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UNIFORM SUPERAPPROXIMATION OF THE DERIVATIVE OF TETRAHEDRAL QUADRATIC FINITE ELEMENT APPROXIMATION *1)

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Abstract

In this paper,we will prove the derivative of tetrahedral quadratic finite element approximation is superapproximate to the derivative of the quadratic Lagrange interpolant of the exact solution in the L^∞-norm, which can be used to enhance the accuracy of the derivative of tetrahedral quadratic finite element approximation to the derivative of the exact solution.

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1. Introduction

Recently, J.H. Brandts and M. Křížek [1] discussed the superconvergence of tetrahedral quadratic finite elements. Their work focused on the superapproximation of the gradient of the quadratic finite element approximation to the gradient of the quadratic Lagrange interpolant of the exact solution in L^2-norm. For the same model problem, utilizing the theory of the discrete Green's function, this paper studies the superapproximation in L^∞-norm.

2. Preliminaries

Let Ω be a convex bounded polyhedral domain in R^3 with Lipschitz boundary and denote by $W^{k,p}(\Omega)$ the usual Sobolev spaces of functions having generalized partial derivatives up to order k in $L^p(\Omega)$ and their usual norm and seminorm by $\|\cdot\|_{k,p}$ and $|\cdot|_{k,p}$, respectively. In addition, we denote by $W_0^{1,p}(\Omega)$ the subspace of $W^{1,p}(\Omega)$ with $\operatorname{supp} u \subset \Omega$ for each $u \in W_0^{1,p}(\Omega)$. In particular, we set

$$H^k(\Omega) = W^{k,2}(\Omega), \qquad H_0^1(\Omega) = W_0^{1,2}(\Omega)$$
$$\|\cdot\|_k = \|\cdot\|_{k,2}, \qquad |\cdot|_k = |\cdot|_{k,2}.$$

In this paper, let $\mathcal{T}^h$ be the same uniform partition of $\bar{\Omega}$ into tetrahedra as in [1], and h be the largest diameter of all element E from the partition $\mathcal{T}^h$. Relative to the partition $\mathcal{T}^h$, let S_h^k be the k-order finite element subspace of $H^1(\Omega)$, and set $S_{0h}^k = S_h^k \cap H_0^1(\Omega)$. Let $L_h : H^2(\Omega) \to S_h^1$ be the linear Lagrange interpolation operator on the vertices of the tetrahedra, and $Q_h : H^2(\Omega) \to S_h^2$ be the quadratic Lagrange interpolation operator on the vertices and midpoints of edges of the tetrahedra.

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Now we introduce the subspace $B_{0h}^2 \subset S_{0h}^2$ of so-called quadratic bubble functions, defined by

$$B_{0h}^2 = \left\{ (I - L_h)v \mid v \in S_{0h}^2 \right\}.$$

This definition induces the following space-decomposition

$$S_{0h}^2 = S_{0h}^1 \oplus B_{0h}^2,$$

which expresses that each $v \in S_{0h}^2$ can be uniquely written as $l + b$ with $l \in S_{0h}^1$ and $b \in B_{0h}^2$ (cf. [1]). This decomposition will be used in our main results. Obviously, B_{0h}^2 is spanned by the basis ψ_i, $(i = 1, \cdots, M)$, where each $\psi_i \in S_{0h}^2$ has a positive value at the midpoint of the internal edge e_i, has norm $|\psi_i|_1 = 1$, and vanishes at all other edges.

Next, we define *discrete* δ function $\delta_z^h \in S_{0h}^2(\Omega)$, *discrete derivative* δ function $\partial_z \delta_z^h \in S_{0h}^2(\Omega)$, L^2 *projection* $Pu \in S_{0h}^2(\Omega)$ of $u \in L^2(\Omega)$, *discrete derivative* Green's function $\partial_z G_z^h \in S_{0h}^2(\Omega)$, and *derivative zhun* Green's function $\partial_z G_z^* \in H_0^1(\Omega)$ as follows [2]:

$$\begin{aligned}
(v,\, \delta_z^h) &= v(z), && \forall v \in S_{0h}^2(\Omega) \\
(u - Pu,\, v) &= 0, && \forall v \in S_{0h}^2(\Omega) \\
(v,\, \partial_z \delta_z^h) &= \partial v(z), && \forall v \in S_{0h}^2(\Omega) \\
(\nabla \partial_z G_z^h,\, \nabla v) &= \partial v(z), && \forall v \in S_{0h}^2(\Omega) \\
(\nabla \partial_z G_z^*,\, \nabla v) &= (\partial_z \delta_z^h,\, v), && \forall v \in H_0^1(\Omega)
\end{aligned}$$

where $S_{0h}^2(\Omega) \subset H_0^1(\Omega)$ is the quadratic tetrahedral finite element space. Obviously, $\partial_z G_z^h$ is the finite element approximation to $\partial_z G_z^*$.

In addition, for $u \in H_0^1(\Omega)$, we can easily obtain

$$(\nabla \partial_z G_z^*,\, \nabla u) = (\partial_z \delta_z^h,\, u) = (\partial_z \delta_z^h,\, Pu) = \partial_z Pu(z).$$

Further, the following stability estimate holds

$$\|Pu\|_{1,\,q} \le C\|u\|_{1,\,q} \quad for \;\; 3 < q \le \infty,$$

which can be similarly proved as Corollary 2 in Zhu,Lin[2, pp104].

Finally, we will give the following two fundamental assumptions which are needed in next sections (cf. [2, 3]):

(A1). For the model problem (1) considered in Section 3, there exist $1 < q_0 \le \infty$ and a constant $C(p)$ such that the following a priori estimate holds

$$\|u\|_{2,\,p,\,\Omega} \le C(p)\|f\|_{0,\,p,\,\Omega},\ \forall\, 1 < p < q_0,\ u \in W^{2,\,p}(\Omega) \cap W_0^{1,\,p}(\Omega).$$

(A2). For each $v \in W^{2,\,q}(\Omega) \cap W_0^{1,\,q}(\Omega)$ there exists a $\chi \in S_{0h}^2$ such that

$$\|v - \chi\|_{1,\,q} \le Ch\|v\|_{2,\,q} \quad for\ 1 \le q \le \infty.$$

In this paper we shall use letter C to denote a generic constant which may not be the same in each occurrence.

3. The Tetrahedral Quadratic Finite Element Method

Let us consider the following boundary value problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and the associated weak formulation is

$$(\nabla u,\, \nabla v) = (f,\, v),\ \forall v \in H_0^1(\Omega).$$

The finite element method is to find $u_h \in S_{0h}^2$ such that

$$(\nabla u_h,\, \nabla v) = (f,\, v),\ \forall v \in S_{0h}^2.$$

Clearly, there is Galerkin orthogonality relation

$$(\nabla(u - u_h),\, \nabla v) = 0,\ \forall v \in S_{0h}^2. \tag{2}$$

4. Some Propositions and Lemmas

In this section, we will introduce some propositions and lemmas that are needed in the proof of our main theorem.

Proposition 1. *Suppose $\mathcal{T}^h$ is a uniform tetrahedral partition, and Q_h is defined as in Section 2. Then for all $v \in W^{3,\infty}(\Omega)$ and each element E from the partition $\mathcal{T}^h$, we have*

$$|v - Q_h v|_{1,\,\infty,\,E} \le Ch^2 |v|_{3,\,\infty,\,E}. \tag{3}$$

Proposition 2[2]**.** **(Sobolev integral identity)** *Let $\Omega \subset R^n$ be a bounded open domain, $S \subset \Omega$ a closed ball such that Ω is star-shaped with respect to S, and $u \in C^m(\Omega)$. Then $u(x)$ can be expressed by*

$$u(x) = \sum_{|\alpha| \le m-1} l_\alpha(u) x^\alpha + \int_\Omega \frac{1}{r^{n-m}} \sum_{|\alpha|=m} Q_\alpha(x,\, y) D^\alpha u(y)\, dy,$$

where $l_\alpha(u)$ is a linear functional on $C^m(\Omega)$ defined by

$$l_\alpha(u) = \int_\Omega \zeta_\alpha(y) u(y)\, dy,$$

and $\zeta_\alpha(y)$ is a continuous bounded function with respect to variable y with $|\alpha| \le m-1$. Moreover, $Q_\alpha(x,\, y)$ with $|\alpha| = m$ is a bounded infinite-times differentiable function with respect to variables x and y. In addition,

$$r = |x - y| = \left(\sum_{j=1}^n |x_j - y_j|^2 \right)^{\frac{1}{2}} \quad for\ x,\, y \in \Omega.$$

Proposition 3. *Let u_h be the finite element approximation of $u \in H^2(\Omega)$, then*

$$\|u - u_h\|_0 \le Ch^2 \|u\|_2,$$

and

$$\|u - u_h\|_1 \le Ch \|u\|_2.$$

Lemma 1. *Suppose $\partial_z \delta_z^h$ is defined as in Section 2, then*

$$|\partial_z \delta_z^h(x)| \le Ch^{-4} e^{-Ch^{-1}|x-z|},\ \forall x,\, z \in \Omega, \tag{4}$$

and

$$\|\partial_z \delta_z^h\|_{0,\,q} \le Ch^{-4+\frac{3}{q}}, \quad for\ 1 \le q \le \infty, \tag{5}$$

where C is a positive constant independent of x,z and h.

With the same argument as in [2] (cf. [2], Theorem 3.6, 100-103), Lemma 1 can be easily proved.

Lemma 2. *Suppose $k \geq 1$, $q_0 > 2$, and $\partial_z G_z^*$ and $\partial_z G_z^h$ are defined as in Section 2, then we have*

$$\|\partial_z G_z^* - \partial_z G_z^h\|_{1,\,p} \leq Ch^{\frac{3}{p}-3}, \tag{6}$$

where C is a positive constant independent of z and h, and $2 \leq p < q_0$.

Proof. Let $g = \partial_z G_z^*$, $g_h = \partial_z G_z^h$, and g_I be the interpolant of g. Then by (5), interpolation error estimate, and a priori estimate, i.e., assumption (A1), we obtain

$$\|g - g_I\|_{1,\,p} \leq Ch\|\nabla^2 g\|_{0,\,p} \leq Ch\|\partial_z \delta_z^h\|_{0,\,p} \leq Ch^{\frac{3}{p}-3}.$$

Further, by inverse estimate we have

$$\|g_I - g_h\|_{1,\,p} \leq Ch^{\frac{3}{p}-\frac{3}{2}}\|g_I - g_h\|_{1,\,2}.$$

However, by Proposition 3, Lemma 1, and the triangular inequality, we obtain

$$\begin{aligned}\|g_I - g_h\|_{1,\,2} &\leq \|g_I - g\|_{1,\,2} + \|g - g_h\|_{1,\,2}\\ &\leq Ch\|g\|_{2,\,2} + Ch\|g\|_{2,\,2}\\ &\leq Ch\|\partial_z \delta_z^h\|_{0,\,2}\\ &\leq Ch^{-\frac{3}{2}}.\end{aligned}$$

Thus,

$$\|g_I - g_h\|_{1,\,p} \leq Ch^{\frac{3}{p}-3}.$$

As a result,

$$\|g - g_h\|_{1,\,p} \leq \|g - g_I\|_{1,\,p} + \|g_I - g_h\|_{1,\,p} \leq Ch^{\frac{3}{p}-3}.$$

Hence, the proof of Lemma 2 is completed.

Lemma 3. $\|\partial_z G_z^*\|_0 \leq Ch^{-\frac{1}{2}}|\ln h|^{\frac{2}{3}}$.

Proof. Setting $g = \partial_z G_z^*$, $g_h = \partial_z G_z^h$ and taking $w \in H_0^1(\Omega)$ such that

$$(\nabla v,\, \nabla w) = (v,\, g),\ \forall v \in H_0^1(\Omega),$$

by the stability estimate we obtain

$$\|g\|_0^2 = (g,\, g) = (\nabla g,\, \nabla w) = \partial_z P w(z) \leq |w|_{1,\,\infty}, \tag{7}$$

where Pw is the L^2-projection of w.

By Proposition 2, we derive

$$|w|_{1,\,\infty} \leq C(q)\|w\|_{2,\,q},$$

where $C(q) \leq C(q-3)^{-\frac{2}{3}},\ (q \to 3+0)$.

Hence, by a priori estimate, we have

$$|w|_{1,\,\infty} \leq C(q-3)^{-\frac{2}{3}}\|g\|_{0,\,q}, \quad \textit{for } 3 < q < q_0. \tag{8}$$

For $3 < q < q_0$, taking $1 < q' = \frac{q}{q-1} < \frac{3}{2}$, then by assumption (A2), there exist $v \in W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega)$ and $\chi \in S_{0h}^2$ such that

$$\begin{aligned} \|g - g_h\|_{0,q}^q &= \big(|g - g_h|^{q-1} sgn(g - g_h),\, g - g_h\big) \\ &= (\nabla v,\, \nabla(g - g_h)) \\ &= (\nabla(v - \chi),\, \nabla(g - g_h)) \\ &\le C\|v - \chi\|_{1,q'} \|g - g_h\|_{1,q} \\ &\le Ch\|v\|_{2,q'} \|g - g_h\|_{1,q}. \end{aligned}$$

By a priori estimate again, we have

$$\|v\|_{2,q'} \le C \left\| |g - g_h|^{q-1} \right\|_{0,q'} \le C\|g - g_h\|_{0,q}^{q-1}.$$

Thus, it follows that

$$\|g - g_h\|_{0,q} \le Ch\|g - g_h\|_{1,q}.$$

By Lemma 2 and inverse estimate,

$$\|g - g_h\|_{0,q} \le Ch\|g - g_h\|_{1,q} \le Ch^{\frac{3}{q}-2}, \tag{9}$$

and

$$\|g_h\|_{0,q} \le Ch^{\frac{3}{q}-\frac{3}{2}} \|g_h\|_{0,2}. \tag{10}$$

From (9) and (10), using the triangular inequality, we derive

$$\|g\|_{0,q} \le Ch^{\frac{3}{q}-2} + Ch^{\frac{3}{q}-\frac{3}{2}} \|g_h\|_{0,2}. \tag{11}$$

Therefore, from (7), (8) and (11), we obtain

$$\|g\|_0^2 \le C(q-3)^{-\frac{2}{3}} h^{\frac{3}{q}-2} + C(q-3)^{-\frac{2}{3}} h^{\frac{3}{q}-\frac{3}{2}} \|g_h\|_0.$$

However, by Proposition 3, Lemma 1, and the triangular inequality, we have

$$\begin{aligned} \|g_h\|_0 &\le \|g_h - g\|_0 + \|g\|_0 \\ &\le Ch^2\|g\|_2 + \|g\|_0 \\ &\le Ch^2\|\partial_z \delta_z^h\|_0 + \|g\|_0 \\ &\le Ch^{-\frac{1}{2}} + \|g\|_0 \end{aligned}$$

Thus,

$$\|g\|_0^2 \le C(q-3)^{-\frac{2}{3}} h^{\frac{3}{q}-2} + C(q-3)^{-\frac{2}{3}} h^{\frac{3}{q}-\frac{3}{2}} \|g\|_0$$

By Young inequality, we have

$$\|g\|_0 \le C(q-3)^{-\frac{2}{3}} h^{\frac{3}{q}-\frac{3}{2}}.$$

Since C is independent of q, in particular, taking $q = 3 + \left(\ln \frac{1}{h}\right)^{-1}$, we obtain

$$\|g\|_0 \le Ch^{-\frac{1}{2}} |\ln h|^{\frac{2}{3}}.$$

Therefore, Lemma 3 is proved.

Remark 1. In fact, for a general convex polyhedral domain, we have known $q_0 > 2$. However, if the biggest dihedra of the boundary of a convex polyhedron is smaller than $\frac{\sqrt{2}}{2}\pi$, one can discover $q_0 > 3$ (cf. [4]).

Lemma 4. $\|\partial_z G_z^* - \partial_z G_z^h\|_0 \le Ch^{-\frac{1}{2}}$.

Proof. By Lemma 1, a priori estimate, and L^2 estimate,

$$\|\partial_z G_z^* - \partial_z G_z^h\|_0 \le Ch^2\|\partial_z G_z^*\|_2 \le Ch^2\|\partial_z \delta_z^h\|_0 \le Ch^2 \cdot Ch^{-4+\frac{3}{2}} \le Ch^{-\frac{1}{2}}.$$

Thus, the proof is completed.

Lemma 5. *Let $\mathcal{T}^h$ be a uniform partition, $\{\phi_i\}$ and $\{\psi_i\}$ be the basis functions sets of S_{0h}^1 and B_{0h}^2, respectively. Then for all cubic polynomials p, we have*

$$(\nabla(p - Q_h p),\, \nabla\phi_i)_{T_i} = 0 \tag{12}$$

and

$$(\nabla(p - Q_h p),\, \nabla\psi_i)_{S_i} = 0, \tag{13}$$

where $T_i = supp\ \phi_i$, and $S_i = supp\ \psi_i$.

Remark 2. (13) has been proved in [1], and (12) can be similarly proved.

Lemma 6. *Let $v_h \in S_{0h}^2$. Then $v_h = l_h + b_h$,*

$$|b_h|_0 \le C|v_h|_0 \tag{14}$$

and

$$|l_h|_0 \le C|v_h|_0, \tag{15}$$

where $l_h = L_h v_h \in S_{0h}^1$ and $b_h = (I - L_h)v_h \in B_{0h}^2$.

Proof. By the interpolation error estimate, there exists a constant $C > 0$ such that

$$|(I - L_h)v_h|_{0,\,E} \le C|v_h|_{0,\,E},$$

i.e.,

$$|b_h|_{0,\,E} \le C|v_h|_{0,\,E}.$$

Summing over all elements in the partition $\mathcal{T}^h$ proves (14). Applying the triangular inequality and $l_h = v_h - b_h$, we immediately obtain (15).

Lemma 7. *Under the conditions of Lemma 5 and Lemma 6, let $l_h = \sum_i \beta_i\phi_i \in S_{0h}^1$ and $b_h = \sum_i \alpha_i\psi_i \in B_{0h}^2$. Then,*

$$\sum_i |\beta_i| \le Ch^{-\frac{3}{2}}|l_h|_0 \tag{16}$$

and

$$\sum_i |\alpha_i| \le Ch^{-\frac{3}{2}}|b_h|_0. \tag{17}$$

Proof. First define an affine transformation by

$$F : \hat{x} \in \hat{E} \longrightarrow x = B\hat{x} + b \in E$$

such that

$$E = F(\hat{E}),$$

where $B = (b_{ij})$ is a matrix of order 3×3. Then, writing $\hat{v}(\hat{x}) = v(F\hat{x})$, for all $v \in L^2(E)$, we have

$$|\hat{v}|_{0,\,\hat{E}} \le C|detB|^{-\frac{1}{2}}|v|_{0,\,E} \tag{18}$$

and

$$|v|_{0,\,E} \le C|detB|^{\frac{1}{2}}|\hat{v}|_{0,\,\hat{E}}, \tag{19}$$

moreover,

$$|detB| \le Ch^3 \tag{20}$$

(cf. [2] 79-81).

By the equivalence of norms in the finite-dimensional space, we have

$$\sum_i |\beta_i| \leq C|\hat{l}_h|_{0,\,\hat{E}} \tag{21}$$

and

$$\sum_i |\alpha_i| \leq C|\hat{b}_h|_{0,\,\hat{E}}. \tag{22}$$

From (18), (20), (21), and taking $v = l_h$, we derive

$$\sum_i |\beta_i| \leq Ch^{-\frac{3}{2}}|l_h|_{0,\,E},$$

which proves (16) by summing over all elements. (17) can be similarly proved.

5. The Main Theorem

Theorem. *Let $u \in W^{4,\,\infty}(\Omega)$, u_h be its tetrahedral quadratic finite element approximation, and $Q_h u$ the quadratic Lagrange interpolant of u. Then we have*

$$|u_h - Q_h u|_{1,\,\infty,\,\Omega} \leq C(u)h^3|\ln h|^{\frac{2}{3}},$$

where $C(u)$ is a positive constant independent of h.

Proof. Since $\partial_z G_z^h \in S_{0h}^2$ having decomposition $\partial_z G_z^h = l_h + b_h$ with $l_h = L_h \partial_z G_z^h \in S_{0h}^1$ and $b_h = (I - L_h)\partial_z G_z^h \in B_{0h}^2$ is the finite element approximation of $\partial_z G_z^*$, we have

$$\begin{aligned} \partial(u_h - Q_h u)(z) &= \big(\nabla\partial_z G_z^h,\, \nabla(u_h - Q_h u)\big) \\ &= \big(\nabla\partial_z G_z^h,\, \nabla(u - Q_h u)\big) \\ &= (\nabla l_h,\, \nabla(u - Q_h u)) + (\nabla b_h,\, \nabla(u - Q_h u))\,. \end{aligned} \tag{23}$$

Let $l_h = \sum_i \beta_i \phi_i$ and $b_h = \sum_i \alpha_i \psi_i$, then by (14) and (16), we obtain

$$\begin{aligned} |(\nabla l_h,\, \nabla(u - Q_h u))| &\leq \textstyle\sum_i |\beta_i|\,|(\nabla\phi_i,\, \nabla(u - Q_h u))| \\ &\leq Ch^{-\frac{3}{2}}|l_h|_0 \cdot |(\nabla\phi_j,\, \nabla(u - Q_h u))| \\ &\leq Ch^{-\frac{3}{2}}|\partial_z G_z^h|_0 \cdot \left|(\nabla\phi_j,\, \nabla(I - Q_h)u)_{T_j}\right|, \end{aligned} \tag{24}$$

where $T_j = supp\ \phi_j$.

By Proposition 1 and Lemma 5, for all cubic polynomials p, we obtain

$$\begin{aligned} \left|(\nabla\phi_j,\, \nabla(I - Q_h)u)_{T_j}\right| &= \left|(\nabla\phi_j,\, \nabla(I - Q_h)(u - p))_{T_j}\right| \\ &\leq |\nabla(I - Q_h)(u - p)|_{0,\,\infty,\,T_j} \cdot |\nabla\phi_j|_{0,\,1,T_j} \\ &\leq Ch^2|u - p|_{3,\,\infty,\,T_j} \cdot Ch^2 \\ &\leq Ch^4|u - p|_{3,\,\infty,\,T_j}. \end{aligned} \tag{25}$$

Let p be the cubic Lagrange interpolant of u, then

$$\left|(\nabla\phi_j,\, \nabla(I - Q_h)u)_{T_j}\right| \leq Ch^5|u|_{4,\,\infty,\,T_j} \leq Ch^5|u|_{4,\,\infty,\,\Omega}. \tag{26}$$

Further, applying Lemma 3, Lemma 4, and the triangular inequality, we derive

$$|\partial_z G_z^h|_0 \leq Ch^{-\frac{1}{2}}|\ln h|^{\frac{2}{3}}. \tag{27}$$

From (24), (26) and (27), it follows that

$$|(\nabla l_h,\, \nabla(u - Q_h u))| \le Ch^3\, |\ln h|^{\frac{2}{3}}\, |u|_{4,\,\infty,\,\Omega}. \tag{28}$$

Similarly, we can obtain

$$|(\nabla b_h,\, \nabla(u - Q_h u))| \le Ch^3\, |\ln h|^{\frac{2}{3}}\, |u|_{4,\,\infty,\,\Omega}. \tag{29}$$

Finally, Theorem follows from (23), (28) and (29).

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