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STABILITY OF GENERAL LINEAR METHODS FOR SYSTEMS OF FUNCTIONAL-DIFFERENTIAL AND FUNCTIONAL EQUATIONS *1)

Si-qing Gan
(*School of Mathematical Science and Computing Technology, Central South University, Changsha 410075, China*)

Wei-min Zheng
(*Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China*)

Abstract

This paper is concerned with the numerical solution of functional-differential and functional equations which include functional-differential equations of neutral type as special cases. The adaptation of general linear methods is considered. It is proved that A-stable general linear methods can inherit the asymptotic stability of underlying linear systems. Some general results of numerical stability are also given.

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Key words: Hybrid systems, Functional-differential equations, Functional equations, General linear methods, Numerical stability.

1. Introduction

Neutral functional differential equations with one state-independent time delay are usually formulated in the form

$$y'(t) = f(t, y(t), y(\phi(t)), y'(\phi(t))), \quad t \geq 0, \tag{1.1}$$

where f and ϕ are given functions with $\phi(t) \leq t$ for $t \geq 0$.

In contrast to (1.1) there are neutral functional differential equations of the form

$$[z(t) - g(t, z(\phi(t)))]' = f(t, z(t), z(\phi(t))), \quad t \geq 0, \tag{1.2}$$

where f, g and ϕ are given functions with $\phi(t) \leq t$ for $t \geq 0$. To distinguish (1.2) from (1.1), Liu [17] calls (1.1) an explicit neutral equation and (1.2) an implicit neutral equation. It is obvious that (1.1) and (1.2) are equivalent to

$$\begin{cases} y'(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \\ z(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} \quad t \geq 0 \tag{1.3}$$

and

$$\begin{cases} y'(t) = f(t, y(t) + g(t, z(\phi(t))), z(\phi(t))), \\ z(t) = y(t) + g(t, z(\phi(t))), \end{cases} \quad t \geq 0, \tag{1.4}$$

respectively.

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The numerical solution for neutral functional-differential equations (1.1) has been studied extensively in recent years (cf. [1, 7, 12, 13, 19, 22]). There seems to be little difference between solving (1.1) numerically and solving (1.3) numerically. However, the situation with the implicit neutral equation (1.2) is very different. Even in the case where (1.2) can be transformed into the form of (1.1), there are certain advantages in solving it by formulating numerical schemes based on its equivalent form (1.4). This has been discussed by Liu [17].

More general form than (1.3) and (1.4) is

$$\begin{cases} y'(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \\ z(t) = g(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} \quad t \geq 0. \tag{1.5}$$

It is easily seen that

$$\begin{cases} y'(t) = f(t, y(t), z(t), y(\phi(t)), z(\phi(t))), \\ z(t) = g(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} \quad t \geq 0 \tag{1.6}$$

can be transformed into the form of (1.5). Systems of the form (1.5) are sometimes called hybrid systems [11] or systems of functional-differential and functional equations [18]. The form of (1.5) includes functional-differential equations of neutral type (1.1) and (1.2) as special cases.

In order to investigate the linear stability of numerical methods to (1.5), Liu [18] considers the following test problems

$$\begin{cases} y'(t) + A_1 y(t) + A_2 y(t-\tau) + B_1 z(t-\tau) = 0, \\ z(t) + A_3 y(t) + A_4 y(t-\tau) + B_2 z(t-\tau) = 0, \end{cases} \quad t \geq 0, \tag{1.7a}$$

with the initial conditions

$$y(t) = \varphi(t), z(t) = \psi(t), \quad t \leq 0, \tag{1.7b}$$

where $\tau > 0$, and $A_1, A_2 \in C^{d_1 \times d_1}, A_3, A_4 \in C^{d_2 \times d_1}, B_1 \in C^{d_1 \times d_2}, B_2 \in C^{d_2 \times d_2}$ are the coefficient matrices, φ, ψ are given vectors of complex functions that satisfy the consistency condition

$$\psi(0) + A_3\varphi(0) + A_4\varphi(-\tau) + B_2\psi(-\tau) = 0.$$

We introduce some notations. $\sigma(A), \rho(A)$ and $\alpha(A)$ for a matrix A designate the spectrum, spectral radius and maximal real parts of the eigenvalues of A, respectively,

$$P(z,\xi) = \begin{vmatrix} zI_{d_1} + A_1 + A_2\xi & B_1\xi \\ A_3 + A_4\xi & I_{d_2} + B_2\xi \end{vmatrix}.$$

Other notation include

$$C^+ = \{z \in C \big| \mathrm{Re} z > 0\}, C^0 = \{z \in C \big| \mathrm{Re} z = 0\}, C^- = \{z \in C \big| \mathrm{Re} z < 0\},$$
$$D = \{z \in C \big| |z| < 1\}, \quad \Gamma = \{z \in C \big| |z| = 1\}.$$

In [18], it is shown that the initial-value problem (1.7) is asymptotically stable for every $\tau > 0$ if and only if

$$\rho(B_2) < 1, \tag{1.8a}$$

$$P(z,\xi) \neq 0 \text{ for all } z \in C^0 \setminus \{0\} \text{ and } \xi \in \Gamma, \tag{1.8b}$$

$$\alpha(B_1(I_{d_2} + B_2)^{-1}(A_3 + A_4) - A_1 - A_2) < 0. \tag{1.8c}$$

Moreover, the asymptotical stability of (1.7) implies the following are true [18]:
(1)$\alpha(-A_1) < 0$,
(2)$P(z,\xi) \neq 0$ for all $z \in C^0$ and $\xi \in D$,

(3)$P(z,\xi)\neq 0$ for all $z\in C^+$ and $\xi\in D\cup\Gamma$.

Further, it is easy to obtain the following result:

Lemma 1.1[8]**.** *The conditions (1.8) implies*

(1) $\sigma(A_1+A_2\xi-B_1\xi(I_{d_2}+B_2\xi)^{-1}(A_3+A_4\xi))\subset C^+$ *for all* $\xi\in D$,

(2) $\sigma(A_1+A_2\xi-B_1\xi(I_{d_2}+B_2\xi)^{-1}(A_3+A_4\xi))\subset C^+\cup\{0\}$ *for all* $\xi\in\Gamma$.

The stability of difference formulas for the system of functional-differential and functional equations (1.7) has been studied by several authors. Recently, Liu [18] investigated the numerical stability of Runge-Kutta-Collocation methods with a constrained grid and linear $\theta-$methods with a uniform grid for linear systems of the form (1.7). Huang and Chang [8, 9] considered the adaptation of linear multistep methods, one-leg methods and Runge-Kutta methods to (1.7) and investigated the linear stability of these methods. In this paper, the adaptation of general linear methods to (1.7) is considered. It is proved that A-stable general linear methods can inherit the asymptotic stability of underlying linear systems. Some general results of numerical stability are also given. General linear methods [2, 3, 15] contain not only Runge-Kutta methods, one-leg methods, linear multistep methods, multistep Runge-Kutta methods, but also a wide range of hybrid methods. Specializing Theorem 3.1 and Theorem 3.6 in this paper to linear multistep methods, one-leg methods and Runge-Kutta methods, our results are in accord with those obtained by Huang and Chang [8, 9]. Moreover, the sufficient and necessary condition of stability of multistep Runge-Kutta methhds is also given.

2. Adaptation of General Linear Method to the System (1.7)

An adaptation of $r-$step and $s-$stage general linear method to the system (1.7) leads to the following numerical process:

$$y_i^{(n)}=\sum_{j=1}^{r}c_{ij}^{22}y_j^{(n-1)}+h\sum_{j=1}^{s}c_{ij}^{21}f_j^{(n)},\quad i=1,2,\cdots,r, \tag{2.1a}$$

$$z_i^{(n)}=\sum_{j=1}^{s}\bar{c}_{ij}^{21}g_j^{(n)},\quad i=1,2,\cdots,r, \tag{2.1b}$$

$$f_i^{(n)}+A_1(\sum_{j=1}^{r}c_{ij}^{12}y_j^{(n-1)}+h\sum_{j=1}^{s}c_{ij}^{11}f_j^{(n)})+A_2\bar{Y}_i^{(n)}+B_1\bar{Z}_i^{(n)}=0,\quad i=1,2,\cdots,s, \tag{2.1c}$$

$$g_i^{(n)}+A_3(\sum_{j=1}^{r}c_{ij}^{12}y_j^{(n-1)}+h\sum_{j=1}^{s}c_{ij}^{11}f_j^{(n)})+A_4\bar{Y}_i^{(n)}+B_2\bar{Z}_i^{(n)}=0,\quad i=1,2,\cdots,s, \tag{2.1d}$$

$$\bar{Y}_i^{(n)}=\sum_{k=-\mu}^{\nu}L_k(\delta)(\sum_{j=1}^{r}c_{ij}^{12}y_j^{(n-1-m+k)}+h\sum_{j=1}^{s}c_{ij}^{11}f_j^{(n-m+k)}),\quad i=1,2,\cdots,s, \tag{2.1e}$$

$$\bar{Z}_i^{(n)}=\sum_{k=-\mu}^{\nu}L_k(\delta)g_i^{(n-m+k)},\quad i=1,2,\cdots,s, \tag{2.1f}$$

where $h>0$ is the fixed stepsize, $(c_{ij}^{11})\triangleq C_{11}\in R^{s\times s},(c_{ij}^{12})\triangleq C_{12}\in R^{s\times r},(c_{ij}^{21})\triangleq C_{21}\in R^{r\times s},(\bar{c}_{ij}^{21})\triangleq\bar{C}_{21}\in R^{r\times s},(c_{ij}^{22})\triangleq C_{22}\in R^{r\times r}$, $f_i^{(n)}$ and $g_i^{(n)}$ are approximations of $y'(t_n+\mu_ih)$ and $z(t_n+\mu_ih)$, respectively, $y_i^{(n)}$ and $z_i^{(n)}$ are approximations of $H_i(t_n+\nu_ih)$ and $G_i(t_n+\nu_ih)$, respectively, where $H_i(t+\nu_ih)$ and $G_i(t+\nu_ih)$ denote a piece of information about the true solutions $y(t)$ and $z(t)$, respectively, $t_n=nh$, μ_i and ν_i are some real constants (let $\bar{\mu}=(\mu_1,\mu_2,\cdots,\mu_s)^T,\bar{\nu}=(\nu_1,\nu_2,\cdots,\nu_r)^T$), $\bar{Y}_i^{(n)}$ and $\bar{Z}_i^{(n)}$ are approximations of $y(t_n+\mu_ih-\tau)$ and $z(t_n+\mu_ih-\tau)$, respectively. $\bar{Y}_i^{(n)}$ and $\bar{Z}_i^{(n)}$ are obtained by initial conditions φ and ψ

whenever $t_n+\mu_i h-\tau\le 0$,

$$L_i(\delta)=\prod_{\substack{q=-\mu\\ q\neq i}}^{\nu}\left(\frac{\delta-q}{i-q}\right),\qquad \delta\in[0,1), i=-\mu,-\mu+1,\cdots,\nu \tag{2.2}$$

and

$$\tau=(m-\delta)h, \tag{2.3}$$

m is a positive integer, $\delta\in[0,1)$, μ,ν are nonnegative integers and $\nu+1\le m$. $f_i^{(n)}$ and $g_i^{(n)}$ are obtained by the initial conditions φ and ψ, respectively, whenever $t_n+\mu_i h\le 0$, $y_i^{(n)}$ is obtained by the initial condition φ whenever $t_n+\nu_i h\le 0$. We assume $m\ge\nu+1$ so as to guarantee that, in the interpolation procedure, no unknown values $f_i^{(l)}$ and $g_i^{(l)}$ with $l\ge n$ are used.

Lemma 2.1[10,20]**.** *$\left|\sum_{i=-\mu}^{\nu}L_i(\delta)z^{\mu+i}\right|\le 1$ (whenever $|z|=1, 0\le\delta<1$) if and only if $\mu\le\nu\le\mu+2$.*

Lemma 2.2[10,20]**.** *If $\mu\le\nu\le\mu+2, \mu+\nu>0$,$|z|=1, 0<\delta<1$, then $\left|\sum_{i=-\mu}^{\nu}L_i(\delta)z^{\mu+i}\right|=1$ if and only if $z=1$.*

Lemma 2.3[4]**.** *For any matrices $H\in C^{i\times j}$ and $G\in C^{p\times q}$, there exist permutation matrices $P(i,p)\in C^{ip\times ip}, P(j,q)\in C^{jq\times jq}$ such that*

$$G\otimes H=P(i,p)^T(H\otimes G)P(j,q)$$

holds, where $P(I,J)$ only depends on the dimensions I and J and satisfies

$$P(I,J)=P(J,I)^T=P(J,I)^{-1}.$$

Application of general linear methods in the case of the following scale test problem

$$\begin{cases} w'(t)=\lambda w(t), & \mathrm{Re}\lambda<0,\\ w(0)=w_0, & w_0\in C,\end{cases}$$

yields

$$w^{(n)}=\Phi(\bar h)w^{(n-1)}=\Phi^n(\bar h)w^{(0)}, \tag{2.4}$$

where $\bar h=h\lambda, w^{(n)}=(w_1^{(n)},w_2^{(n)},\cdots,w_r^{(n)})^T(w_i^{(n)}\sim w(t_n+\nu_i h))$,

$$\Phi(\bar h)=C_{22}+\bar hC_{21}(I_s-\bar hC_{11})^{-1}C_{12}. \tag{2.5}$$

By (2.5) and matrix theory in [14], we have the following lemma

Lemma 2.4. *A general linear method*

$$\begin{bmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$$

is A-stable if and only if $(I_s-\bar hC_{11})$ is nonsingular and $\rho[\Phi(\bar h)]<1$ for $Re\bar h<0$.

3. Stability Analysis

Theorem 3.1. *Assume $\mu\le\nu\le\mu+2$, the system (1.7) satisfies (1.8), and one of the following conditions holds:*

(i) $det(zI_r-C_{22})\neq 0$ for all $z\in\Gamma\setminus\{1\}$,
(ii) $\delta\neq 0$ and $\mu+\nu>0$.

Then difference equation (2.1) is asymptotically stable if and only if the underlying general linear method for ordinary differential equations is A-stable.

Proof. Let

$$x_n = (y_1^{(n)^T}, y_2^{(n)^T}, \cdots, y_r^{(n)^T}, f_1^{(n)^T}, f_2^{(n)^T}, \cdots, f_s^{(n)^T}, g_1^{(n)^T}, g_2^{(n)^T}, \cdots, g_s^{(n)^T})^T.$$

We can express (2.1) in matrix form

$$\begin{aligned} &\begin{bmatrix} I_r \otimes I_{d_1} & -h(C_{21} \otimes I_{d_1}) & 0 \\ 0 & I_s \otimes I_{d_1} + h(C_{11} \otimes A_1) & 0 \\ 0 & h(C_{11} \otimes A_3) & I_s \otimes I_{d_2} \end{bmatrix} x_n + \begin{bmatrix} -C_{22} \otimes I_{d_1} & 0 & 0 \\ C_{12} \otimes A_1 & 0 & 0 \\ C_{12} \otimes A_3 & 0 & 0 \end{bmatrix} x_{n-1} + \\ &\sum_{i=-\mu}^{\nu} L_i(\delta) \begin{bmatrix} 0 & 0 & 0 \\ 0 & h(C_{11} \otimes A_2) & I_s \otimes B_1 \\ 0 & h(C_{11} \otimes A_4) & I_s \otimes B_2 \end{bmatrix} x_{n-m+i} \\ &+ \sum_{i=-\mu}^{\nu} L_i(\delta) \begin{bmatrix} 0 & 0 & 0 \\ C_{12} \otimes A_2 & 0 & 0 \\ C_{12} \otimes A_4 & 0 & 0 \end{bmatrix} x_{n-m+i-1} = 0, \end{aligned} \tag{3.1}$$

where the symbol $\otimes$ denote the Kronecker product.

The characteristic equation of the above difference equation turns out to be

$$z^{(m+\mu-1)(rd_1+sd_1+sd_2)+sd_1+sd_2} \det(Q(z)) = 0, \tag{3.2}$$

where

$$Q(z) = \begin{bmatrix} (I_r \otimes I_{d_1})z - C_{22} \otimes I_{d_1} & -h(C_{21} \otimes I_{d_1}) & 0 \\ C_{12} \otimes (A_1 + q(z)A_2) & I_s \otimes I_{d_1} + hC_{11} \otimes (A_1 + q(z)A_2) & q(z)(I_s \otimes B_1) \\ C_{12} \otimes (A_3 + q(z)A_4) & hC_{11} \otimes (A_3 + q(z)A_4) & I_s \otimes (I_{d_2} + q(z)B_2) \end{bmatrix},$$

$$q(z) = \sum_{i=-\mu}^{\nu} L_i(\delta) z^{-m+i}.$$

First, we consider the 'if' part. We will prove by contradiction that the characteristic equation (3.2) has no solution outside the open unit disc D. Suppose that there exists $\hat{z} \in C$ such that $\det(Q(\hat{z})) = 0$ and $|\hat{z}| \geq 1$. We note that $m \geq \nu + 1$, therefore, $\lim_{z \to \infty} q(z) = 0$. By Lemma 2.1 and maximum modulus principle, we have

$$|q(\hat{z})| \leq 1. \tag{3.3}$$

The condition (1.8a) implies that the matrix $I_{d_2} + q(\hat{z})B_2$ is nonsingular. Let

$$S = A_1 + q(\hat{z})A_2 - q(\hat{z})B_1(I_{d_2} + q(\hat{z})B_2)^{-1}(A_3 + q(\hat{z})A_4). \tag{3.4}$$

Since

$$\begin{aligned} &\begin{bmatrix} I_r \otimes I_{d_1} & 0 & 0 \\ 0 & I_s \otimes I_{d_1} & -q(\hat{z})I_s \otimes (B_1(I_{d_2} + q(\hat{z})B_2)^{-1}) \\ 0 & 0 & I_s \otimes I_{d_2} \end{bmatrix} Q(\hat{z}) \\ &= \begin{bmatrix} \hat{z}I_r \otimes I_{d_1} - C_{22} \otimes I_{d_1} & -hC_{21} \otimes I_{d_1} & 0 \\ C_{12} \otimes S & I_s \otimes I_{d_1} + hC_{11} \otimes S & 0 \\ C_{12} \otimes (A_3 + q(\hat{z})A_4) & hC_{11} \otimes (A_3 + q(\hat{z})A_4) & I_s \otimes (I_{d_2} + q(\hat{z})B_2) \end{bmatrix}, \end{aligned}$$

therefore,

$$\det \begin{bmatrix} \hat{z}I_r \otimes I_{d_1} - C_{22} \otimes I_{d_1} & -hC_{21} \otimes I_{d_1} \\ C_{12} \otimes S & I_s \otimes I_{d_1} + hC_{11} \otimes S \end{bmatrix} = 0. \tag{3.5}$$

On the other hand, using(3.3), (3.4) and Lemma 1.1, we obtain that $\sigma(S) \subset C^+ \cup \{0\}$. In view of A-stability of the method and Lemma 2.4, we have $\sigma(C_{11}) \subset C^+ \cup C^0$. Therefore, $I_s \otimes I_{d_1} + hC_{11} \otimes S$ is nonsingular. Consequently, we have

$$\begin{bmatrix} I_r \otimes I_{d_1} & (hC_{21} \otimes I_{d_1})(I_s \otimes I_{d_1} + hC_{11} \otimes S)^{-1} \\ 0 & I_s \otimes I_{d_1} \end{bmatrix} \begin{bmatrix} \hat{z} I_r \otimes I_{d_1} - C_{22} \otimes I_{d_1} & -hC_{21} \otimes I_{d_1} \\ C_{12} \otimes S & I_s \otimes I_{d_1} + hC_{11} \otimes S \end{bmatrix}$$
$$= \begin{bmatrix} \hat{z} I_r \otimes I_{d_1} - C_{22} \otimes I_{d_1} + (hC_{21} \otimes I_{d_1})(I_s \otimes I_{d_1} + hC_{11} \otimes S)^{-1}(C_{12} \otimes S) & 0 \\ C_{12} \otimes S & I_s \otimes I_{d_1} + hC_{11} \otimes S \end{bmatrix},$$

which shows

$$\det(\hat{z} I_r \otimes I_{d_1} - C_{22} \otimes I_{d_1} + (hC_{21} \otimes I_{d_1})(I_s \otimes I_{d_1} + hC_{11} \otimes S)^{-1}(C_{12} \otimes S)) = 0. \tag{3.6}$$

It follows from (3.6) and Lemma 2.3 that

$$\det(\hat{z} I_{d_1} \otimes I_r - I_{d_1} \otimes C_{22} + h(I_{d_1} \otimes C_{21})(I_{d_1} \otimes I_s + hS \otimes C_{11})^{-1}(S \otimes C_{12})) = 0. \tag{3.7}$$

Let J denote Jordan's normal form of matrix S, i.e., there exists a nonsingular matrix T such that $S = T^{-1} J T$. It follows from (3.7) that

$$\det((I_{d_1} \otimes I_r)\hat{z} - I_{d_1} \otimes C_{22} + h(I_{d_1} \otimes C_{21})(I_{d_1} \otimes I_s + hJ \otimes C_{11})^{-1}(J \otimes C_{12})) = 0, \tag{3.8}$$

which yields

$$\prod_{i=1}^{d_1} \det(\hat{z} I_r - C_{22} + h\lambda_i C_{21}(I_s + h\lambda_i C_{11})^{-1} C_{12}) = 0, \tag{3.9}$$

where $\lambda_i (i = 1, 2, \cdots, d_1)$ are the eigenvalues of S. By (3.9), there exists at least one $\lambda_i \in \sigma(S) \subset C^+ \cup \{0\}$ such that

$$\det(\hat{z} I_r - \Phi(-h\lambda_i)) = 0. \tag{3.10}$$

If $|\hat{z}| > 1$ then (3.10) contradicts A-stability of the methods. Therefore,

$$|\hat{z}| = 1. \tag{3.11}$$

If $\mathrm{Re}\lambda_i > 0$ then it also contradicts A-stability. Therefore,

$$\lambda_i = 0. \tag{3.12}$$

On the other hand, if $|q(\hat{z})| < 1$, It follows from Lemma 1.1 that $\sigma(S) \subset C^+$, It contradicts (3.12). Therefore,

$$|q(\hat{z})| = 1. \tag{3.13}$$

If condition (ii) holds, then from (3.11), (3.13) and Lemma 2.2, we have

$$\hat{z} = 1, \tag{3.14}$$

which gives

$$q(\hat{z}) = 1. \tag{3.15}$$

If condition (i) holds, then from (3.10), (3.11) and (3.12), we also get (3.14). Therefore, we have $q(\hat{z}) = 1$ and

$$0 = \lambda_i \in \sigma(S) = \sigma(A_1 + A_2 - B_1(I_{d_2} + B_2)^{-1}(A_3 + A_4)). \tag{3.16}$$

This contradicts the assumption (1.8c). Consequently, the characteristic equation (3.2) has no solution outside the open unit disc D. From the theory of difference equation it follows that

$\lim_{n\to\infty} x_n = 0$, accordingly, $\lim_{n\to\infty} y^{(n)} = 0$. By (2.1b), we also have $\lim_{n\to\infty} z^{(n)} = 0$. Therefore, difference equation (2.1) is asymptotically stable.

The 'only if' part follows directly from setting $d_1 = 1, A_2 = 0, B_1 = 0, A_3 = 0, A_4 = 0, B_2 = 0$ in (2.1).

Remark 3.2. In view of A-stability of the method and Lemma 2.4, we have $\sigma(C_{11}) \subset C^+ \cup C^0$. We also note that $\alpha(-A_1) < 0$. Therefore, $I_s \otimes I_{d_1} + hC_{11} \otimes A_1$ is nonsingular. Consequently, (2.1c) possesses an unique solution. Hence, the system (2.1) possesses an unique solution.

Remark 3.3. Specializing Theorem 3.1 to linear multistep methods, one-leg methods and Runge-Kutta method, we can obtain the numerical stability results obtained by [8, 9]. Especially, in the case of Runge-Kutta methods, condition (i) automatically holds. Further, according to the prove process of Theorem 3.1, the condition (ii) is not necessarily required in the case of Runge-kutta methods. Therefore, the conditions (i) and (ii) can be removed in the case of Runge-Kutta methods. Our results are accordant with that obtained by [8, 9]

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_r)^T, \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_s)^T$, $\nu_1 = 1, \nu_2 = 2, \cdots, \nu_r = r$,

$$C_{22} = \left[\begin{array}{cc} 0 & I_{r-1} \\ \hline \multicolumn{2}{c}{\alpha^T} \end{array}\right], \qquad C_{21} = \left[\begin{array}{c} 0 \\ \hline \gamma^T \end{array}\right].$$

Specializing Theorem 3.1 to the case of multistep Runge-Kutta methods, we can obtain immediately the following result

Corollary 3.4. *Assume $\mu \le \nu \le \mu+2$, the system (1.7) satisfies (1.8), and one of the following conditions holds:*

(I) $\alpha_1 + \alpha_2 z + \ldots + \alpha_r z^{r-1} - z^r \neq 0$ for all $z \in \Gamma \setminus \{1\}$,
(II) $\delta \neq 0$ and $\mu + \nu > 0$.

Then the adaptation of multistep multistep Runge-Kutta methods to (1.7), i.e. the corresponding difference equation of (2.1), is asymptotically stable if and only if the underlying multistep Runge-Kutta method for ordinary differential equations is A-stable.

Remark 3.5. Condition (i) in the assumptions of Theorem 3.1 is very weak. In fact, in addition to Runge-Kutta methods, multistep methods of Admas type and BDF methods, there are many methods which satisfy the assumption, such as many of multistep Runge-Kutta methods appeared in [15, 16].

For general case of general linear methods, we have the following result.

Theorem 3.6. *Suppose $\mu \le \nu \le \mu+2, \rho(B_2) < 1$ and $\sigma(h(B_1\xi(I_{d_2}+B_2\xi)^{-1}(A_3+A_4\xi) - A_1 - A_2\xi)) \subset S_{GLM}$ for all $\xi \in D \cup \Gamma$, then the difference equation (2.1) is asymptotically stable. Where*

$$S_{GLM} = \{\bar{h} \in C \text{ s.t. the matrix } (I_s - \bar{h}C_{11}) \text{ is regular and } \rho[\Phi(\bar{h})] < 1\}.$$

Proof. We only need to prove that the matrix $Q(z)$ is nonsingular for any $z \in C, |z| \ge 1$. In fact, for any $z \in C, |z| \ge 1$, we have

$$|q(z)| \le 1. \tag{3.17}$$

Let $S = A_1 + q(z)A_2 - q(z)B_1(I_{d_2} + q(z)B_2)^{-1}(A_3 + q(z)A_4)$. In view of (3.17) and the assumptions of the theorem, we have

$$\sigma(-hS) \subset S_{GLM}. \tag{3.18}$$

This shows that, for every $\lambda_i \in \sigma(S)$,

$$\det(zI_r - \Phi(-h\lambda_i)) \neq 0,$$

which yields

$$\prod_{i=1}^{d_1} \det(zI_r - C_{22} + h\lambda_i C_{21}(I_s + h\lambda_i C_{11})^{-1} C_{12}) \neq 0.$$

It follows from (3.18) that $I_s \otimes I_{d_1} + hC_{11} \otimes S$ is nonsingular. Similar to the reverse process from (3.5) to (3.9), we have

$$\det(Q(z)) \neq 0 \text{ for any } z \in C, |z| \geq 1$$

Therefore, the matrix $Q(z)$ is nonsingular for any $z \in C, |z| \geq 1$.

Specializing Theorem 3.6 to the case of functional-differential equations of neutral type, we obtain immediately

Corollary 3.7. *Consider the methods (2.1) applied to the system*

$$\begin{aligned} y'(t) + A_1 y(t) + A_2 y(t-\tau) + B_1 z(t-\tau) = 0, \\ z(t) + A_1 y(t) + A_2 y(t-\tau) + B_1 z(t-\tau) = 0, \end{aligned} \tag{3.19}$$

i.e., functional-differential equations of neutral type

$$y'(t) + A_1 y(t) + A_2 y(t-\tau) + B_1 y'(t-\tau) = 0.$$

Suppose $\mu \leq \nu \leq \mu+2, \rho(B_1) < 1$ *and* $\sigma(h(I_{d_2}+B_1\xi)^{-1}(-A_1-A_2\xi)) \subset S_{GLM}$ *for all* $\xi \in D\cup\Gamma$. *Then the corresponding difference equation is asymptotically stable.*

Remark 3.8. Specializing Theorem 3.6 to linear multistep methods, one-leg methods and Runge-Kutta method, we can obtain the numerical stability results obtained by [9]. Furthermore, specializing Corollary 3.7 to linear multistep methods and Runge-Kutta methods, our results are in accord with the those presented in the literature [1, 5, 6, 7, 19, 21, 23]

4. Numerical Results

In order to illustrate the results of this paper obtained in Section 3, we consider the following problem

$$\begin{cases} y'(t) + y(t) + y(t-1) + z(t-1) = 0, \\ z(t) + y(t) + 0.5y(t-1) + B_2 z(t-1) = 0, \end{cases} \quad t \geq 0, \tag{4.1}$$

with the initial conditions

$$y(t) = \mathrm{sin}t, \quad z(t) = t + \frac{B_2 + 0.5\mathrm{sin}1}{1 + B_2}, \qquad t \leq 0$$

where $B_2 \neq -1$ is a parameter. It is obvious that the initial functions satisfy the consistency condition: $z(0) + y(0) + 0.5y(-1) + B_2 z(-1) = 0$. We can easily verify that the initial value problem (4.1) satisfies conditions (1.8) for $B_2 = 0.5$ and doesn't for $B_2 = 1.5$. Therefore, problem (4.1) with $B_2 = 0.5$ is asymptotically stable.

Consider 2 step 1 stage Runge-Kutta methods[15]:

$$\begin{cases} C_{11} = (c), \quad C_{12} = (\frac{2a}{1+a}, \frac{1-a}{1+a}), \quad C_{21} = \begin{pmatrix} 0 \\ 1+a \end{pmatrix}, \\ C_{22} = \begin{pmatrix} 0 & 1 \\ a & 1-a \end{pmatrix}, \quad \bar{\mu} = (u), \quad \bar{\nu} = (1,2)^T, \end{cases} \tag{4.2}$$

where $0 < a \leq 1, c = \frac{1+3a}{2(1+a)}$ and $u = c + \frac{1-a}{1+a}$. The methods have order 2 and are algebraically stable[15], therefore, they are A-stable. It is easily verified that the methods satisfy the assumption condition (i) in Theorem 3.1.

Table 1. IMR for problem (4.1) with $B_2 = 0.5$

$h = 0.2$	t	5	10	20	30	50	100
	y	5.5×10^{-3}	-1.4×10^{-4}	-1.4×10^{-7}	-1.4×10^{-10}	-1.3×10^{-16}	-1.2×10^{-31}
	z	-2.2×10^{-2}	6.5×10^{-4}	6.4×10^{-7}	6.2×10^{-10}	5.9×10^{-16}	5.3×10^{-31}
$h = 0.3$	t	6	12	24	36	60	99.9
	y	-1.7×10^{-3}	-2.1×10^{-5}	-2.0×10^{-9}	-1.3×10^{-13}	-7.1×10^{-23}	1.4×10^{-35}
	z	4.2×10^{-3}	2.9×10^{-5}	3.4×10^{-10}	-1.2×10^{-13}	-1.6×10^{-21}	1.0×10^{-35}

Table 2. MRK for problem (4.1) with $B_2 = 0.5$

$h = 0.2$	t	5	10	20	30	50	100
	y	5.3×10^{-3}	-1.3×10^{-4}	-1.4×10^{-7}	-1.3×10^{-10}	-1.3×10^{-16}	-1.1×10^{-31}
	z	-1.8×10^{-2}	5.3×10^{-4}	5.2×10^{-7}	5.1×10^{-10}	4.8×10^{-16}	4.3×10^{-31}
$h = 0.3$	t	6	12	24	36	60	99.9
	y	-2.1×10^{-3}	-2.4×10^{-5}	-2.2×10^{-9}	-1.6×10^{-13}	-1.7×10^{-22}	1.6×10^{-35}
	z	3.4×10^{-3}	1.6×10^{-5}	-1.0×10^{-9}	-2.2×10^{-13}	-1.7×10^{-21}	2.0×10^{-35}

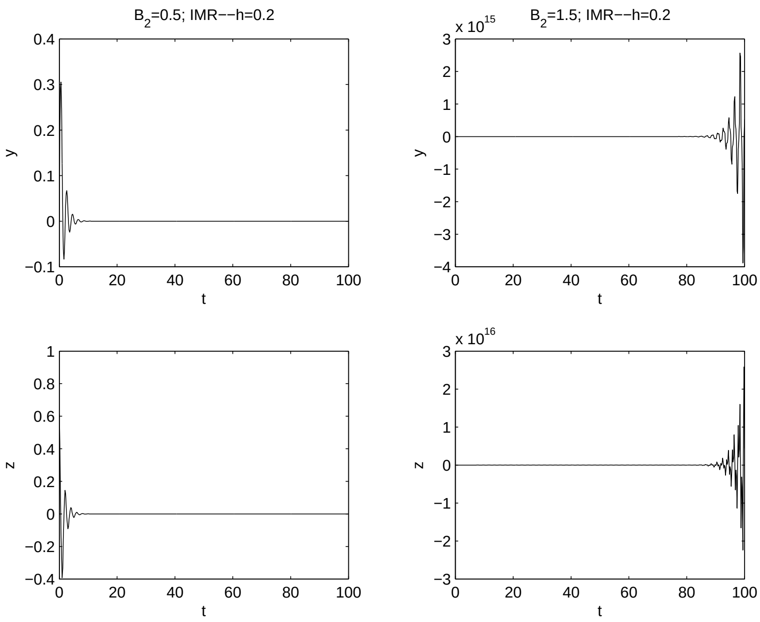

Fig. 1: Numerical results of IMR with $\delta = 0$

We can write implicit midpoint rule[3] as general linear method with

$$C_{11} = (\frac{1}{2}) \in R^1, \quad C_{12} = C_{21} = C_{22} = (1) \in R^1, \quad \bar{\mu} = \bar{\nu} = \frac{1}{2}.$$

It is well known that implicit midpoint rule has order 2 and is A-stable. It also satisfies the assumption condition (i) in Theorem 3.1.

We apply the multistep Runge-Kutta methods (4.2)(MRK) and implicit midpoint rule (IMR) to the problems (4.1), respectively, where the delay terms are evaluated using linear interpolation procedure (i.e., $\mu = 0, \nu = 1$). In addition to starting values $y(0)$ and $z(0)$, we require the other starting values for MRK, which are evaluated using implicit midpoint rule. Numerical results of IMR and MRK(a=0.5) are shown in Fig. 1-4 and Table 1-2.

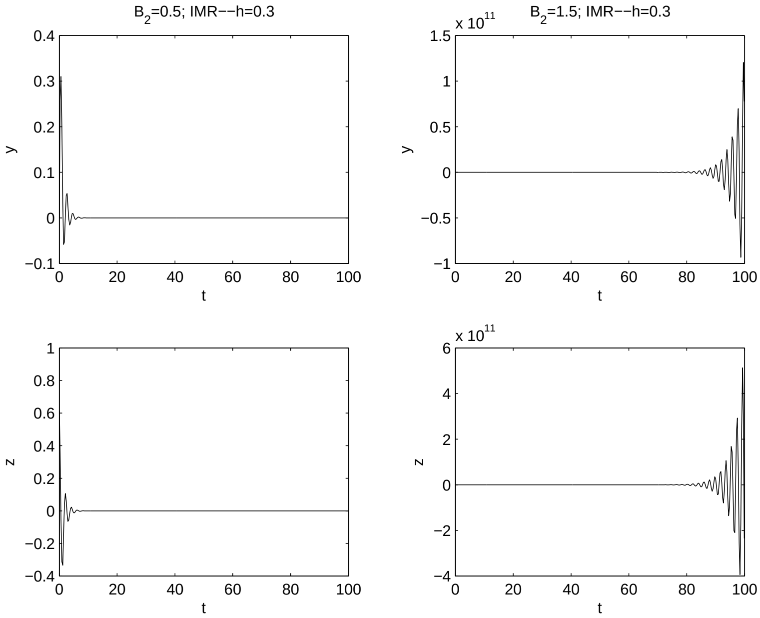

Fig. 2: Numerical results of IMR with $\delta \neq 0$

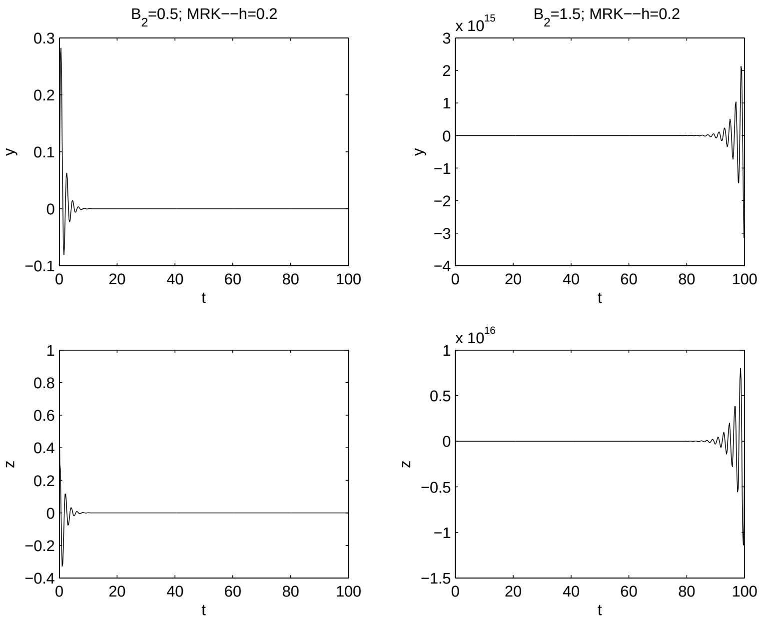

Fig. 3: Numerical results of MRK with $\delta = 0$

For B_2=0.5, the numerical results of IMR and MRK are displayed in left pictures of Fig. 1-4. The related data is listed in Table 1-2. The pictures and the tables show that IMR and MRK inherit the asymptotic stability of problem (4.1), and confirm theorem 3.1. Meanwhile, we note that the problem (4.1) and the both methods satisfy the assumption conditions in

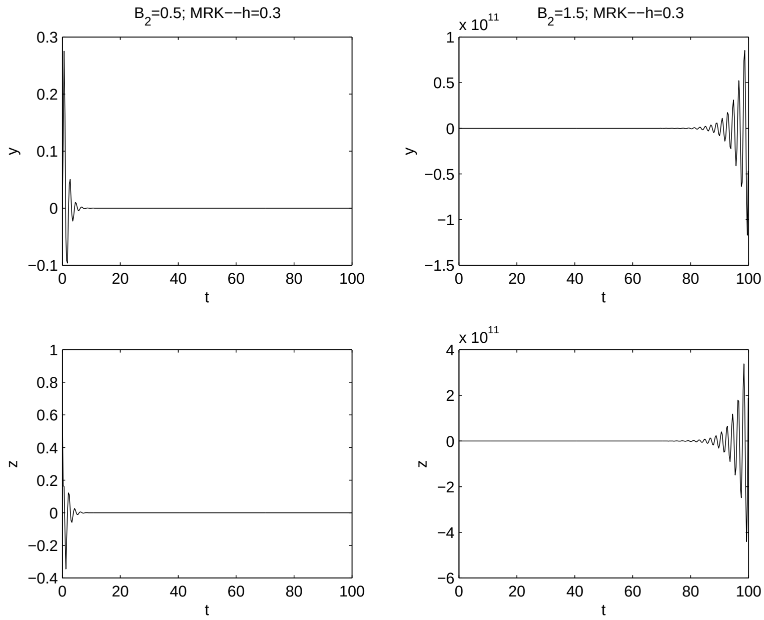

Fig. 4: Numerical results of MRK with $\delta \neq 0$

Theorem 3.6. The results mentioned above also confirm Theorem 3.6.

For B_2=1.5, problem (4.1) doesn't satisfy the conditions (1.8). The numerical solutions of (4.1) are displayed in right pictures of Fig.1-4. The pictures show that the numerical solutions are unlikely to be asymptotically stable.

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