \\
\ge \ & \sum_{e\in T_h(i)}\left\{|\Omega_i^e| - \tfrac{M_a\tau}{2}4\tfrac{|\Omega_i^e|}{\kappa_e^2} - 2L\tau\tfrac{|\Omega_i^e|}{\kappa_e}\right\} \\
= \ & \sum_{e\in T_h(i)}\left[\left(1 - \tfrac{2M_a\tau}{\kappa_e^2} - \tfrac{2L\tau}{\kappa_e}\right)|\Omega_i^e|\right]
\end{aligned}$$

Hence if τ satisfy the condition (7.1), then

$$m_{ii} - \frac{\tau}{2}(a_{ii}^n + 2b_{ii}^n) \ge 0$$

(iii) $m_{ij} \ge 0$

(iv) if $i \neq j$, $m_{ij} + \frac{\tau}{2}(a_{ij}^n + b_{ij}^n) = \frac{\tau}{2}(a_{ij}^n + b_{ij}^n) \leq 0$

The remains proof is the same as that of Theorem 1.

Theorem 8. *Let u be the exact solution of problem (P) which is sufficient smooth, conditions (A1), (A2), (A3), (A4), (A5) hold, and h be the space step and τ be the time step which are sufficient small, U be the solution of problem (P_h1), then we have*

$$\|U - u\|_{\tilde{L}^\infty(0,T;L^2(\Omega))} \leq C(h+\tau)$$

where C is a constant independent of h and τ.

Proof. The theorem will be proved by the same process as Theorem 2 with one minor change. Only the estimation $(A4)$ has to be insert one term : If h is small enough, we get

$$\begin{aligned}
& |R(\vec{B}^n, U^n, \theta^{n+\frac{1}{2}}) - R(\vec{B}^n, U^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}})| \\
= & \ |\sum_{i=1}^{N} \theta_i^{n+\frac{1}{2}} \sum_{j\in\Lambda_i} \sigma_{ji}^n \left[(U_j^n - U_i^n) - (U_j^{n+\frac{1}{2}} - U_i^{n+\frac{1}{2}})\right] \int_{\Gamma_{ij}} \vec{B}_{ij}^n \cdot \vec{n}_{ij} \mathrm{d}s| \\
\leq & \sum_{e\in T_h} \sum_{p_i,p_j\in e, i<j} |\theta_i^{n+\frac{1}{2}} - \theta_j^{n+\frac{1}{2}}||\sigma_{ji}^n| \left(|U_j^n - U_i^n| + |U_j^{n+\frac{1}{2}} - U_i^{n+\frac{1}{2}}|\right) |\Gamma_{ij}^e||\vec{B}_{ij}^n| \\
\leq & \ Ch\|\nabla\theta^{n+\frac{1}{2}}\|_{0,\Omega}\left(\|\nabla U^n\|_{0,\Omega} + \|\nabla U^{n+\frac{1}{2}}\|_{0,\Omega}\right) \\
\leq & \ Ch^2 + \varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 + Ch|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2 \\
\leq & \ Ch^2 + 2\varepsilon|\theta^{n+\frac{1}{2}}|_{1,\Omega}^2
\end{aligned}$$

8. Numerical Tests

Example 1. *First, we test a problem on the unit square $\Omega = (0,2)^2$, with $a(u) = \varepsilon = 0.01$, $b(u) = (0.4u^2, 0.4u^2)$, and choose the right-hand side f in such way that*

$$u(x,y,t) = \frac{1}{4t+1} e^{-[(x-0.8t-0.5)^2+(y-0.8t-0.5)^2]/[\varepsilon(4t+1)]} \tag{8.1}$$

is the exact solution. The initial condition and the boundary values are obtained directly from (8.1). The solution at t=0.0025, t=0.5 and t=1.25 are illustrated in Figure 1.

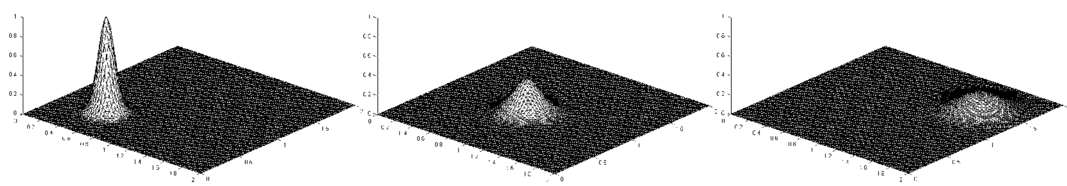

Figure 1. The solution (8.1) at t=0.0025, t=0.5 and t=1.25

We choose space step $h = 0.025$ as [12], time step $\tau = 0.0025$ which satisfied the condition (4.4) and (7.1). As [2], we define norm:

$$\|e\|_\varepsilon := \sqrt{\varepsilon|I_h u - U|_{1,\Omega}^2 + \|I_h u - U\|_{0,\Omega}}$$

When $T = 0.0025$ tests are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|e\|_\varepsilon$
P_h	0.016725	0.000165	0.500000	0.003102
Scheme (6.2)	0.013289	0.000170	0.500000	0.002859
GFEM	0.013772	0.000179	0.500000	0.002993
P_h1	0.023869	0.000230	0.500000	0.004341

where GFEM denotes the standard Galerkin Finite Element Method.

Test final results when $T = 1.25$ are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|\|e\|\|_\varepsilon$
(P_h)	0.001078	0.000114	0.166667	0.000538
Scheme (6.2)	0.001923	0.000151	0.166667	0.000710
GFEM	0.001349	0.000129	0.166667	0.000589
(P_h1)	0.002831	0.000209	0.166667	0.001057

This tabular says, our schemes can compute general example as well as the standard Galerkin finite element method.

Example 2. $\Omega = (0,1)^2$, *with* $a(u) = \varepsilon = 0.01$, $b(u) = (0.5u^2, 0.5u^2)$, *and choose the right-hand side* f *in such way that*

$$u(x,y,t) = 2 - e^{-(x+y)*(t+0.01)/\varepsilon} \tag{8.2}$$

is the exact solution. The initial condition and the boundary values are obtained directly from (8.2). The solution at t=0.01, t=0.1 and t=1.0 are illustrated in Figure 2.

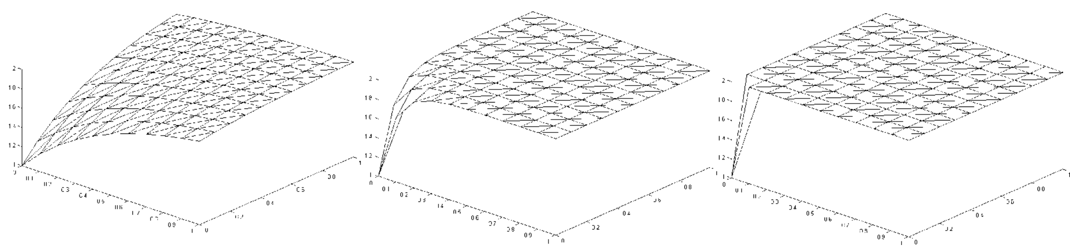

Figure 2. The solution (8.2) at t=0.01, t=0.1 and t=1.0

We choose space step $h = 0.1$, time step $\tau = 0.01$ which satisfied the condition (4.4) and (7.1). At this time, the standard Galerkin finite element method got oscillation.

Test results when $T = 1.0$ are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|\|e\|\|_\varepsilon$
(P_h)	0.000873	0.000092	2.000000	0.000228
Scheme (6.2)	0.000873	0.000092	2.000000	0.000228
GFEM	0.587784	0.062196	2.000000	0.175511
(P_h1)	0.001071	0.000109	2.000000	0.000278

Example 3. $\Omega = (0,1)^2$, $a(u) = \varepsilon = 0.01$, $b(u) = (0.5u^2, 0.5u^2)$, *and choose the right-hand side* f *in such way that*

$$u(x,y,t) = xy\left(1 - e^{(x-1)/\varepsilon - t}\right)\left(1 - e^{(y-1)/\varepsilon - t}\right) \tag{8.3}$$

is the exact solution. The initial condition and the boundary values are obtained directly from (8.3). The solution at t=0.01, t=0.5 and t=1.0 are illustrated in Figure 3.

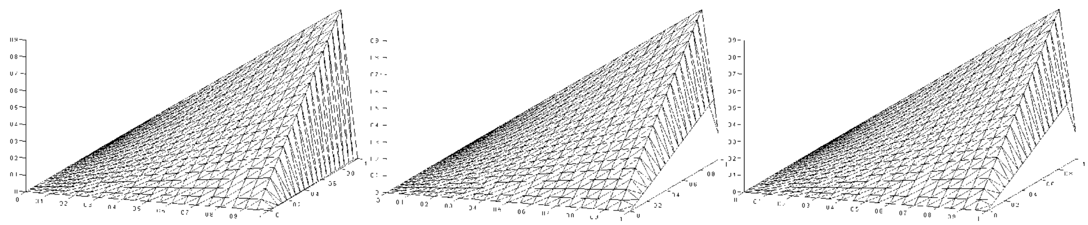

Figure 3. The solution (8.3) at t=0.01, t=0.5 and t=1.0

We choose space step $h = 0.05$, time step $\tau = 0.01$ which satisfied the condition (4.4) and (7.1). At this time, the standard Galerkin finite element method gets oscillation.

Test results when $T = 1$ are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|e\|_\varepsilon$
(P_h)	0.021100	0.002462	0.898031	0.008910
Scheme (6.2)	0.021097	0.002461	0.898031	0.008904
GFEM	0.221252	0.005373	0.898031	0.050288
(P_h1)	0.021220	0.002466	0.898031	0.008987

We test by $a(u) = \varepsilon = 0.001, 0.0001, 0.000001$. We find when ε is lower, the error of the partial upwind schemes are lower, and the standard Galerkin finite element method gets more oscillation. The $\|e\|_\varepsilon$ of these tests results when $T = 1$ are presented in tabular form:

scheme	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.0001$	$\varepsilon = 0.000001$
(P_h)	0.008910	0.007597	0.007851	0.007826
Scheme (6.2)	0.008904	0.007583	0.007836	0.007811
GFEM	0.050288	0.172671	0.231119	0.235908
(P_h1)	0.008987	0.007585	0.007845	0.007820

Test results when $T = 1$, $\varepsilon = 0.000001$ using (P_h1) are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|e\|_\varepsilon$
h=0.1	0.027163	0.010205	0.810000	0.014223
h=0.05	0.014120	0.006282	0.902500	0.007845
h=0.025	0.007185	0.003488	0.950625	0.004099

Example 4. $\Omega = (0,1)^2$, $a(u) = \varepsilon = 0.000001$, $b(u) = (0.5u^2, 0.5u^2)$, *and choose the right-hand side f in such way that*

$$u(x,y,t) = arctg\frac{x+y-t-0.5}{0.1} \tag{8.4}$$

is the exact solution. The initial condition and the boundary values are obtained directly from (8.4). The solution at t=0.01, t=0.5 and t=1.0 are illustrated in Figure 4.

We choose space step $h = 0.025$, time step $\tau = 0.01$ which satisfied the condition (4.4) and (7.1). At this time, the standard Galerkin finite element method got oscillation.

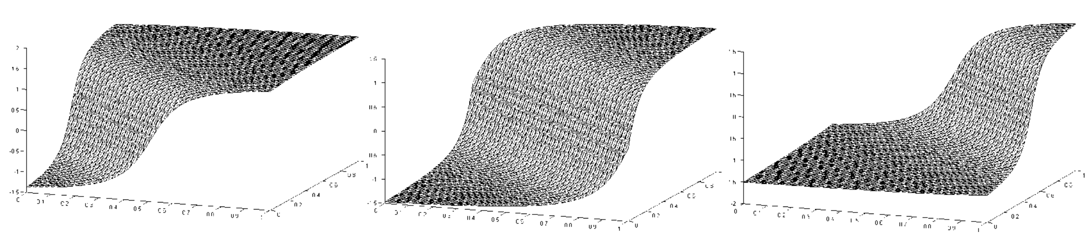

Figure 4. The solution (8.4) at t=0.01, t=0.5 and t=1.0

Test results when $T=1$ are presented in tabular form:

scheme	Maximum |error|	Average |error|	Maximum |u|	$\|e\|_\varepsilon$
(P_h)	0.082885	0.014508	1.373401	0.020481
Scheme (6.2)	0.051696	0.031305	1.373401	0.035792
GFEM	NAN	NAN	1.373401	NAN
(P_h1)	0.095585	0.021565	1.373401	0.030143

where NAN denotes that the compute is overflow.

From Example 2-4, we can see that when the solution has a abrupt slope, our schemes is more accurate than the standard Galerkin finite element method.

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