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THE STRUCTURAL CHARACTERIZATION AND LOCALLY SUPPORTED BASES FOR BIVARIATE SUPER SPLINES *1)

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Abstract

Super splines are bivariate splines defined on triangulations, where the smoothness enforced at the vertices is larger than the smoothness enforced across the edges. In this paper, the smoothness conditions and conformality conditions for super splines are presented. Three locally supported super splines on type-1 triangulation are presented. Moreover, the criteria to select local bases is also given. By using local supported super spline function, a variation-diminishing operator is built. The approximation properties of the operator are also presented.

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Key words: Spline, Local Bases, Super Spline.

1. Introduction

Let D be a polygonal domain in R^2 and Δ a triangulation of D consisting of finitely straight lines or line segments defined by $\Gamma_i : y - a_i x - b_i = 0, i = 1, \cdots, N$. Denote by $v_i, i = 1, \cdots, V_I$ all the vertices of Δ. Denote by $D_i, i = 1, \cdots, T$, all the cells of Δ. For integers $k \geq \rho \geq r \geq 0$, we say that

$$S_k^{r,\rho}(\Delta) = \{s \in S_k^r(\Delta) : s \in C^\rho(v_i), i = 1, \cdots, V_I\},$$

is a *super spline space of degree k and smoothness r, ρ* (cf.[5,13,12]), where $C^\rho(v)$ denotes the set of functions defined on D which are ρ times continuously differentiable at the point v and $S_k^r(\Delta)$ is an ordinary spline space defined as

$$S_k^r(\Delta) = \{s \in C^r(\Omega) : s|_{D_i} \in \mathbf{P}_k(x, y) \forall i\}.$$

Throughout the paper, $\mathbf{P}_k(x, y)$ and $\mathbf{P}_k(x)$ denote the collection of polynomials

$$\mathbf{P}_k(x, y) := \{\sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij} x^i y^j | c_{ij} \in R\}, \mathbf{P}_k(x) := \{\sum_{i=0}^{k} c_i x^i | c_i \in R\},$$

respectively. Moreover, if $k < 0$, $\mathbf{P}_k(x, y)$ and $\mathbf{P}_k(x)$ are both equal to zero. If $S_k^{r,\rho}(\Delta) \neq S_k^\rho(\Delta)$, the super spline space $S_k^{r,\rho}(\Delta)$ is called a *nontrivial super spline space of degree k and smoothness r, ρ.* Super splines have strongly applied background in finite elements, vertex spline and Hermite interpolation. In [13], the relation between super spline theory and finite element theory was introduced. In [10,2,11], by using super spline, bivariate macro element was built. In [9], based on super spline spaces, Hermite interpolation was discussed. In [1,4,13,8,12], the

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dimensions of $S_k^{r,\rho}(\Delta)$ were given. In these papers, super splines were discussed by a B net method. In this paper, the smooth cofactors method for studying super splines is presented. The method is more effective for solving some problems about super splines.

By using Bezout's theorem from algebraic geometry, Wang discovered the following smooth conditions and the conformality conditions for bivariate splines (cf.[15]).

Theorem 1[15]**.** *The function $s(x,y)$ is a bivariate spline belonging to $S_k^\mu(\Delta)$ if and only if the following conditions are satisfied.*

(i) For any grid-segment Γ_i defined by $l_i(x,y)=0$, there exists the so-called smoothing cofactor $q_i(x,y)$ such that

$$p_{i1}(x,y)-p_{i2}(x,y)=l_i^{\mu+1}(x,y)q_i(x,y), \tag{1}$$

where the polynomials p_{i1} and p_{i2} are determined by the restriction of $s(x,y)$ to the two cells D_{i1} and D_{i2} with Γ_i as common edge and $q_i\in\mathbf{P}_{k-(\mu+1)}(x,y)$.

(ii) For any interior vertex v_j of Δ, the following conformality conditions are satisfied

$$\sum_i (l_i^{(j)}(x,y))^{\mu+1}q_i^{(j)}(x,y)\equiv 0, \tag{2}$$

where the summation is taken over all the interior edges $\Gamma_i^{(j)}$ passing through v_j and the signs of the smoothing cofactors $q_i^{(j)}$ are refixed in such a way that when a point crosses $\Gamma_i^{(j)}$ from D_{i2} to D_{i1} it goes around v_j in a counter-clockwise manner.

Smooth conditions and the conformality conditions are very effective tools for studying bivariate splines (cf.[16]). The purpose of this paper is to present the smooth conditions and the conformality conditions for super splines. Using the smooth conditions and conformality conditions , the locally supported bases of super splines on type-1 triangulation are also discussed. The local supported bases of super splines have a wide range of applications in approximation, interpolation, numerical analysis and finite element methods. We shall only discuss some approximation properties arising from the variation-diminishing super spline series.

2. The Smooth Conditions and the Conformality Conditions for Super Spline

To obtain the smooth conditions and the conformality conditions for super splines, we firstly introduce a lemma. One can find a similar result in [14].

Lemma 1. *Denote by $l(x,y)$ the straight line $y-ax-b=0$. Let $p(x,y)\in\mathbf{P}_k(x,y)$ and $(x_1,y_1),(x_2,y_2)$ be two distinct points lying on l. Then $\frac{\partial^n p(x,y)}{\partial^j x\partial^{n-j}y}|_{(x_1,y_1)}=0$, $\frac{\partial^n p(x,y)}{\partial^j x\partial^{n-j}y}|_{(x_2,y_2)}=0, j\le n\le\mu$, if and only if there exist $q(x,y)\in\mathbf{P}_{k-\mu-1}(x,y)$ and $c_m(x)\in\mathbf{P}_{k-\mu-m-1}(x)$ such that*

$$p(x,y)=(y-ax-b)^{\mu+1}q(x,y)+\sum_{m=1}^{\mu+1}(y-ax-b)^{\mu+1-m}(\prod_{i=1}^{2}(x-x_i))^m c_m(x) \tag{3}$$

where $c_m(x)\equiv 0$ provided $k-\mu-m-1<0$.

Proof. There exist $q(x,y)\in\mathbf{P}_{k-1}(x,y)$ and $c(x)\in\mathbf{P}_k(x)$, such that

$$p(x,y)=(y-ax-b)q(x,y)+c(x). \tag{4}$$

When $\mu=0$, then $c(x_1)=0, c(x_2)=0$, i.e., there exists $c_1(x)$ such that $c(x)=(x-x_1)(x-x_2)c_1(x)$. So, the theorem holds for $\mu=0$. Suppose that the lemma holds for $\mu=g-1$, i.e. there exist $q_0(x,y)$ and $c_m^{(0)}(x)$ such that

$$p(x,y)=(y-ax-b)^g q_0(x,y)+\sum_{m=1}^{g}(y-ax-b)^{g-m}(\prod_{i=1}^{2}(x-x_i))^m c_m^{(0)}(x). \tag{5}$$

To prove the lemma for $\mu = g$. Since $\frac{\partial^g p(x,y)}{\partial y^g}|_{(x_i,y_i)} = 0$ and $\frac{\partial^g p(x,y)}{\partial y^g}|_{(x_i,y_i)} = g!q_0(x_i,y_i)$, we have $q_0(x_i,y_i) = 0, i = 1,2$. By $q_0(x_i,y_i) = 0, i = 1,2$ and the result for $\mu = 0$, there exist $q(x,y)$ and $c_1(x)$ such that

$$q_0(x,y) = (y - ax - b)q(x,y) + (x - x_1)(x - x_2)c_1(x). \tag{6}$$

Substituting (6) into (5), we have

$$\begin{aligned} p(x,y) &= (y - ax - b)^{g+1}q(x,y) + (y - ax - b)^g(x - x_1)(x - x_2)c_1(x) \\ &+ \sum_{m=1}^{g}(y - ax - b)^{g-m}(\prod_{i=1}^{2}(x - x_i))^m c_m^{(0)}(x). \end{aligned} \tag{7}$$

Denote by $S_m(x,y)$ the $(y - ax - b)^{g-m}(\prod_{i=1}^2(x - x_i))^m c_m^{(0)}(x), 1 \le m \le g$. Obviously, $\frac{\partial^g S_m(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)} \ne 0, 0 \le t \le g, i = 1,2$, if and only if $m \le t$. When $t = 1$, $\frac{\partial^g S_m(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)} \ne 0, i = 1,2$, if and only if $m = 1$. When $t = 1$, $\frac{\partial^g p(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)} = \frac{\partial^g S_1(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)}$. Because $\frac{\partial^g p(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)} = 0$, we have $c_1^{(0)}(x_i) = 0, i = 1,2$, i.e., there exists $c_2(x)$ such that $c_1^{(0)}(x) = \prod_{i=1}^2(x - x_i)c_2(x)$. Suppose $c_j^{(0)}(x_i) = 0, 1 \le j \le h - 1, i = 1,2, h \in Z$. Then when $t = h$, $\frac{\partial^g S_m(x,y)}{\partial^t x \partial^{g-t} y}|_{(x_i,y_i)} \ne 0, i = 1,2$, if and only if $m = h$, which implies $c_h^{(0)}(x_i) = 0, i = 1,2$, i.e. there exists $c_{h+1}(x)$ such that $c_h^{(0)}(x) = \prod_{i=1}^2(x - x_i)c_{h+1}(x)$. We now continue this process and obtain that there exists $c_{m+1}(x)$ such that $c_m^{(0)}(x) = \prod_{i=1}^2(x - x_i)c_{m+1}(x), 1 \le m \le g$.

Substituting $q_0(x,y)$ and $c_m^{(0)}(x)$ into (6), we have

$$p(x,y) = (y - ax - b)^{g+1}q(x,y) + \sum_{m=1}^{g+1}(y - ax - b)^{g+1-m}(\prod_{i=1}^{2}(x - x_i))^m c_m(x). \tag{8}$$

Hence, the lemma holds.

Theorem 2. *$s(x,y) \in S_k^{r,\rho}(\Delta)$ if and only if the following conditions are satisfied:*
(i) For each interior edge of Δ, defined by $\Gamma_i : l(x,y) = y - a_i x - b_i = 0$, there exist $q_i(x,y)$ and $c_{im}(x), 1 \le m \le \rho - r$ such that

$$p_{i1}(x,y) - p_{i2}(x,y) = l_i^{\rho+1}(x,y)q_i(x,y) + \sum_{m=1}^{\rho-r} l_i^{\rho-m+1}(x,y)((x - x_{i1})(x - x_{i2}))^m c_{im}(x), \tag{9}$$

where (x_{i1},y_{i1}) and (x_{i2},y_{i2}) are two vertices lying on Γ_i, the polynomials p_{i1} and p_{i2} are determined by the restriction of $s(x,y)$ to the two cells D_{i1} and D_{i2} with Γ_i as the common edge and $q(x,y) \in \mathbf{P}_{k-\rho-1}(x,y), c_{im}(x) \in \mathbf{P}_h(x)$ $h = k - \rho - m - 1$. $c_{im}(x) \equiv 0$ provided $h < 0$.
(ii)For any interior vertex $v_j : (x_j,y_j)$ of Δ,the following conformality conditions are satisfied:

$$\sum((l_i^{(j)})^{\rho+1}q_i^{(j)}(x,y) + \sum_{m=1}^{\rho-r}(l_i^{(j)})^{\rho-m+1}((x - x_{i1})(x - x_j))^m c_{im}^{(j)}(x)) \equiv 0, \tag{10}$$

where the summation is taken over all the interior edges $\Gamma_i^{(j)}$ passing through v_j and (x_{i1},y_{i1}) and (x_j,y_j) are two vertices lying on $\Gamma_i^{(j)}$ and the signs of the $q_i^{(j)}$ and $c_{im}^{(j)}$ are refixed in such a way that when a point crosses $\Gamma_i^{(j)}$ from D_{i2} to D_{i1}, it goes around v_j in a counter-clockwise manner.

Proof. By using Theorem 1, we have

$$p_{i1}(x,y) - p_{i2}(x,y) = l_i^{r+1}(x,y)h_i(x,y), \tag{11}$$

where $h_i(x,y) \in \mathbf{P}_{k-r-1}(x,y)$. According to the definition of a super spline, we have $\frac{\partial^n h_i(x,y)}{\partial^j x \partial^{n-j} y}|_{(x_{i1},y_{i1})} = 0, \frac{\partial^n h_i(x,y)}{\partial^j x \partial^{n-j} y}|_{(x_{i2},y_{i2})} = 0$, where $0 \le j \le n, n \le \rho - r - 1$. Using the Lemma

1, we have

$$h_i(x,y) = (y-ax-b)^{\rho-r}q(x,y) + \sum_{m=1}^{\rho-r}(y-ax-b)^{\rho-r-m}(\prod_{i=1}^{2}(x-x_i))^m c_m(x).$$

Substituting $h_i(x,y)$ into (11), we can prove (i). The proof of (ii) is similar to the proof of (ii) in Theorem 1. Hence,we omit it.

In general, (9) and (10) are called smooth conditions and conformality conditions of super spline space $S_k^{r,\rho}(\Delta)$, respectively. Using the smooth conditions, we obtain the following theorems.

Theorem 3. *When* $r<\rho$, $S_k^{r,\rho}(\Delta) = S_k^{\rho}(\Delta)$ *if and only if* $\rho \geq k-1$.

proof. We use the same notations as in Theorem 2. Using the smooth condition (9), we have $c_{im}(x) \in \mathbf{P}_{k-\rho-m-1}(x,y), 1 \leq m \leq \rho-r$. Obviously, $k-\rho-m-1 \leq k-\rho-2$. $k-\rho-2<0$ if and only if $\rho \geq k-1$. So, if and only if $\rho \geq k-1$, $k-\rho-m-1<0$, where $1 \leq m \leq \rho-r$. Hence, if and only if $\rho \geq k-1$, $c_{im}(x) \equiv 0$, where $1 \leq m \leq \rho-r$, i.e. $S_k^{r,\rho}(\Delta) \subset S_k^{\rho}(\Delta)$. Because of $S_k^{\rho}(\Delta) \subset S_k^{r,\rho}(\Delta)$, we have $S_k^{\rho}(\Delta) = S_k^{r,\rho}(\Delta)$ if and on if $\rho \geq k-1$.

Theorem 4. *For any triangulation* Δ, *if* $k \leq 2\rho+1$, *then* $S_k^{-1,\rho}(\Delta) = S_k^{2\rho+1-k,\rho}(\Delta)$, *where*

$$S_k^{-1,\rho}(\Delta) = \{s \in C^{\rho}(v_j), : s|_{D_i} \in \mathbf{P}_k(x,y), \forall i,j\}.$$

Proof. We use the same notations with Theorem 2. Let $s(x,y) \in S_k^{-1,\rho}(\Delta)$. Using the smooth conditions

$$p_{i1}(x,y) - p_{i2}(x,y) = l_i^{\rho+1}(x,y)q_i(x,y) + \sum_{m=1}^{\rho+1} l_i^{\rho-m+1}(x,y)((x-x_{i1})(x-x_{i2}))^m c_{im}(x),$$

we have when $m > k-\rho-1$, $c_{im}(x) \equiv 0$. Hence,

$$p_{i1}(x,y) - p_{i2}(x,y) = l_i^{\rho+1}(x,y)q_i(x,y) + \sum_{m=1}^{k-\rho-1} l_i^{\rho-m+1}(x,y)((x-x_{i1})(x-x_{i2}))^m c_{im}(x).$$

So,

$$\begin{aligned} p_{i1}(x,y) - p_{i2}(x,y) &= l_i^{2\rho+2-k}(x,y)(l_i^{k-\rho-1}(x,y)q_i(x,y) \\ &+ \sum_{m=1}^{k-\rho-1} l_i^{k-\rho-1-m}(x,y)((x-x_{i1})(x-x_{i2}))^m c_{im}(x)). \end{aligned}$$

By Theorem 1, $s(x,y) \in S_k^{2\rho-k+1,\rho}(\Delta)$. Hence $S_k^{-1,\rho}(\Delta) \subset S_k^{2\rho-k+1,\rho}$ (Δ). Because of $S_k^{2\rho-k+1,\rho}$ $(\Delta) \subset S_k^{-1,\rho}$ (Δ), we have $S_k^{2\rho-k+1,\rho}(\Delta) = S_k^{-1,\rho}(\Delta)$.

Remark.

1. If the grid line l_i is defined by $x - a_i y - b_i = 0$ then (9) can be replaced by the following smooth condition :

$$p_{i1}(x,y) - p_{i2}(x,y) = l_i^{\rho+1}(x,y)q_i(x,y) + \sum_{m=1}^{\rho-r} l_i^{\rho-m+1}(x,y)((y-y_{i1})(y-y_{i2}))^m c_{im}(y). \quad (12)$$

The proof is similar to that of Lemma 1.

2. If the degrees of super-smooth at the vertex (x_{i1}, y_{i1}) and (x_{i2}, y_{i2}) are ρ_{i1} and ρ_{i2} respectively then (9) can be replaced by the following smooth condition:

$$\begin{aligned} p_{i1}(x,y) - p_{i2}(x,y) &= l_i^{\rho_i+1}(x,y)q_i(x,y) \quad (13) \\ + \sum_{m=1}^{\rho_i-r} l_i^{\rho_i-m+1}(x,y)&(x-x_{i1})^{(m+\rho_{i1}-\rho_i)_+}(x-x_{i2})^{(m+\rho_{i2}-\rho_i)_+}c_{im}(x), \end{aligned}$$

where $\rho_i = max\{\rho_{i1}, \rho_{i2}\}$ and $(\cdot)_+ = max\{\cdot, 0\}$. The proof is also similar as Lemma 1.

3. The Spaces of Super Spline on Type-1 Triangulation

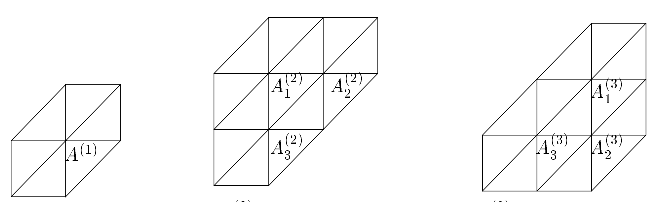
We begin with the necessary notations. Let

$$D_{mn} = [0, m+1] \otimes [0, n+1],$$

where m and n are positive integers. Partition D_{mn} first by drawing in the vertical lines $x - i = 0$ and horizontal lines $y - j = 0, i = 1, \cdots, m$ and $j = 1, \cdots, n$. Then by drawing in the diagonals with positive slopes to the rectangles $[i, i+1] \times [j, j+1]$, we obtain a type-1 triangulation $\Delta_{mn}^{(1)}$ of D_{mn}. By Theorem 3, when $r < \rho$, the necessary condition for the super spline space $S_k^{r,\rho}(\Delta_{mn}^{(1)})$ being nontrivial is $k \geq \rho + 2$. In practice, super spline spaces with the lowest possible degree k and the highest possible smooth degree r are the most useful. Hence, the important spaces to study are

$$S_3^{0,1}(\Delta_{mn}^{(1)}), S_4^{1,2}(\Delta_{mn}^{(1)}), S_5^{2,3}(\Delta_{mn}^{(1)}) \cdots .$$

The space $S_3^{0,1}(\Delta_{mn}^{(1)})$ is trivial from the mathematical point of view. In this section, we will discuss various local support bases of the bivariate super spline space $S_4^{1,2}(\Delta_{mn}^{(1)})$. By using the smooth conditions and conformality conditions, we obtain three locally supported super spline functions, write as $B^{(i)}, i = 1, 2, 3$ respectively.



$A^{(1)}: (\frac{1}{2}, 0, 0, -6, 3, -6)$
$B^{(1)}$

$A_1^{(2)}: (\frac{1}{2}, 1, -1, 0, -3, 0)$
$A_2^{(2)}: (\frac{1}{2}, -1, 0, 0, 3, -6)$
$A_3^{(2)}: (\frac{1}{2}, 0, 1, -6, 3, 0)$
$B^{(2)}$

$A_1^{(3)}: (\frac{1}{2}, 0, -1, -6, 3, 0)$
$A_2^{(3)}: (\frac{1}{2}, -1, 1, 0, -3, 0)$
$A_3^{(3)}: (\frac{1}{2}, 1, 0, 0, 3, -6)$
$B^{(3)}$

Fig.1

In Fig.1, the supports of $B^{(i)}, 1 \leq i \leq 3$ are shown. The vertices $A^{(1)}, A_1^{(i)}, A_2^{(i)}, A_3^{(i)}, i = 2, 3$ inside the support of $B^{(i)}$ are labelled and the values of $B^{(i)}, D_x B^{(i)}, D_y B^{(i)}, D_x^2 B^{(i)}, D_{xy}^2 B^{(i)}, D_y^2 B^{(i)}$, respectively, at these vertices are also given. These values completely determine $B^{(i)}$ with the exception of a translation. To determine $B^{(i)}, i = 1, 2, 3$ uniquely, we place the vertex $A^{(1)}, A_3^{(2)}, A_3^{(3)}$ in Fig.1 at the origin respectively.

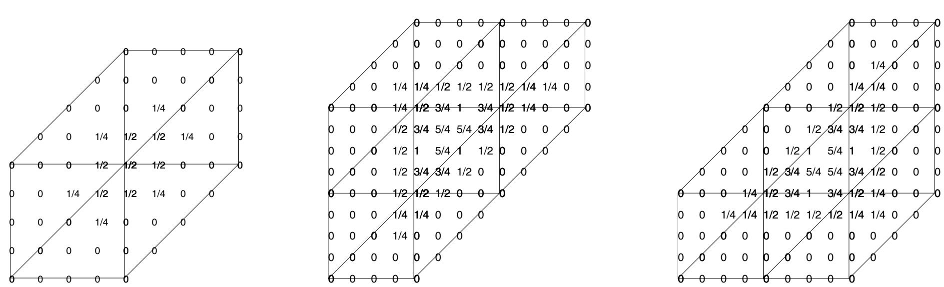


Fig.2

In Fig.2, the B net coordinates of $B^{(i)}, i=1,2,3$ are also presented.

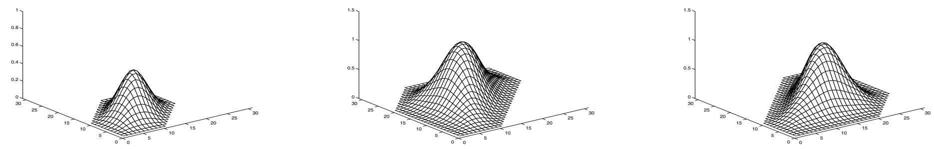


Fig.3

In Fig.3, the graphs of the three locally supported super splines are shown. By using the conformality conditions of bivariate super splines, it can be shown that the supports of $B^{(i)}$ are minimal.

We now translate $B^{(i)}, i=1,2,3$ to obtain bases of $S_4^{1,2}(\Delta_{mn}^{(1)})$. That is, we consider

$$B_{ij}^{(p)}(x,y)=B^{(p)}(x-i,y-j), p=1,2,3.$$

To facilitate our presentation, we introduce the index sets

$$\Omega_p=\{(i,j): B_{ij}^{(p)}\text{does not vanish identically on } D_{mn}\}.$$

It is clear that the cardinality of $\bigcup_{p=1}^{3}\Omega_p$ is $3mn+8(m+n)+20$. From [4] we also know that the dimension of $S_4^{1,2}(\Delta_{mn}^{(1)})$ is

$$dim S_4^{1,2}(\Delta_{mn}^{(1)})=3mn+8(m+n)+19.$$

Hence the collection

$$\mathcal{B}=\bigcup_{p=1}^{3}\{B_{ij}^{(p)}:(i,j)\in\Omega_p\}$$

must be linearly dependent on D_{mn}. We will give criteria to determine which element can be deleted from $\mathcal{B}$ to give a local basis of $S_4^{1,2}(\Delta_{mn}^{(1)})$.

Theorem 5. *For any $f\in\bigcup_{p=2}^{3}\{B_{ij}^{(p)}:(i,j)\in\Omega_p\}$, the elements of $\mathcal{B}\setminus f$ are linear independent.*

Proof. Let $D_1=[i_1,i_1+1]\bigotimes[j_1,j_1+1]$ and

$$\Omega_p^0=\{(i,j): B_{ij}^{(p)}\text{does not vanish identically on } D_1\}.$$

For any $(i_0,j_0)\in\Omega_2^0\cup\Omega_3^0$, write

$$F(x,y)=\sum_{(i,j)\in\Omega_1^0}c_{i,j}B_{ij}^{(1)}(x,y)+\sum_{(i,j)\in\Omega_2^0}d_{i,j}B_{ij}^{(2)}(x,y)+\sum_{(i,j)\in\Omega_3^0}e_{i,j}B_{ij}^{(3)}(x,y),$$

where d_{i_0,j_0} or e_{i_0,j_0} is equal to zero. We have to show that if $F(x,y)=0$ for all $(x,y)\in D_1$ then all the other $c_{i,j}, d_{i,j}$ and $e_{i,j}$ are equal to zero. We assume $F(x,y)\equiv 0$ on D_1. Then using the equations

$$\begin{aligned}
F(i_1+\mu,j_1+v)&=0\\
D_xF(i_1+\mu,j_1+v)&=0\\
D_yF(i_1+\mu,j_1+v)&=0\\
D_x^2F(i_1+\mu,j_1+v)&=0\\
D_{xy}^2F(i_1+\mu,j_1+v)&=0\\
D_y^2F(i_1+\mu,j_1+v)&=0\\
\mu,v&\in\{0,1\},
\end{aligned}$$

and the values in Fig.1, we can arrive at the following linear systems:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ -6 & -6 & 0 & 0 & 0 & 0 & -6 \\ 3 & 3 & 3 & -3 & 3 & -3 & 3 \\ -6 & 0 & -6 & 0 & -6 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{i_1+\mu,j_1+v} \\ d_{i_1+\mu,j_1+v} \\ d_{i_1-1+\mu,j_1-1+v} \\ d_{i_1+\mu,j_1-1+v} \\ e_{i_1+\mu,j_1+v} \\ e_{i_1-1+\mu,j_1+v} \\ e_{i_1-1+\mu,j_1-1+v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mu, v \in \{0,1\}.$$

Using these linear systems, we can show if d_{i_0,j_0} or e_{i_0,j_0} is equal to zero, all the other $c_{i,j}, d_{i,j}$ and $e_{i,j}$ are equal to zero. Suppose $f = B^{(2)}_{i_2,j_2}, (i_2,j_2) \in \Omega_2$. Write

$$G(x,y) = \sum_{(i,j)\in\Omega_1} c_{i,j} B^{(1)}_{ij}(x,y) + \sum_{(i,j)\in\Omega_2} d_{i,j} B^{(2)}_{ij}(x,y) + \sum_{(i,j)\in\Omega_3} e_{i,j} B^{(3)}_{ij}(x,y),$$

where, $d_{i_2,j_2} = 0$. Hence, the coefficients of $B^{(1)}, B^{(2)}$ and $B^{(3)}$ on sub-rectangle $[i_2, i_2 + 1] \bigotimes [j_2, j_2 + 1]$ are also 0. By Fig.1, it is easy to prove the coefficient of local support super spline on adjacent sub-rectangle is also 0. We now continue this process and obtain that all the $c_{i,j}, d_{i,j}$ and $e_{i,j}$ are equal to zero. Similarity if $f = B^{(3)}_{i_2 j_2}(i_2,j_2) \in \Omega_3$, the result also holds. Hence for any $f \in \bigcup_{p=2}^{3}\{B^{(p)}_{ij}\}$, the elements of $\mathcal{B} - f$ are linear independent.

By the above result and the dimension of $S_4^{1,2}(\Delta^{(1)}_{mn})$, the following theorem can be obtained:

Theorem 6. *For any $f \in \bigcup_{p=2}^{3}\{B^{(p)}_{ij} : (i,j) \in \Omega_p\}$, the elements of $\mathcal{B} \setminus f$ form a locally supportedd basis of $S_4^{1,2}(\Delta^{(1)}_{mn})$.*

Let $H^{(1)}_{ij}(x,y) = B^{(2)}_{ij}(x,y) - B^{(1)}_{ij}(x,y), H^{(2)}_{ij}(x,y) = B^{(3)}_{ij}(x,y) - B^{(1)}_{ij}(x,y)$. Using the values shown in Fig.1 or B net coordinates in Fig.2, we have

Theorem 7. *For all $(x,y) \in R^2$,*

$$\sum_{i,j} H^{(1)}_{ij}(x,y) \equiv 1, \qquad \sum_{i,j} H^{(2)}_{ij}(x,y) \equiv 1.$$

The $H^{(1)}_{ij}(x,y)$ and $H^{(2)}_{ij}(x,y)$ have a partition of unity, but the function values of $H^{(1)}_{ij}(x,y)$, $H^{(2)}_{ij}(x,y)$ are not nonnegative. To build a "variation diminishing" operator, we introduce another local supported super spline function, denote B. In Fig.4, the support of B is shown. For $B^{(i)}$, we need explicit values of B. In Fig.3, the vertices inside the support of B are labelled A_1, A_2, A_3, A_4, and the values of $B, D_xB, D_yB, D_x^2B, D_{xy}^2B, D_y^2B$, respectively, at these vertices are also given as 6-tuples. We also assume that A_3 is located at the origin. It is clear that the location of A_3 and the given values in Fig.4 uniquely determine B.

In Fig.5, the B net coordinates of B are presented. In Fig.6, the graph of B is shown.

Let $B_{ij}(x,y) = B(x-i, y-j)$. Using the values shown in Fig.4 or B net coordinates in Fig.5, we have

Theorem 8.

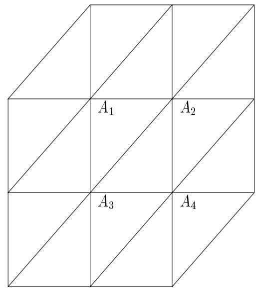
(1) For any $(x,y) \in R^2$,

$$\sum_{i,j} B_{ij}(x,y) = 1.$$

(2) Inside the support of B, the function values of $B(x,y)$ are strictly positive.

Let L be the "variation-diminishing" operator that map $C(D_{mn})$ into $S_4^{1,2}(\Delta^{(1)}_{mn})$ defined by

$$(Lf)(x,y) = \sum_{i,j} f(i + \frac{1}{2}, j + \frac{1}{2}) B_{ij}(x,y).$$



$$A_1 : (\tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}, 0, -\tfrac{3}{2}, 0)$$
$$A_2 : (\tfrac{1}{4}, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{3}{2}, 0)$$
$$A_3 : (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{3}{2}, 0)$$
$$A_4 : (\tfrac{1}{4}, -\tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{3}{2}, 0)$$

Fig.4

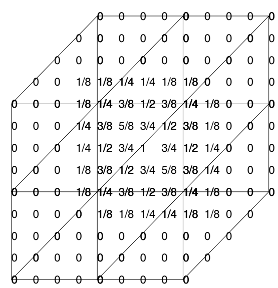


Fig.5

Now, we discuss some approximation properties of the variation diminishing operator. We first introduce a lemma.

Lemma 2. *L(f)=f for all* $f \in \mathbf{P}_1(x,y)$.

We remark that the above theorem does not hold for $f(x,y) = x^2, xy$, and y^2, and that for $f(x,y) = 1$ it was already shown in Theorem 8. Since a polynomial in $\mathbf{P}_1(x,y)$ on a triangle with vertices A, B, C vanishes identically if its values at A, B, C and the values of its two first partial derivatives, three second partial derivatives at A, B, and C are all equal to zero, the result follows by verifying that $L(f) - f, f \in \mathbf{P}_1(x,y)$, satisfies these conditions on each triangular cell of the partition $\Delta_{mn}^{(1)}$. This can be shown by using the values given in Fig.4.

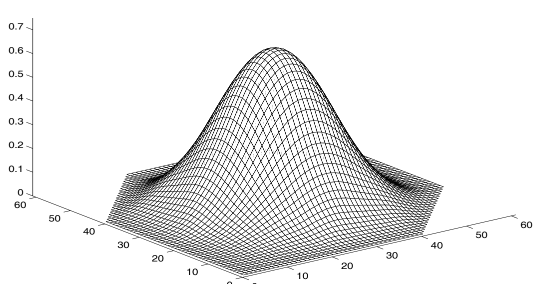


Fig.6

Suppose K is a closed set in R^2 and $f \in C(K)$. Let

$$\omega_K(f;\delta) = sup\{|f(x,y) - f(u,v)| : (x,y),(u,v) \in K, |(x,y)-(u,v)| < \delta\},$$

and δ_0 is the radius of a cell in $\Delta_{mn}^{(1)}$ and δ_0' is the radius of the support of B. Suppose $D_{mn} \subset K$ and the centers of the support of B_{ij} lie in the interior of K. Let $\|\cdot\|_{D_{mn}}$ denote the maxim value on D_{mn}. We have

Theorem 9. *If $f \in C(K)$, then*

$$\|f - V(f)\|_{D_{mn}} \le \omega_K(f, \delta_0'). \tag{14}$$

If $f \in C^1(K)$, then

$$\|f - V(f)\|_{D_{mn}} \le \delta_0 max(\omega_K(D_x f, \delta_0/2), \omega_K(D_y f, \delta_0/2)). \tag{15}$$

If $f \in C^2(K)$, then

$$\|f - V(f)\|_{D_{mn}} \le \delta_0^2 \|D^2 f\|, \tag{16}$$

where, the linear operator $D^2 f(x,y)(\cdot,\cdot) : \mathbf{R^2} \times \mathbf{R^2} \to \mathbf{R}$ is defined as

$$\begin{aligned} & D^2 f(x,y)((u_1,u_2),(v_1,v_2)) \\ = \ & D_x^2 f(x,y)u_1v_1 + D_{xy}^2 f(x,y)u_1v_2 + D_{yx}^2 f(x,y)u_2v_1 + D_y^2 f(x,y)u_2v_2. \end{aligned}$$

Proof.

Since the property of partition of unity, (14) obviously holds.

For $f \in C^1(K)$, suppose F be such a closure of a triangular cell that

$$\|f - L(f)\|_{D_{mn}} = \|f - L(f)\|_F.$$

Suppose (x_0, y_0) be equal to $(i, j - \frac{1}{2})$ or $(i - \frac{1}{2}, j)$. By the mean-value theorem, we have

$$f(x,y) = p_1(x,y) + (D_x f(u,v) - D_x f(x_0,y_0))(x - x_0) + (D_y f(u,v) - D_y f(x_0,y_0))(y - y_0), \tag{17}$$

where $(u,v) = t(x,y) + (1-t)(x_0,y_0), 0 \le t \le 1,$

$$p_1(x,y) = f(x_0,y_0) + D_x f(x_0,y_0)(x - x_0) + D_y f(x_0,y_0)(y - y_0). \tag{18}$$

By Lemma 2 and $\|L\| = 1$, we have

$$\|f - L(f)\|_F \le \|f - p_1\|_F + \|L(f - p_1)\|_F \le 2\|f - p_1\|_F.$$

Hence, by (17), equation (15) can be obtained.

For $f \in C^2(K)$, by Taylor's formula

$$f(x,y) = p_1(x,y) + \frac{1}{2} D^2 f(u,v)(x - x_0, y - y_0)^2, \tag{19}$$

where $(u,v) = t(x,y) + (1-t)(x_0,y_0), t \in [0,1]$, and $(x - x_0, y - y_0)^2 = ((x - x_0, y - y_0), (x - x_0, y - y_0))$. By (19), (16) can be proved easily.

Remark.

1. Using the smooth conditions and conformality conditions, similar as the ordinary multivariate spline spaces, the dimensions of bivariate super spline spaces can be constructed. Moreover, some other problems, such as constructing a bivariate macro-element [10], can also be investigated by the smooth conditions and the conformality conditions.
2. For nonuniform triangulated rectangles, since the grid lines determined by $0 = x_0 < \cdots < x_{m+1} = m+1$ and $0 = y_0 < \cdots < y_{n+1} = n+1$ are arbitrary, they can be moved appropriately to fit the given data. In fact, adaptive schemes can be developed and the problems of approximation by super spline spaces of degree 4 and smoothness $1, 2$ with variable grid partitions can be investigated by using the bivariate local supported super spline functions $B_{ij}(x,y)$. The study of these problems will be delayed to a later date.

References

[1] P. Alfeld, L.L. Schumaker, Upper and lower bounds on the dimension of super spline spaces, *Constructive Approx,* **19** (2003), 145-161.

[2] P. Alfeld, L.L. Schumaker, Smooth Macro-elements based on Powell-Sabin triangle splits, *Advances in Comp. Math.,* **16** (2002), 29-46.

[3] C. de Boor, K. Hö llig, S. Riemenschneider, Boxs Splines, Springer-Verlag, New York, (1993).

[4] C.K. Chui, T.X. He, On the dimension of bivariate super spline spaces, *Math. Comp.,* **53**:187 (1989), 219-234.

[5] C.K. Chui and M.J. Lai, On bivariate super vertex splines, *Constr. Approx.*, **6** (1990), 399-419.

[6] G. Nürnberger, Approximation by spline functions, Springer-Verleg, Berlin, (1989).

[7] G. Nürnberger and F. Zeilfelder, Developments in bivariate spline interpolation, *J. Comp. Appl. Math.*, **121** (2000), 125-152.

[8] Ibrahim, A. and L.L. Schumaker, Super spline spaces of smoothness r and degree $d \geq 3r+2$, *Constr. Approx.,* **7** (1991), 401-423.

[9] Lai, M.-J. and L.L. Schumaker, Scattered data interpolation using C^2 super splines of degree six, *SIAM J. Numer. Anal.*, **34** (1997), 905–921.

[10] Lai, M.-J. and L.L. Schumaker, Macro-elements and stable local bases for splines on Clough-Tocher triangulations, *Numer. Mathe.*, **88** (2001), 105-119.

[11] Lai, M.-J. and L.L. Schumaker, Macro-elements and stable local bases for splines on Powell-Sabin triangulations, *Math. Comp.*, **72** (2003), 335–354.

[12] L.L. Schumaker, Dual bases for spline spaces on cells, CAGD, (1988), 277-284.

[13] L.L. Schumaker, On Super splines and finite elements, *J. Numer. Anal.*, **26**:4 (1989), 997-1005.

[14] Ren-Hong Wang, Multivariate Weak Spline, The National Conference on Numerical Analysis. GuangZhou, 1979.

[15] Ren-Hong Wang, The structural characterization and interpolation for multivariate splines, *Acta Math. Sinica*, **18** (1975), 91-106 (English transl.ibid. **18** (1975), 10-39).

[16] Ren-Hong Wang etc., Multivariate Splines Functions and Their Application, Science Press, Beijing, (1994).