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CLOSED SMOOTH SURFACE DEFINED FROM CUBIC TRIANGULAR SPLINES *1)

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Abstract

In order to construct closed surfaces with continuous unit normal, we introduce a new spline space on an arbitrary closed mesh of three-sided faces. Our approach generalizes an idea of Goodman and is based on the concept of 'Geometric continuity' for piecewise polynomial parametrizations. The functions in the spline space restricted to the faces are cubic triangular polynomials. A basis of the spline space is constructed of positive functions which sum to 1. It is also shown that the space is suitable for interpolating data at the midpoints of the faces.

Mathematics subject classification: 41A05, 41A25.
Key words: Closed triangular mesh, Triangular Bernstein polynomial, Smooth spline, Geometric continuity.

1. Introduction

In computer-aided geometric design it is often useful to use surfaces defined parametrically from box splines on a regular mesh. Those box splines usually used are tensor product B-splines on a square mesh. However, such a representation cannot give a simple closed surface. It therefore seems natural to attempt to define a "geometrically smooth" simple closed surface. Attempts to do this have used the notion of subdivision for box splines to construct surfaces by a process of recursive subdivision of the mesh, see for examples [1],[2] and [3]. [6],[7] and [9] pioneered the idea of geometrically continuous spline spaces. [8] gives local bases of G^2 continuous G-splines. They consider only quadrilateral meshes. The other more general constructions based on irregular meshes may be seen in [5] and [10]. It is generally agreed that [6]-[9] are a global method in that one needs to solve large linear, irregularly sparse systems to match data, while [5] and [10] are a local method in that the coefficients of the parametrization in Bernstein-Bézier form are generated by applying averaging masks to the input mesh, but before computing the coefficients, several earlier algorithms contribute the idea of mesh refinement to parametrizations. In this paper we consider a global method similar to [9]. We take a closed polyhedral mesh M of three-sided faces and consider the space $S(M)$ of all functions on it whose restrictions to each face are certain polynomials and which satisfy certain matching conditions across the edges. These matching conditions ensure that a parametrically defined surface in R^3 whose components lie in $S(M)$ is G^1, i.e., is continuous and has a continuous unit normal vector.

After giving the G^1 continuous conditions between two adjacent triangular Bèzier patches in section 2, we consider in section 3 a way defining spline space $S(M)$ and discuss the dimension

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of $S(M)$. In section 4 we study in detail the special case when M is a tetrahedron and the technique used here should extend to other meshes. We construct a basis for $S(M)$ of positive functions which sum to 1 and are thus useful for the design of surfaces. We also show that $S(M)$ can be used for interpolating data at the midpoints of the faces of M.

2. The G^1 Continuous Conditions between Two Adjacent Triangular Bèzier Patches

2.1. Triangular Patches

In the section, we will present the G^1 continuous conditions between two adjacent triangular Bézier patches. Given conditions are used as the matching conditions of spline space $S(M)$ in next section. As for the G^1 continuous conditions, the discussion can also be seen in [4,11].

Triangular polynomial patches can be expressed in a Bernstein-Bèzier form,

$$\varphi(u,v,w) = \sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} G_{i,j,k} \frac{n!}{i!j!k!} u^i v^j w^k,\ u+v+w=1, u,v,w \geq 0, \tag{2.1}$$

where coefficients $G_{i,j,k} \in R^3$. The parameters u, v, w are called barycentric coordinates of a triangle; φ can be viewed as a map of this triangle into R^3 (see Fig.1). Again, we use a shorthand notation for the coefficients:

$$T_i = G_{1,i,n-i-1},\ i = 0,\dots,n-1,\ S_i = G_{0,i,n-i},\ i = 0,\dots,n.$$

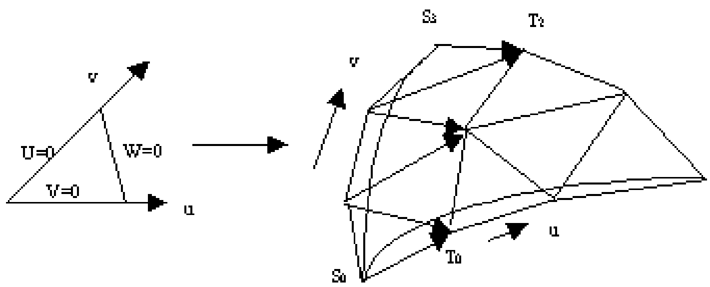

Fig.1. Triangular polynomial patch

The boundary $\Gamma(v) = \varphi(0, v, 1-v)$ has the form

$$\Gamma(v) = \sum_{0}^{n} S_i B_i^n(v)$$

and hence its derivative is given by

$$[D\Gamma](v) = n\sum_{i=0}^{n-1}(S_{i+1} - S_i)B_i^{n-1}(v). \tag{2.2}$$

We shall consider a particular cross-boundary derivative, namely,

$$[D\varphi](v) = (1-v)(\varphi_u - \varphi_v) + v(\varphi_w - \varphi_v),$$

where $\varphi_u = \varphi_u(0, v, 1-v), \varphi_v = \varphi_v(0, v, 1-v), \varphi_w = \varphi_w(0, v, 1-v)$. Expressed in terms of Bernstein polynomials,

$$[D\varphi](v) = n(1-v)\sum_{i=0}^{n-1}(T_i - S_i)B_i^{n-1}(v) + nv\sum_{i=0}^{n-1}(T_i - S_{i+1})B_i^{n-1}(v).$$

Simple algebra yields

$$[D\varphi](v) = n\sum_{i=0}^{n}(\frac{n-i}{n}T_i + \frac{i}{n}T_{i-1} - S_i)B_i^n(v). \tag{2.3}$$

Let us now consider a special case: assume the boundary curve with cofficients $S_i, i = 0, \ldots, n$, is only of degree $n-1$. This implies the existence of $\tilde{S}_i, i = 0, \ldots, n-1$, with

$$S_i = \frac{n-i}{n}\tilde{S}_i + \frac{i}{n}\tilde{S}_{i-1}, i = 0, \ldots, n. \tag{2.4}$$

Invoking the degree elevation of Bernstein polynomial [4] and (2.4), we see that (2.3) is equivalent to

$$[D\varphi](v) = n\sum_{i=0}^{n-1}(T_i - \tilde{S}_i)B_i^{n-1}(v). \tag{2.5}$$

2.2. The G^1 continuous conditions between two adjacent triangular Bèzier patches

Let ϕ and φ be two adjacent triangular patches of degree n, all of whose boundaries are degree $n-1$ and who share a common boundary cure Γ of the form($0 \le v = \tilde{v} \le 1$)(see Fig.2)

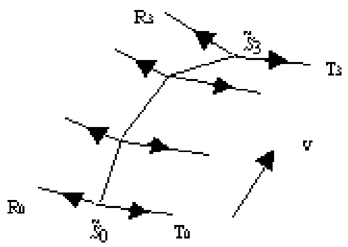

Fig.2. Coefficients for cross-boundary derivatives

$$\Gamma(v) = \sum_{i=0}^{n} S_i B_i^n(v) = \sum_{i=0}^{n-1}\tilde{S}_i B_i^{n-1}(v)$$

with the derivative

$$[D\Gamma](v) = n\sum_{i=0}^{n-1}(S_{i+1} - S_i)B_i^{n-1}(v) = (n-1)\sum_{i=0}^{n-2}(\widetilde{S}_{i+1} - \widetilde{S}_i)B_i^{n-2}(v).$$

From (2.5) in section 2.1, we assume that ϕ possesses a cross-boundary derivative of the form

$$[D_1\phi](v) = n\sum_{i=0}^{n-1}(R_i - \widetilde{S}_i)B_i^{n-1}(v)$$

and φ possesses a cross-boundary derivative of the form

$$[D_2\varphi](v) = n\sum_{i=0}^{n-1}(T_i - \widetilde{S}_i)B_i^{n-1}(v).$$

The G^1 continuous condition[4,11] is equivalent to

$$\mu(v)[D_1\phi](v) + \alpha(v)[D_2\varphi](v) + \lambda(v)[D\Gamma](v) = 0,\ \mu, \alpha, \lambda \ne 0.$$

In order to arrive at a manageable G^1 construction, we specify that μ and α must be constants while λ must be linear:

$$\lambda(v) = (1-v)\lambda_0 + v\lambda_1.$$

Since $\alpha \neq 0$, we can assume without loss of generality that $\alpha = 1$ and get

$$0 = \sum_{i=0}^{n-1} \Big\{ \mu(R_i - \widetilde{S}_i) + (T_i - \widetilde{S}_i) + \frac{n-1}{n}\left(\lambda_0 \frac{n-1-i}{n-1}(\widetilde{S}_{i+1} - \widetilde{S}_i) + \lambda_1 \frac{i}{n-1}(\widetilde{S}_i - \widetilde{S}_{i-1})\right)\Big\} B_i^{n-1}(v).$$

This gives the desired G^1 continuous condition

$$T_i = \frac{n-1-i}{n-1}\left((1 + \frac{n-1}{n}\lambda_0 + \mu)\widetilde{S}_i - \frac{n-1}{n}\lambda_0\widetilde{S}_{i+1} - \mu R_i\right) + \frac{i}{n-1}\left(\frac{n-1}{n}\lambda_1\widetilde{S}_{i-1} + (1 - \frac{n-1}{n}\lambda_1 + \mu)\widetilde{S}_i - \mu R_i\right), i = 0, 1, \ldots, n-1. \quad (2.6)$$

3. The Spline Space

Let M be a closed triangular mesh in which every face has three edges and there is no constraint on the number of triangles meeting at a vertex. On each face of M we take barycentric coordinates u, v and w so that the face has the form $\{(u, v, w) : 0 \leq u, v, w, u + v + w = 1\}$. We shall construct a vector space $S(M)$ of scalar functions on M whose restrictions to each face of M are triangular Bernstein polynomials in u, v and w. This shall be constructed so that any surface $f : M \longrightarrow R^3$ whose components lie in $S(M)$ is guaranteed to be G^1.

To ensure that, it is sufficient to require that the elements of $S(M)$ satisfy some conditions similar to (2.6) across the edges of M. We assume that any function S in $S(M)$ coincides with a cubic triangular Berstein polynomial on any face $\{(u, v, w) : 0 \leq u, v, w, u + v + w = 1\}$

$$\phi(u, v, w) = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} g_{i,j,k} \frac{3!}{i!j!k!} u^i v^j w^k, \; u + v + w = 1, u, v, w \geq 0, \quad (3.1)$$

where coefficients $g_{i,j,k} \in R$ and along any edge of the face ϕ is a Bernstein polynomial of degree 2.

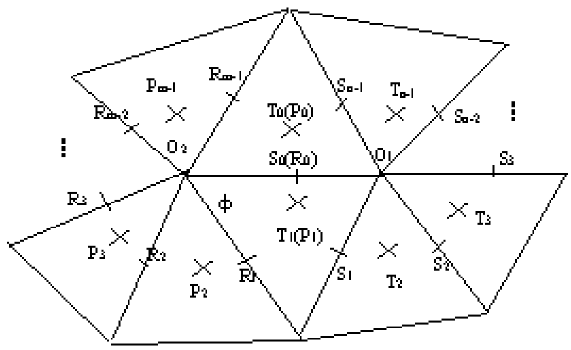

Fig.3. The coefficients around vertexes O_1 and O_2

We now consider how to match S on two adjacent faces of M. Suppose that on one face S is given by (3.1) and on an adjacent face it is given by $\psi(\widetilde{u}, \widetilde{v}, \widetilde{w})$ similar to (3.1). These two faces have a common edge

$$(0, v, 1-v) = (0, \widetilde{v}, 1-\widetilde{v}),\ 0 \le v = \widetilde{v} \le 1.$$

Let two end points of the common edge be o_1, o_2, respectively, and the number of edges meeting at o_1 be n, the number at o_2 be m (see Fig.3). The coefficients of the polynomials on the faces meeting at o_1 and o_2 are labelled in Fig.3. Note that in Fig.3 we denote the coefficients of these polynomials on the faces as (3.1) in capitals, these coefficients are real numbers, for example, O_1, O_2 for coefficients, but o_1, o_2 for the vertices of parameteric triangular domain. Suppose that o_2 corresponds to $v = 0$ and o_1 to $v = 1$. In order to ensure that ϕ and ψ is G^1 continuous along common edge $o_1 o_2$, according to (2.6), these relations between coefficients $\tilde{R}_1, T_1, \tilde{S}_1$ and $\tilde{R}_{m-1}, R_0, \tilde{S}_{n-1}$ are

$$\begin{cases} \widetilde{R}_1 = & (1+\frac{2}{3}\lambda_0+\mu)O_2 - \frac{2}{3}\lambda_0 S_0 - \mu\widetilde{R}_{m-1}, \\ T_1 = & \frac{1}{2}\left((1+\frac{2}{3}\lambda_0+\mu)S_0 - \frac{2}{3}\lambda_0 O_1 - \mu T_0\right) + \frac{1}{2}\left((1-\frac{2}{3}\lambda_1+\mu)S_0 + \frac{2}{3}\lambda_1 O_2 - \mu T_0\right), \\ \widetilde{S}_1 = & \frac{2}{3}\lambda_1 S_0 + (1-\frac{2}{3}\lambda_1+\mu)O_1 - \mu\widetilde{S}_{n-1}, \end{cases} \tag{3.2}$$

where

$$\tilde{R}_1 = \frac{2}{3}R_1 + \frac{1}{3}O_2, \tilde{R}_{m-1} = \frac{2}{3}R_{m-1} + \frac{1}{3}O_2, \tilde{S}_1 = \frac{2}{3}S_1 + \frac{1}{3}O_1, \tilde{S}_{n-1} = \frac{2}{3}S_{n-1} + \frac{1}{3}O_1. \tag{3.3}$$

(3.3) can be obtained by the degree elevation. In order to consider the G^1 conditions of other adjacent polynomials at o_1 and o_2, we take

$$\mu = 1, \lambda_0 = -2\cos\frac{2\pi}{m}, \lambda_1 = 2\cos\frac{2\pi}{n} \tag{3.4}$$

in G^1 condition (3.2). Substituting $\tilde{R}_1, \tilde{R}_{m-1}, \tilde{S}_1, \tilde{S}_{n-1}, \mu, \lambda_0, \lambda_1$ in (3.2) by (3.3) and (3.4), we have

$$\begin{cases} (1-\cos\frac{2\pi}{m})O_2 + \cos\frac{2\pi}{m}S_0 = \frac{1}{2}R_1 + \frac{1}{2}R_{m-1}, & (3.5) \\ \cos\frac{2\pi}{n}O_2 + \cos\frac{2\pi}{m}O_1 - (\cos\frac{2\pi}{n} + \cos\frac{2\pi}{m} - 3)S_0 = \frac{3}{2}(T_0+T_1), & (3.6) \\ (1-\cos\frac{2\pi}{n})O_1 + \cos\frac{2\pi}{n}S_0 = \frac{1}{2}S_1 + \frac{1}{2}S_{n-1}. & (3.7) \end{cases}$$

In order to guarantee G^1 continuity for n adjacent polynomials at o_1, we require that the relations similar to (3.5)-(3.7) exist among any two adjacent polynomials surrounding some common vertex. Hence there exists following relations between the coefficients around o_1

$$(1-\cos\frac{2\pi}{n})O_1 + \cos\frac{2\pi}{n}S_i = \frac{1}{2}S_{i-1} + \frac{1}{2}S_{i+1},\ i = 0, \cdots, n-1, \tag{3.8}$$

where all subscripts are taken modulo n. Adding the equations in (3.8) up, we have

$$O_1 = \frac{1}{n}\sum_{i=0}^{n-1} S_i. \tag{3.9}$$

According to (3.5),(3.6),(3.7) and (3.9), we have

$$\begin{cases} O_1 = \frac{1}{n}\sum_{i=0}^{n-1} S_i, & (3.10) \\ 2\left((1-\cos\frac{2\pi}{n})\cos\frac{2\pi}{n} + (1-\cos\frac{2\pi}{m})\cos\frac{2\pi}{m} \right. \\ \left. -3(1-\cos\frac{2\pi}{n})(1-\cos\frac{2\pi}{m})\right) S_0 - \cos\frac{2\pi}{n}(1-\cos\frac{2\pi}{n})\cdot \\ (R_1+R_{m-1}) - \cos\frac{2\pi}{m}(1-\cos\frac{2\pi}{m})(S_1+S_{n-1}) & (3.11) \\ = -3(1-\cos\frac{2\pi}{n})(1-\cos\frac{2\pi}{m})(T_0+T_1), \end{cases}$$

where $R_0 = S_0$. (3.10),(3.11) are the G^1 compatible conditions of polynomials on these faces meeting at a common vertex. We call the coefficients S_i and R_i edge coefficient labelled by "—" in Fig.3, O_1 and O_2 vertex coefficient, T_i face coefficient labelled by "x", the condition (3.10) vertex condition, and the condition (3.11) edge condition.

Let the closed mesh M has e edges, f faces and v vertices. Then the sum of polynomial coefficients on all faces is $e+f+v$. From (3.10) and (3.11) we see that the vertex conditions and the edge conditions can afford $e+v$ independent equations at most. Thus this gives

Theorem 1. *If $S(M)$ is defined as before and M has f faces, then*

$$dim S(M) \geq f.$$

4. Splines on a Tetrahedron

Theorem 2. *If M is a tetrahedron, then*

$$dim S(M) = 4.$$

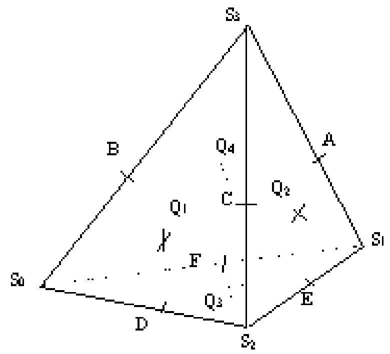

Fig.4. Coefficients of polynomials on each face of tetrahedron

The coefficients of polynomials on each face of tetrahedron are labelled in Fig.4, Q_1 for face$S_0S_2S_3$ coefficient, Q_2 for face $S_2S_1S_3$ coefficient, Q_3 for face $S_0S_2S_1$ coefficient and Q_4 for face $S_0S_1S_3$ coefficient. Because the number of edges meeting at every vertex is 3, n and m in section 3 are 3. From vertex condition (3.10) and edge condition (3.11) on each face of tetrahedron, we have

$$AV = Q, \tag{4.1}$$

where

$$A = \begin{pmatrix} \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{8}{3} & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix},$$

$$V = \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, Q = \begin{pmatrix} Q_2 + Q_4 \\ Q_1 + Q_4 \\ Q_1 + Q_2 \\ Q_1 + Q_3 \\ Q_2 + Q_3 \\ Q_3 + Q_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to verify that $detA \neq 0$*, thus, Theorem 2 come into existence.*

We now return to Fig.4 and consider the case when $Q_1 = 1$ and all other face coefficients vanish. Resolving system (4.1), we obtain

$$A = E = F = \frac{1}{22}, B = C = D = \frac{5}{11}, S_0 = S_2 = S_3 = \frac{7}{22}, S_1 = \frac{1}{22}.$$

For each other face we have a corresponding function such that the coefficient for that face is 1 and the other face coefficients are 0. It is easily seen that these four functions sum to 1. Thus we have constructed a basis for $S(M)$ of positive functions that sum to 1.

These functions can be used to design a closed G^1 surface in the following standard fashion. Denote the functions by $B_i, i = 1, 2, 3, 4$. Given any points V_i in $R^3, i = 1, 2, 3, 4$. Consider the parametrically defined surface

$$f = \sum_{i=1}^{4} V_i B_i. \tag{4.2}$$

Then the surface lies in the convex hull of V_1, V_2, V_3, V_4 and moving the points V_i correspondingly alters the shape of the surface. These basis functions are also useful for interpolation.

Theorem 3. *Let* t_1, t_2, t_3, t_4 *be the points at the barycenters of the faces of the tetrahedron. Then given* y_1, y_2, y_3, y_4 *in* R*, there is a unique element* q *of* $S(M)$ *satisfying*

$$q(t_i) = y_i, \ i = 1, 2, 3, 4.$$

Proof. Let B_1 be the function as above with Bèzier coefficient $Q_1 = 1$ in the face with barycenter t_1. Let $q = \sum_1^4 a_j B_j$. Then

$$q(t_1) = \frac{1}{2}a_1 + \frac{1}{6}a_2 + \frac{1}{6}a_3 + \frac{1}{6}a_4 = y_1.$$

There are corresponding equations for $q(t_i), i = 1, 2, 3, 4$. Since the determinant of the coefficient matrix of the system is not null, there is a unique solution for a_1, a_2, a_3, a_4.

Now the space $S(M)$ is of value only in constructing parametrically defined surfaces. Take points $X_i = (x_i, y_i, z_i) \in R^3, i = 1,2,3,4$. Then, by Theorem 3, there are $a_j, b_j, c_j \in, j = 1,2,3,4$, with

$$\sum_{j=1}^{4}(a_j, b_j, c_j)B_j(t_i) = (x_i, y_i, z_i), i = 1,2,3,4,$$

giving a surface of form (4.2) passing through X_1, X_2, X_3, X_4.

The above discussion of spline on tetrahedron is also suitable for hexahedron and octahedron composed of triangular face, we can obtain similar results.

Remarks. The discussion of the paper is only done under the condition of closed triangular mesh. The study of the spline spaces defined on open triangular meshes is also significant.

Example. The closed surface in Fig.5 is generated by freely given four space points whose components are regarded as face coefficients.

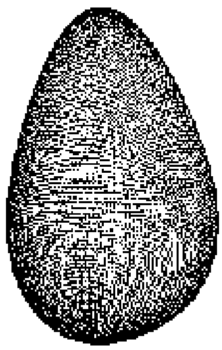

Fig.5. Closed surface generated by four space points

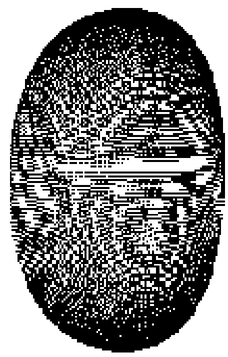

Fig.6. Closed surface generated by six space points

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