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THE DERIVATIVE ULTRACONVERGENCE FOR QUADRATIC TRIANGULAR FINITE ELEMENTS *1)

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Abstract

This work concerns the ultraconvergence of quadratic finite element approximations of elliptic boundary value problems. A new, discrete least-squares patch recovery technique is proposed to post-process the solution derivatives. Such recovered derivatives are shown to possess ultraconvergence. The keys in the proof are the asymptotic expansion of the bilinear form for the interpolation error and a "localized" symmetry argument. Numerical results are presented to confirm the analysis.

Mathematics subject classification: 65N30.
Key words: Ultra-closeness, Superconvergence patch recovery (SPR), Ultraconvergence.

1. Introduction

The superconvergence of the derivatives for quadratic triangular finite element has been studied in the pointwise sense since 1981. And based on the theory of the discrete Green's function and two basic estimates[15]~[20], a usual superconvergence order, $O(h^3)$, has been obtained. However, the *superconvergence patch recovery (SPR)* introduced by *Zienkiewics and Zhu*[21]~[23] indicates that, by using the least-squares and the interpolation process, the recovered nodal values of the derivatives are superconvergent, in particular those for quadratic elements are ultraconvergent, i.e., with order of $O(h^4)$. The above results for the two-point boundary value problem and rectangular elements have been discussed a great deal in [9] ∼ [14], but the ultraconvergence for quadratic triangular finite elements is so challenging that nobody has proved it up to now.

Our paper obtained the ultra-closeness results by the asymptotic expansion proposed in [2], and proceeded to verify the important problem.

For simplicity, we consider the model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,\, v),\ \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

where $a(u,v) = (\nabla u, \nabla v)$, $(f,v) = \displaystyle\int_\Omega f \cdot v dxdy$, Ω is a smooth or convex polygonal domain. It is easy to see from the PDE theory that there exists $2 < q_0 < \infty$ such that the mapping

$$\Delta:\quad W^{2,\,q}(\Omega) \cap W_0^{1,p}(\Omega) \to L^q(\Omega) \tag{1.2}$$

is a homeomorphic one for any $q \in (1, q_0)$.

Let $\mathcal{T}^h$ denote a uniform triangulation of Ω (only a local uniform triangulation is needed while considering the local superconvergence in this paper), $S_0^h(\Omega)$ a finite element space over $\mathcal{T}^h$ consisting of piecewise polynomials of degree 2. For each triangular element $e \in \mathcal{T}^h$, it suffices to let $e = \Delta z_1 z_2 z_3$, and define by $P_k(e)$ the set of the polynomials of degree k on e, and define $P_k^0(e) = \{p \in P_k(e) : p|_{\partial e} = 0\}$. Furthermore, to distinguish we add "′" to the

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corresponding term depending on e'. For a point z, $U_d(z)$ denotes the neighborhood of the point z, whose center is z and whose radius is d.

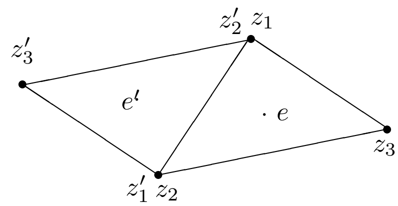


Fig. 1

Point z is said to be locally symmetric for the subdivision $\mathcal{T}^h$ if there exists a neighborhood, $U_d(z)$, of z such that the mesh covering this neighborhood is symmetric with respect to the point z. Generally, denote by E_z the union of the elements surrounding the point z.

It is clear that the vertexes of each element and the middle point of each side are the locally symmetric points.

Let ∂_z denote the direction derivative operator in the oriented direction l, and

$$\bar{\partial}_z v(z) = \frac{1}{2}[\lim_{t\to+0} \partial_z v(z+tl) + \lim_{t\to+0} \partial_z v(z-tl)],$$

is said to be the average derivative in the direction l for v. In the meantime, we may similarly define the gradient average operator $\bar{\nabla}$. And $\int_\Omega' f dxdy = \sum_{e\in\mathcal{T}^h} \int_e f dxdy$, then $\|f\|_m' = (\sum_{e\in\mathcal{T}^h} \|f\|^2_{m,e})^{\frac{1}{2}}$.

Throughout this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for *Sobolev* space on $\Omega \subset R^2$ with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. Particularly, we denote $W^{m,2} = H^m$. In addition, C denotes a nonnegative constant independent of h, u unless additional explanation, which can have different value in different place.

It is well known that all superconvergent estimates are in relation to pollution of boundary condition, but these pollution can be controlled by the negative norm

$$\|u-u^h\|_{-s,\Omega} = \sup_{v\in H_0^s(\Omega)} \frac{(u-u^h,\, v)}{\|v\|_s}, s>0,$$

that is

Proposition 1. *Suppose that* $w \in S^h(\Omega)$ *satisfy*

$$a(w,v)=0,\ \forall v\in S_0^h(U_d(z)),$$

then

$$|\bar{\nabla} w(z)| \le C\|w\|_{-s,U_d(z)}.$$

Strang[6], *Nitsche* and *Schatz*[3] have achieved the following negative norm estimates:

$$\|u-u^h\|_{-s,\Omega} \le Ch^{2k}\|u\|_{k+1,\Omega},$$

where k is the degree of finite element, in particular, $k=2$ in this paper. This shows the pollution does rarely affect interior and local superconvergence, and only local ultraconvergence is considered in this paper, so it is sufficient to consider the case that Ω be rectangular domain.

We have proved the following in the monograph [2] (cf. Theorem 3.5.1):

Proposition 2. *Let* $\mathcal{T}^h$ *be a uniform triangulation of* Ω, $S_0^h(\Omega)$ *a quadratic finite element space over* $\mathcal{T}^h$, *then*

$$\begin{aligned} a(u-u^I,v) = \quad & h^4 \int_\Omega' (C\cdot D^4u\cdot D^2v + C\cdot D^5u\cdot Dv)dxdy \\ & +O(h^5)\|u\|_{5,\infty}\|v\|_{2,1}',\ \forall v\in S_0^h(\Omega), \end{aligned} \tag{1.3}$$

where $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$, $u^I \in S_0^h(\Omega)$ (the interpolation of u) and $C \cdot D^m$ denotes a linear combination of the m-th order differential operators with constant coefficients.

Moreover, in this paper, we shall use the Green's function $G_z \in W_0^{1,p}(\Omega), 1 \le p < 2$, its derivative $\partial_z G_z$ and their finite solution G_z^h, $\partial_z G_z^h \in S_0^h(\Omega)$ (see [20] for details). they satisfy

$$a(G_z,\, v) = v(z),\ \forall v \in C_0^\infty(\Omega), \quad a(\partial_z G_z,\, v) = \partial_z v(z),\ \forall v \in C_0^\infty(\Omega),$$

and

$$v(z) = a(G_z^h,\, v),\ \forall v \in S_0^h(\Omega), \quad \partial_z v(z) = a(\partial_z G_z^h,\, v),\ \forall v \in S_0^h(\Omega),$$

In addition, we may define the quasi-Green's function $G_z^\star \in H^2(\Omega) \bigcap H_0^1(\Omega)$ and its derivative $\partial_z G_z^\star$, and their finite solution are still G_z^h, $\partial_z G_z^h \in S_0^h(\Omega)$ (see [20] Chap. 3 §5 (5.8) for details). they satisfy

$$a(G_z^\star,\, v) = (\delta_z^h,\, v),\ \forall v \in H_0^1(\Omega), \quad a(\partial_z G_z^\star,\, v) = (\partial_z \delta_z^h,\ v),\ \forall v \in H_0^1(\Omega),$$

where δ_z^h, $\partial_z \delta_z^h \in S_0^h(\Omega)$ are such that

$$(\delta_z^h\ v) = v(z),\ \ (\partial_z \delta_z^h\ v) = \partial_z v(z),\ \forall v \in S_0^h(\Omega).$$

Then we may derive the following (cf. [20] Chap. 3):

Proposition 3. *The following estimates for the Green's function and discrete Green's function hold:*

$$\begin{aligned} &\|G_z^h\|_1^2 + \|G_z^h\|_{0,\infty} + \|G_z^h\|_{2,1} + \|\partial_z G_z^h\|_{1,1} \le C \log \frac{1}{h}, \\ &\|\partial_z G_z^h\|_{2,\sigma^{1+\varepsilon}} \le C h^{-1+\varepsilon}, \end{aligned} \tag{1.4}$$

where the weighted function $\sigma = |x-z|^2$, $0 < \varepsilon < 1 - \frac{2}{q_0} = \varepsilon_0$, and the weighted norm is defined by

$$\|v\|_{\sigma^t} = [\int_\Omega \sigma^t |v|^2 dxdy]^{\frac{1}{2}},\ \ \|v\|_{m,\,\sigma^t} = \{\sum_{|\alpha| \le m} [\|D^\alpha v\|_{\sigma^t}]^2\}^{\frac{1}{2}},\ v \in H^m(\Omega).$$

Proof. We only prove the second estimate in (1.4) because the proof of the first one can be found in [20] (cf. Chap. 3).

Let $g = \partial_z G_z^\star$, then $g^h = \partial_z G_z^h$. Notice the following facts:

1). $\|g\|_{2,\,\phi^{-1-\varepsilon}} \le C h^{-1+\varepsilon}$, where $\phi = [|x-z|^2 + (\gamma h)^2]^{-1}$ (cf. [20] Chap. 3. §5. (5.22));

2). $\|g - g^h\|_{1,\,\phi^{-1-\varepsilon}} \le C h^\varepsilon$ (cf. [20] Chap. 3. Theorem 3.10.), and $\|g - g^I\|_{m,\phi^{-1-\varepsilon}} \le C h^{2-m} \|\nabla^2 g\|_{\phi^{-1-\varepsilon}} \le C h^{-m+1+\varepsilon}$ for $m = 1, 2$ hold (cf. [20] Chap. 3. Lemma 3.10.);

3). It is easy, from (3.15) in Chap. 3 of [20], to show that the following inverse estimate for the weighted norm holds:

$$\|g - g^h\|_{2,\phi^{-1-\varepsilon}} \le C h^{-1} \|g^I - g^h\|_{1,\phi^{-1-\varepsilon}} \le C h^{-1+\varepsilon};$$

Because $\sigma = |x-z|^2 \le \{[|x-z|^2 + (\gamma h)^2]^{-1}\}^{-1} = \phi^{-1}$,

$$\begin{aligned} \|g^h\|'_{2,\sigma^{1+\varepsilon}} &\le \|g^h\|'_{2,\phi^{-1-\varepsilon}} \\ &\le \|g\|_{2,\phi^{-1-\varepsilon}} + \|g - g^I\|'_{2,\phi^{-1-\varepsilon}} + \|g^h - g^I\|'_{2,\phi^{-1-\varepsilon}} \\ &\le C h^{-1+\varepsilon}. \end{aligned}$$

The main result of this paper is as follows:

Theorem 1. *Assume that z is a locally symmetric point for the subdivision $\mathcal{T}^h$ and the neighborhood $U_{2d}(z)$ of the point z is covered by a uniform family of subdivisions. Let u^h and u^I be the finite element approximation and the interpolation of u respectively, then we have the ultra-closeness:*

$$|\bar\nabla(u^h - u^I)(z)| \le C h^{3+\varepsilon} \|u\|_{5,\infty,\,U_{2d}(z)} + C[u - u^h]_{-s,\,\Omega},\ \ (0 < \varepsilon < 1 - \frac{2}{q_0}\ ) \tag{1.5}$$

Note $\varepsilon_0 = 1$ when Ω is a smooth or rectangular domain.

where $[\cdot]_{-s,D}$ *is defined as in* (3.1) *and* s *is an arbitrarily fixed nonnegative integer.*

In Section 4, we shall discuss the new derivative ultraconvergence recovery operator.

2. Some Lemmas

We say that z is a locally odd-symmetric point of the function u if there exists $d > 0$ such that

$$u(x) = -u(z-(x-z)),\ \forall x \in U_d(z).$$

Obviously, $u(z) = 0$.

Lemma 1. *If* $z \in \Omega$ *is a locally odd-symmetric point of the function* $u \in H^5(\Omega)$*, then it is one of* D^4u *too, and*

$$\|D^4u\|_{\sigma^{-1-\varepsilon}} \le C\|u\|_{5,\infty},\ 0 < \varepsilon < 1 - \frac{2}{q_0}. \tag{2.1}$$

Proof. The first result is easily proved, we only prove the second one. It indicates $D^4u(z) = 0$ that z is a locally odd-symmetric point of the function D^4u. Moreover, $u \in W^{5,\infty}$ implies $|D^4u(x)| \le C\sigma^{\frac{1}{2}}|D^5u|_\infty$, $\forall x \in U_d(z)$, hence integral of the left side in (2.1) exists and is finite. Next notice when $|x - z| = O(h)$ and γ is properly large,

$$\begin{aligned} & \|D^4u\|_{\sigma^{-1-\varepsilon}} \\ = \ & [\int_\Omega |x-z|^{-2-2\varepsilon}|D^4u|^2dx]^{\frac{1}{2}} \\ \le \ & C\|u\|_{5,\infty}(0 \le \varepsilon < 1). \end{aligned}$$

Then the second result (2.1) holds.

Lemma 2. *If* $z \in \Omega$ *is a locally odd-symmetric point of the function* u*, a locally symmetric point for the subdivision* $\mathcal{T}^h$*, and the neighborhood* $U_d(z)$ *of* z *is covered by a uniform family of subdivisions, then there exist the ultra-closeness estimates:*

$$|\bar{\nabla}(u^h - u^I)(z)| \le Ch^{3+\varepsilon}\|u\|_{5,\infty,U_{2d}} + \|u-u^h\|_{-s,U_d}. \tag{2.2}$$

Proof. Without lost of generality, we replace $\bar{\nabla}$ by $\bar{\partial}_z$. Construct the smooth function $\omega \in C_0^\infty(U_{2d}(z))$ such that $\omega = 1$ for all $x \in U_d(z)$, and let $u_1 = u\omega,\ u_2 = u - u_1$. Noting that $a(u_2^h, v) = a(u_2, v) = 0$ for all $v \in S_0^h(U_d)$ and recalling proposition 1, we find

$$\begin{aligned} |\bar{\nabla}(u_2^h - u_2^I)(z)| &= |\bar{\nabla}u_2^h(z)| \\ &\le C\|u_2^h\|_{-s,U_d} \\ &= C\|u_2^h - u_2\|_{-s,U_d} \\ &\le C\|u-u^h\|_{-s,U_d} + C\|u_1 - u_1^h\|_{-s,U_d} \\ &\le C\|u-u^h\|_{-s,U_d}. \end{aligned} \tag{2.3}$$

In proposition 2 we replace u by u_1 and set $v_z = \bar{\partial}_z G_z^h$, hence

$$\begin{aligned} |\bar{\partial}_z(u_1^h - u_1^I)(z)| &= |a(u_1 - u_1^I, v_z)| \\ &\le Ch^4|D^4u_1|_{\sigma^{-1-\varepsilon}}\|D^2v_z\|_{\sigma^{1+\varepsilon}} + Ch^4|D^5u_1|_{0,\infty}\|Dv_z\|'_{2,1}. \end{aligned}$$

Because u_1 is oddly symmetric, we derive from proposition 3 and lemma 1

$$\begin{aligned} |\bar{\partial}_z(u_1^h - u_1^I)(z)| &\le Ch^{3+\varepsilon}\|u_1\|_{5,\infty,U_{2d}} \\ &\le Ch^{3+\varepsilon}\|u\|_{5,\infty,U_{2d}}, \end{aligned} \tag{2.4}$$

Notice that $u = u_1 + u_2$, the lemma follows from (2.3) and (2.4).

Lemma 3. *For* $d > 0$ *and a fixed point* z*, there exists a* C^5 *locally symmetric transformation* $T : \Omega \to \Omega$*:*

$$T_dx = \begin{cases} 2z - x, & |x-z| \le d, \\ x, & |x-z| \ge 2d. \end{cases} \tag{2.5}$$

Proof. Construct the one-variable function

$$\phi(x)=\begin{cases}-x, & 0\le x\le d,\\ g(x), & d<x<2d,\\ x, & x\ge 2d,\end{cases}$$

where $g\in P_{11}(d,2d)$ satisfies $g(d)=-d,\ g(2d)=2d,\ g'(d)=-1,\ g'(2d)=1$ and $g^{(i)}(d)=g^{(i)}(2d)=0$ for $1<i\le 5$.

So

$$T_d x = z+\phi(|x-z|)\frac{x-z}{|x-z|},\ \forall x\in\Omega$$

is the desired one.

3. The Proof of the Main Theorem

Denote $\hat{u}(x)=-u(T_d x)$, and define

$$[u-u^h]_{-s,D}=\|u-u^h\|_{-s,D}+\|\hat{u}-\hat{u}^h\|_{-s,D}. \tag{3.1}$$

Obviously, if Ω is sufficiently smooth, $s\ge 1$ and $D\subset\Omega$, then

$$[u-u^h]_{-s,D}\le C(d)h^4\|u\|_{4,\Omega} \tag{3.2}$$

where $C(d)$ is a constant independent of h.

Let $u\in W^{5,\infty}(\Omega)$, z be a locally symmetric point for the subdivision $\mathcal{T}^h$. Construct a locally odd symmetric function

$$\tilde{u}(x)=\frac{1}{2}[u(x)-u(T_d x)]=\frac{1}{2}[u(x)+\hat{u}(x)].$$

It is clear that $\bar\partial_z\tilde{u}(z)=\bar\partial_z u(z)$, $\bar\partial_z\tilde{u}^I(z)=\bar\partial_z u^I(z)$ and $\bar\partial_z\widetilde{u^h}(z)=\bar\partial_z u^h(z)$, hence

$$\begin{aligned}&\bar\partial_z u^h(z)-\bar\partial_z u^I(z)\\ &=\bar\partial_z\widetilde{u^h}(z)-\bar\partial_z\tilde{u}^I(z)\\ &=(\bar\partial_z\widetilde{u^h}(z)-\bar\partial_z\tilde{u}^h(z))+(\bar\partial_z\tilde{u}^h(z)-\bar\partial_z\tilde{u}^I(z))\\ &\equiv A+B.\end{aligned} \tag{3.3}$$

From Lemma 2 we derive

$$\begin{aligned}|B| &\le Ch^{3+\varepsilon}\|\tilde{u}\|_{5,\infty,U_{2d}}+\|\tilde{u}-\tilde{u}^h\|_{-s,U_d}\\ &\le Ch^{3+\varepsilon}\|u\|_{5,\infty,U_{2d}}+[u-u^h]_{-s,U_d}.\end{aligned} \tag{3.4}$$

In the last, we note that $a(\widetilde{u^h}-\tilde{u}^h,v)=0$ holds for all $v\in S_0^h(U_d)$ and recall proposition 1, thus

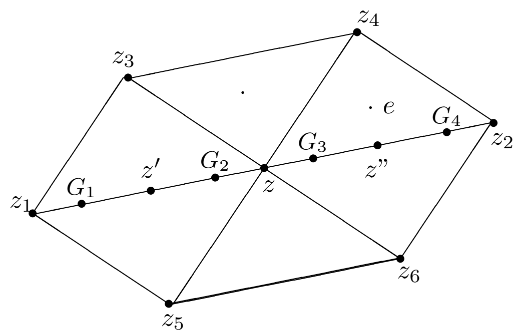
$$|A|\le\|\widetilde{u^h}-\tilde{u}^h\|_{-s,U_d}\le[u-u^h]_{-s,U_d} \tag{3.5}$$

hold.

The theorem follows from (3.2) $\sim$ (3.4).

4. The Derivative Recovery Operator

We assume that z is a locally symmetric point for the subdivision $\mathcal{T}^h$, it is sufficient to let z be a vertex of an element. Denote by E_z an element patch which consists of 6 congruent triangles sharing the point z (cf. Fig. 2).



(Fig. 2)

We choose arbitrarily a subdividing line through the point z, for example, $l = z_1z_2 = z_1z + zz_2$, which consists of sides z_1z and zz_2 of two elements. Taking 4 Gauss points $G_i(i = 1, 2, 3, 4)$ with 2 order on the line l (cf. Fig. 2). It is easy to show that, for each $w \in S^h(l)$, there exists a unique $v_0 \in P_2(l)$ such that

$$J(v_0) = \min_{v\in P_2(l)} J(v), \quad J(v) = \sum_{i=1}^{4}[v(G_i) - \partial_l w(G_i)]^2.$$

Construct the derivative recovery operator in the direction l at the point z:

$$R_l : w \to v_0,$$

then, it is easy to verify directly the following properties:

a). $|R_lw(z)| \leq C|\bar{\partial}_l w|_{0,\infty,l_h}$, *for all* $w \in S^h(l)$, *where* $l_h = \{z', z'', z\}$;

b). *There exists a constant* $C > 0$ *such that* $|\bar{\partial}u_l(z) - R_lu^I(z)| \leq Ch^4|u|_{5,\infty,l}$.

Proof. Define the inner product $< w, v >= \sum\limits_{i=1}^{4} w(G_i)v(G_i)$ and the corresponding norm $|||v||| = \{\sum\limits_{i=1}^{4}[v(G_i)]^2\}^{\frac{1}{2}}$ in the linear space $S = P_2(l) \oplus S'_h(l) = span\{p + v : p \in P_2(l), v \in S'_h(l)\} \subset L^2(l)$ (notice that S is a finite dimensional space.), so we derive from the projection theorem in the *Hilbert* space.

$$\begin{aligned} |v_0(z)| &\leq C|||v_0||| \leq C|||\partial_l w||| \\ &\leq C\max\{|\partial_l(G_j)| : j = 1, 2, 3, 4\} \\ &\leq C\max\{|\partial w(z + 0)|, |\partial w(z - 0)|, |\partial w(z')|, |\partial w(z'')|\}. \end{aligned} \tag{4.1}$$

To prove the result, let

$$w(x) = \frac{1}{2}[w(x) - w(z - (x - z))] + \frac{1}{2}[w(x) + w(z - (x - z))] \equiv w_1 + w_2.$$

Obviously, $R_lw_2(z) = 0$. in fact, $w_2 = \frac{1}{2}[w(x) + w(z - (x - z))]$ is even function of z, then $R_lw_2(z)$ is odd one, hence $R_lw_2(z) = 0$. For the oddly symmetrical function w_1, we have

$$\begin{aligned} &\partial_l w_1(z \pm 0) = \bar{\partial}_l w(z), \\ &|\partial_l w_1(z')| + |\partial_l w_1(z'')| \leq C\max\{|\partial_l w(z')|, |\partial_l w(z'')|\} \end{aligned}$$

From (4.1),

$$\begin{aligned} |R_lw(z)| &= |R_lw_1(z)| \\ &\leq \max\{|\partial_l w_1((z + 0)|, |\partial w_1(z - 0)|, |\partial_l w_1(z')|, |\partial_l w_1(z'')|\} \\ &\leq C\|\bar{\partial} w\|_{0,\infty,l_h} \end{aligned}$$

which proves a);

Here $S'_h(l) = \{\bar{\partial}_l v : v \in S^h(l)\}$.

Next, direct calculation shows that:

$$\bar{\partial} u(z) = R_l u^I(z), \forall u \in P_4(l). \tag{4.2}$$

Thus we get b) by using the $Bramble - Hilbert$ lemma.

The above derivative recovery operators R_l are named the SPR operators, which are also called the *Z-Z* operators for being introduced first by *Zienkiewicz and Zhu*[21]~[23], and most of which are defined over an element patch. Although their forms are varied, they satisfy conditions a), b). We may summarize the above conclusions in the following theorem:

Theorem 2. *Under the condition of theorem 1, if the derivative recovery operator R_l satisfies conditions* a), b), *then the ultraconvergent estimate is obtained:*

$$|(\partial_l u - R_l u^h)(z)| \leq Ch^{3+\varepsilon} \|u\|_{5,\infty,U_{2d}} + [u - u^h]_{-s,\Omega}.$$

where $0 < \varepsilon < 1 - \frac{2}{q_0} = \varepsilon_0$.

Proof. The triangle inequality, condition b), condition a) and theorem 1 imply

$$\begin{aligned} |(\partial_l u - R_l u^h)(z)| &\leq |(\partial_l u - R_l u^I)(z)| + |R_l(u^I - u^h)(z)| \\ &\leq Ch^4 \|u\|_{5,\infty,l} + |\bar{\partial}_l(u^I - u^h)|_{0,\infty,l_n} \\ &\leq Ch^{3+\varepsilon} \|u\|_{5,\infty,U_{2d}} + [u - u^h]_{-s,U_{2d}}, \end{aligned}$$

where s is an arbitrarily fixed nonnegative integer and $0 < \varepsilon < 1 - \frac{2}{q_0} = \varepsilon_0$.

Remark. Using the symmetric technique, Lin, Zhou and Yan[1] get improved superconvergence (1-3 order higher) on the inner locally symmetric rectangular or triangular meshes.

5. Numerical Test

We again consider the model problem (1.1), where f is constructed to correspond to the exact solution $u = x(1-x)y(1-y)(1+2x+7y)$. Let h be the step, and subdivide the domain Ω uniformly. At the point $z_0 = (0.25, 0.25)$, we obtained the following:

Table 1. SPR means the error of recovery derivatives obtained by the Z-Z method, IMP the error of recovery derivatives obtained by the method in this paper.

	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{12}$	$h=\frac{1}{16}$	$h=\frac{1}{20}$	rate
SPR	4.1504(−3)	2.5940(−4)	4.2513(−5)	1.6212(−5)	5.3107(−5)	4.0208
IMP	2.1558(−3)	1.0598(−4)	2.1576(−5)	6.8996(−6)	2.8399(−6)	4.0819

References

[1] Lin, Q., Zhou, J.M., Yan, N. N., Superconvergence in Hign-order Garlerkin Finite Element Methods, *J. Math. Study,* **32**:3 (1999), 9, 217-231.

[2] Lin, Q., Zhu, Q.D., The Preprocessing and Postprocessing for the Finite Element Method (in Chinese), Shanghai Sci. and Tech. Publishers, China, 1994.

[3] Nitsche, J., Schatz, A.H., Interior estimates for Ritz-Galerkin methods, *Math. Comp*, **28**(1974), 11, 937-958.

[4] Schatz, A.H., Sloan, I.H., Wahlbin, L.B., Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, *SIAM J. Numer. Anal.,* **33**(1996), 505-521.

[5] Schatz, A.H., Wahlbin, L.B., Interior Maximum norm estimates for finite element methods, Part 2, *Math. Comp.,* 1995.

[6] Strang, G., Fix, G.J., An analysis of the finite element methods[M]. Prentice Hall, 1973.

[7] Wahlbin, L.B., Superconvergence in Galerkin finite element methods, Lecture Notes in Mathematics 1605, Springer, 1995.

[8] Wahlbin, L.B., General principles of superconvergence in Galerkin finite element methods, *Lecture Notes in Pure and Applied Mathematics,* **196**(1998), 269-286.

[9] Zhang, T., The derivative patch interpolating recovery technique and superconvergence (in Chinese), *Mahematca numerica sinica*, **23**:1 (2000), 1-8.

[10] Zhang, Z.M., Ultraconvergence of the patch recovery technique, *Math. Comp.*, **65**:216 (1996), 1431-1437.

[11] Zhang, Z.M., Ultraconvergence of the patch recovery technique II, *Math. Comp.*, **69** (2000), 141-158.

[12] Zhang, Z.M., Zhu, J.Z., Analysis of the superconvergence patch recovery techniques and a posteriori error estimator in the finite element method (I), *Comput. Methods Appl. Mech. Engrg.,* **123** (1995), 173-187.

[13] Zhang, Z.M., Zhu, J.Z., Analysis of the supencorvergente patch recovery technique and a posteriori error estimator in the finite element method (II), *Comput. Methods Appl. Mech. Engrg.,* **163** (1998), 159-170.

[14] Zhang, Z.M., Victorg Jr, H.D., Mathematical analysis of Zienkiewicz-zhu's derivative patch recovery technique, *Numer. Methods for PDEs*, **12** (1996), 507-524.

[15] Zhu, Q.D., Natural inner superconvergence for the finite element method, Proceedings of the China-France Symposium on Finite Element Methods, Science Press, Beijing, Gorden and breach, 1983, 935-960.

[16] Zhu, Q.D., Optimal points of the stresses for quadratic triangular element (in Chinese), *Natur. Sci. J.,* Xiangtan University, **3** (1981), 36-45.

[17] Zhu, Q.D., Superconvergence Uniformly of the finite element (in Chinese), *Natur. Sci. J.,* Xiangtan University, **9** (1985), 10-26.

[18] Zhu, Q.D., A survey of superconvergence technique in finite element methods, *Lecture Notes in Pure and Applied Mathematics*, **196** (1998), 287-302.

[19] Zhu, Q.D., A review of two different approaches for superconvergence analysis, *Applications of Mathematics,* **43** (1998), 401-411.

[20] Zhu, Q.D., Lin, Q., The superconvergence theory of finite element methods (in Chinese), Hunan scientific and technical publishers, Changsha, China, 1989.

[21] Zienkiewicz, O.C., Zhu, J.Z., The superconvergence patch recovery and a posterior estimates. Part 1: The recovery technique, *Int. J. Numer. Methods Eng.,* **33** (1992), 1331-1364.

[22] Zienkiewicz, O.C., Zhu, J.Z., The superconvergence patch recovery and a posterior estimates. Part 2: Error estimates and adaptivity, *Int. J. Numer. Methods Engrg.,* **33** (1992), 1365-1382.

[23] Zienkiewicz, O.C. and Zhu, J.Z., The superconvergence patch recovery (SPR) and adaptive finite element refinement, *Comput. Methods Appl. Mech. Engrg.*, **101** (1992), 207-224.