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EXPECTED NUMBER OF ITERATIONS OF INTERIOR-POINT ALGORITHMS FOR LINEAR PROGRAMMING *1)

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Abstract

We study the behavior of some polynomial interior-point algorithms for solving random linear programming (LP) problems. We show that the expected and anticipated number of iterations of these algorithms is bounded above by $O(n^{1.5})$. The random LP problem is Todd's probabilistic model with the Cauchy distribution.

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1. Introduction

Since Karmarkar [4] introduced his $O(nL)$-iteration projective algorithm, the area of interior point algorithms for linear programming have developed rapidly. Many other algorithms have been introduced to the growing literature on interior point algorithms, for examples, path-following algorithms; potential reduction algorithms; and predictor-corrector algorithms, etc. The best known worst-case iteration complexity for interior point algorithms is $O(\sqrt{n}L)$, where n is the number of variables and L is the input data length of the LP problems.

In practice the interior point algorithms also performed compatetive with simplex algorithm. People (e.g.,Lustig et la. [6], Yang and Huang [10]) have observed that the number of iterations needed to solve the LP problems is $O(\ln n)$ using regression. Therefore there is a gap between the theoretical worst case complexity and practical performance of the interior point algorithms. Ye [11] showed that the anticipated number of iterations of interior point algorithms is bounded above by $O(\sqrt{n}\ln n)$. Recently, Anstreicher et al. [1] have obtained expected number of iterations bound of $O(n\ln n)$ for a variant of degenerate random LP model (Model II of Todd [9]) using the infeasible primal-dual algorithms of Potra [8]. Huang [3] has shown that the expected number of iterations of some feasible interior point algorithms (e.g., Kojima et la. [5]) is bounded above by $O(n^{1.5})$ for a nondegenerate random LP model (Model I of Todd [9]).

In this paper, we will show that the expected and anticipated number of iterations of some interior point algorithms is bounded above by $O(n^{1.5})$ for solving a random LP model which is an extension of Todd's model I in [9].

The paper is organized as follows. In section 2, we introduce the random LP model and some useful results. We review the stopping criterion for polynomial interior-point algorithms in section 3. Section 4 derives a bound for the expected number of iterations for solving the random LP model using certain interior point algorithms. We show the anticipated result in section 5. Finally we will give some concluding remarks in section 6.

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2. The Probabilistic Model

We consider the following probabilistic model which is an extension of Todd's model I in [9].

(LP) minimize $e^T x$
subject to $Ax = b,\ x \geq 0,$

where $b = Ae$ and $e \in R^n$ is a vector of all ones, $A = (a_{ij}) \in R^{m\times n}$ is a random matrix whose entries are independently and identically distributed as Cauchy distribution with characteristic function $e^{-c|t|}$ $(c > 0)$. It's dual form can be stated as:

(LD) minimize $e^T s$
subject to $s = e - A^T y, s \geq 0,$

since $b^T y = (Ae)^T y = e^T A^T y = e^T(e - s) = n - e^T s$. Therefore max $b^T y$ is equivalent to min $e^T s$. The following lemma will help us to derive the distribution of the optimal solution of above random LP model.

Lemma 2.1. *Consider the system $Bx = d$, where $B = (a_{ij}) \in R^{m\times m}$ is a random matrix such that a_{ij} $(i, j = 1, \cdots, m)$ are independent and identical(iid) Cauchy random variables, and $d \in R^m$ is a random vector such that $d_i (i = 1, \cdots, m)$ are iid Cauchy random variables. Assume the columns of B and d are independent. Then the random variables x_k $(k = 1, \cdots, m)$ are distributed as $\frac{\lambda_k}{\lambda_0}$ $(k = 1, \cdots, m)$ where λ_k $(k = 0, 1, \cdots, m)$ are independent and identical random variables with Cauchy distribution.*

Proof. The proof is similar to the proof in Girko [2] where the matrix B is not a square matrix. For completion we include it here. It is easy to see that $detB \neq 0$ with probability one. By Cramer's rule we have

$$x_k = \frac{\sum_{i=1}^m d_i B_{ik}}{\sum_{i=1}^m a_{ik} B_{ik}} = \frac{(\sum_{i=1}^m d_i B_{ik}) R^{-1}}{(\sum_{i=1}^m a_{ik} B_{ik}) R^{-1}}, \qquad (*)$$

where B_{ik} $(i = 1, \cdots, m)$ is the cofactor of the a_{ik} in the matrix B, $R = (\sum_{i=1}^m |B_{ik}|)$ and $|B_{ik}|$ is the determinant of B_{ik}.

Next we calculate the joint characteristic function of the numerator and denominator of $(*)$ using conditional expectation:

$$\begin{aligned}
& Eexp\{it\sum_{i=1}^m d_i B_{ik} R^{-1} + i\tau \sum_{i=1}^m a_{ik} B_{ik} R^{-1}\} \\
&= EE[exp\{it\sum_{i=1}^m d_i B_{ik} R^{-1} + i\tau \sum_{i=1}^m a_{ik} B_{ik} R^{-1}\} | a_{\nu\mu}, \nu = 1, \cdots, m, \mu = \{1, \cdots, m\}/k] \\
&= E[\prod_{i=1}^m exp\{-c|tB_{ik}|R^{-1}\} \prod_{i=1}^m exp\{-c|\tau B_{ik}|R^{-1}\}] \\
&= exp\{-c|t|\} exp\{-c|\tau|\}.
\end{aligned}$$

Therefore the numerator and denominator are independent and identically distributed as Cauchy distribution.

Using lemma 2.1 we can obtain following two lemmas which discuss the distribution of the vertices of (LP) and (LD).

Lemma 2.2. *Let λ_0, λ_1, $\cdots$, λ_m be independent and identical Cauchy random variables. Then a vertex (may not be feasible) of (LP) has its basic variables distributed like $1 + d\frac{\lambda_i}{\lambda_0}$ for $i = 1, \cdots, m$ $(d = n - m)$ and its nonbasic variables equal 0. Furthermore, every vertex of (LP) is nondegenerate with probability one.*

Proof. Assume that first m columns of A are basic columns and is denoted by B, and

$x = (x_B, o)^T$ is a vertex of (LP) corresponding to B, then

$$(B, N)(x_B, o)^T = (B, N)(e_B, e_N)^T,$$

which implies

$$x_B = e_B + B^{-1}Ne_N = e_B + (B^{-1}a_{m+1}, \cdots, B^{-1}a_n)e_N = e_B + (y_{m+1}, \cdots, y_n)e_N.$$

Let $Y = (y_{ij}) = B^{-1}N$, we now need to show that each component of every row of Y is distributed like $\frac{\eta_j}{\eta_0}$, where η_j $(j = m+1, \cdots, n)$ and η_0 are identical and independent Cauchy random variables. Let $y_j = B^{-1}a_j$ and $y_l = B^{-1}a_l$ be any two columns of Y $(j \neq l; j, l = m+1, \cdots, n)$, and

$$y_{kj} = \frac{\sum_{i=1}^m a_{ij}B_{ik}}{\sum_{i=1}^m a_{ik}B_{ik}} = \frac{(\sum_{i=1}^m a_{ij}B_{ik})R^{-1}}{(\sum_{i=1}^m a_{ik}B_{ik})R^{-1}} = \frac{\eta_j}{\eta_0}$$

and

$$y_{kl} = \frac{\sum_{i=1}^m a_{il}B_{ik}}{\sum_{i=1}^m a_{ik}B_{ik}} = \frac{(\sum_{i=1}^m a_{il}B_{ik})R^{-1}}{(\sum_{i=1}^m a_{ik}B_{ik})R^{-1}} = \frac{\eta_l}{\eta_0}$$

be two kth components of y_j and y_l $(k = 1, \cdots, m)$, where $R = \sum_{i=1}^m |B_{ik}|$, then similar to the proof in lemma 2.1 we can show that η_j, η_l and η_0 are independently and identically distributed as Cauchy random variables. Note that for any i.i.d. Cauchy random variables X and Y with characteristic function $e^{-c|t|}$ $(c > 0)$, we have

$$X + Y =: 2X$$

Since

$$E[exp\{it(X+Y)\}] = E[exp\{itX\}exp\{itY\}] = e^{-c|2t|}.$$

Therefore, x_k is distributed like $1 + d\frac{\lambda_k}{\lambda_0}$ $(k = 1, \cdots, m)$. It is easy to see that $x = (x_B, 0)$ is nondegenerate with probability one.

A similar result applies to the dual form of (LD).

Lemma 2.3. *Let η_0, η_1, $\cdots$, η_d be independent Cauchy random variables. Then a vertex (may not be feasible) of (LD) has its basic variables distributed like $1 - m\frac{\eta_i}{\eta_0}$ for $i = 1, \cdots, d$ and its nonbasic variables equal 0. Furthermore, every vertex of (LD) is nondegenerate with probability one.*

Proof. Let $s = (0, s_N)^T$ be a vertex of (LD) and (B, N) be a partition of A with respect to s, then

$$(0, s_N)^T = (e_B, e_N)^T - (B^Ty, N^Ty)^T$$

implies that

$$e_B = B^Ty$$

and

$$\begin{aligned} s_N &= e_N - N^Ty \\ &= e_N - N^T(B^T)^{-1}e_B \\ &= e_N - e_B^T(B^{-1}N) \\ &= e_N - e_B^T(B^{-1}a_{m+1}, \cdots, B^{-1}a_n). \end{aligned}$$

By lemma 2.1 we know s_j is distributed like $1 - m\frac{\eta_j}{\eta_0}$ $(j = 1, \cdots, d)$ conditioned on that $s_j \geq 0$, where η_j $(j = 0, \cdots, d)$ are independently and identically distributed as Cauchy distribution. It is also easy to see that $s = (0, s_N)$ is nondegenerate with probability one.

We will use Lemma 2.2 and Lemma 2.3 to analyze the distribution functions of the components of an optimal solution of (LP) and (LD).

3. A Stopping Criterion for Interior Point Algorithms

If the data for a LP problem are rational, then a primal-dual polynomial algorithm for LP stops whenever it finds a pair of primal-dual feasible solutions (x, s) such that $x^Ts \leq 2^{-L}$, where

L is the input data length of the (LP) and (LD). This is the theoretical base of all polynomial algorithms for LP. Since our LP model were randomly drawn from real numbers, therefore we can't use this stopping criterion any more. In this paper we will use a new theoretical termination criterion of Mehrotra and Ye [7] as was used in Huang [3].

Define $\sigma(x)$ to be the index set such that

$$\sigma(x) = \{i : x_i > 0\}.$$

It is well known that if (LP) and (LD) are both feasible, then there is a unique σ^* such that for all strict complementary optimal solutions (x^*, s^*) of (LP) and (LD), we have

$$\sigma(x^*) = \sigma^*, \qquad \sigma(s^*) = \bar{\sigma}^* = \{1, \cdots, n\}/\sigma^*.$$

Let matrix B consists of those columns in A corresponding to the partition σ^* and the rest form matrix N, and (x_B, x_N) and (s_B, s_N) denote those corresponding primal-dual variables respectively. Then, the optimal face for the primal is

$$\Omega_p = \{x : Bx_B = b, \quad x_B \geq 0, \quad x_N = 0\}, \tag{1}$$

and the dual optimal face is

$$\Omega_d = \{s : s_N = c_N - N^T y, \quad s_N \geq 0, \quad s_B = 0\}. \tag{2}$$

Define

$$\xi_p = \min_{j \in \sigma^*} \{\max\ x_j, \quad s.t. \quad x \in \Omega_p\}, \tag{3}$$

and

$$\xi_d = \min_{j \in \bar{\sigma}^*} \{\max\ s_j, \quad s.t. \quad s \in \Omega_d\}. \tag{4}$$

Let

$$\xi = \min(\xi_p, \xi_d). \tag{5}$$

Obviously, $\xi > 0$. Note that if both primal and dual are nondegenerate, then both Ω_p and Ω_d contains only one point. Let $\{(x^k, s^k)\}$ be the solution sequence generated by a primal-dual interior point algorithm (e.g., Kojima et al. [5]) such that

$$\frac{x_j^k s_j^k}{(x^k)^T s^k} \geq \Omega(1/n) \qquad \text{for all j}, \tag{6}$$

where $\Omega(x)$ denotes a function such that $\Omega(x) \geq cx$ for some constant c. We also denote $O(x)$ a function such that $O(x) \leq c'(x)$ for some constants c'. Let

$$\sigma^k = \{j : x_j^k \geq s_j^k\}.$$

Then we have the following theorem.

Theorem 3.1[3,7]. *Let $\{(x^k, s^k)\}$ be a primal-dual solution sequence generated by an interior point algorithm satisfying* (6). *If $(x^k)^T s^k < O(\xi^2/n)$, then*

$$\sigma^k = \sigma^*.$$

Theorem 3.1 states that the optimal partition can be identified whenever the duality gap is less than $O(\xi^2/n)$ for LP problems. A detailed proof can be found in Mehrotra and Ye [7] and Huang [3]. We can find an optimal primal solution at that time by projecting x_B^k onto the hyperplane $\{x_B : Bx_B = b\}$ and find an optimal dual solution similarly. Note that in this paper we will focus on the random LP problem described in Section 2. Since the LP problem is nondegenerate with probability one, the optimal partition B will be an optimal basis with probability one. Thus, an optimal primal and dual solution pair can be obtained immediately by computing $x_B^* = B^{-1}b$ and $y^* = B^{-T}c_B$. Therefore, we can use $x^T s \leq O(\xi^2/n)$ as a stopping criterion to terminate most primal-dual interior-point algorithms which generates sequences satisfying (6). One draw back of theorem 3.1 is that we don't know the value of ξ before hand. Therefore it only has theoretical value.

Now consider applying the stopping criterion in theorem 3.1 to our random model, since $(x^0, s^0) = (e, e)$ is a feasible primal-dual interior point, we can use it as an starting point to solve the random LP. The number of iterations of interior point algorithms satisfying (6) will depend on $\ln \xi$ as stated by following theorem.

Theorem 3.2. *Let $k = k(A)$ denote the number of iterations to solve the random LP problem using an interior point algorithm satisfying* (6) *(e.g., Kojima et al. [5]), then*

$$k = -\sqrt{n}O(\ln \xi) + O(\sqrt{n} \ln n).$$

The proof is easy and can be found in theorem 4.1 of Huang [3]. Hence the expected value of $\ln \xi$ will be a key to obtain the expected number of iterations for interior point algorithms. Next section we will derive a bound for expected value of $\ln \xi$.

4. The Expected Number of Iterations

Let (x, s) be an optimal primal-dual solution of (LP)-(LD). Since the random (LP)-(LD) model is nondegenarate with probability one, hence we can assume, without loss of generality, that $\sigma^* = \{1, \cdots, m\}$. Once σ^* is determined, then partition B and N of columns of A is determined, and x_B and s_N is also determined and their components are distributed as lemma 2.2 and lemma 2.3. Because the random model does not favor any particular index set as a basis, therefore

$$P(\sigma^* = \{1, \cdots, m\}) = \frac{1}{C_n^m}.$$

Let Ω_p and Ω_d be defined as in (1) and (2) of section 3, and $x = (x_B, 0) \in \Omega_p$ be a vertex on primal optimal face of our model and $s = (0, s_N) \in \Omega_d$ be a vertex on dual optimal face. Similar to section 3 we define, for our random model,

$$\xi_p = \min\{x_1, \cdots, x_m | \sigma^* = \{1, \cdots, m\}\},$$

$$\xi_d = \min\{s_{m+1}, \cdots, s_n | \sigma^* = \{1, \cdots, m\}\},$$

and

$$\xi = \min(\xi_p, \xi_d). \tag{7}$$

Now we are ready to derive a bound for expected value of $\ln \xi$. We prove some lemmas first. The following lemma is basically the lemma 3.1 of Huang [3].

Lemma 4.1. *Let x_i $(i = 1, \cdots, n)$ be continuous random variables with probability density functions (p.d.f.) $f_{x_i}(u)$ $(i = 1, \cdots, n)$, and let*

$$y = min\{x_1, \cdots, x_n\}$$

with p.d.f. $f_y(u)$, then

$$f_y(u) \leq \sum_{i=1}^{n} f_{x_i}(u).$$

The following lemma shows that $f_\xi(u)$, the p.d.f. of ξ, is bounded above by a constant.

Lemma 4.2. *Let (x, s) be an optimal vertex of (LP) and (LD) and ξ be defined as in (7), let $f_\xi(u)$ denotes the p.d.f. of ξ, then*

$$f_\xi(u) \leq nC_n^m, \qquad \text{for} \quad 0 < u < \frac{1}{2}.$$

Proof. From Lemma 2.2 and Lemma 2.3 we know x_i $(i = 1, \cdots, m)$ and s_j $(j = m + 1, \cdots, n)$ are distributed as $1+d\lambda_i/\lambda_0$ and $1-m\eta_j/\eta_0$ conditioned on that they are nonnegative respectively, and λ_i/λ_0 and η_j/η_0 are distributed as the ratio of two independent Cauchy distributions and are identically distributed as $\frac{2}{\pi^2}\frac{\ln|u|}{\pi^2-1}$ by [2]. Since $P(x_i \geq 0) = P(\lambda_i/\lambda_0 \geq -\frac{1}{d}) \geq \frac{1}{2}$ and $P(s_j \geq 0) = P(\eta_j/\eta_0 \leq \frac{1}{m}) \geq \frac{1}{2}$, therefore x_i (conditioned on $x_i \geq 0$) has the p.d.f.

$$f_{x_i}(u) = \frac{2c_1}{d\pi^2}\frac{\ln|\frac{u-1}{d}|}{(\frac{u-1}{d})^2-1}, \qquad i = 1, \cdots, m,$$

where $c_1 = P(x_i \geq 0)^{-1}$, and s_j (conditioned on $s_j \geq 0$) has the p.d.f.

$$f_{s_j}(u) = \frac{2c_2}{m\pi^2}\frac{\ln|\frac{1-u}{m}|}{(\frac{1-u}{m})^2-1}, \qquad j = m+1, \cdots, n,$$

where $c_2 = P(s_j \geq 0)^{-1}$. Let $0 < u < \frac{1}{2}$, from Lemma 4.1 we have

$$\begin{aligned} f_\xi(u) &\leq \frac{1}{C_n^{m-1}}\{\sum_{i=1}^{m} f_{x_i}(u) + \sum_{j=m+1}^{n} f_{s_j}(u)\} \\ &= C_n^m\{\frac{2mc_1}{d\pi^2}\frac{\ln|\frac{u-1}{d}|}{(\frac{u-1}{d})^2-1} + \frac{2dc_2}{m\pi^2}\frac{\ln|\frac{1-u}{m}|}{(\frac{1-u}{m})^2-1}\} \\ &\leq C_n^m\{\frac{4m}{d\pi^2} + \frac{4d}{m\pi^2}\} \\ &\leq nC_n^m, \end{aligned}$$

where the third inequality uses the fact $\ln(1 + x) < x, x > -1, x \neq 0$ and $\ln|\frac{u-1}{d}| = \ln(1 + \{|\frac{u-1}{d}| - 1\}) < |\frac{u-1}{d}| - 1$.

Now we are ready to give a lower bound for the expected value of $\ln\xi$.

Lemma 4.3. *Let ξ be defined as in (7), then*

$$E(\ln\xi) \geq -2n - 2\ln n - 1.$$

Proof. Let $a = nC_n^m$, then

$$\begin{aligned} E(\ln\xi) &= \int_0^\infty \ln u f_\xi(u)du \\ &= \int_0^{\frac{1}{a}} \ln u f_\xi(u)du + \int_{\frac{1}{a}}^\infty \ln u f_\xi(u)du \\ &\geq \int_0^{\frac{1}{a}} \ln u f_\xi(u)du - \ln a\int_{\frac{1}{a}}^\infty f_\xi(u)du \\ &\geq \int_0^{\frac{1}{a}} \ln u f_\xi(u)du - \ln a \end{aligned}$$

But from lemma 4.2 we have

$$\begin{aligned} |\int_0^{\frac{1}{a}} \ln u f_\xi(u)du| &\leq \int_0^{\frac{1}{a}} |\ln u| f_\xi(u)du \\ &\leq a\int_0^{\frac{1}{a}} |\ln u|du \\ &= \ln a + 1. \end{aligned}$$

So,

$$\int_0^{\frac{1}{a}} \ln u f_\xi(u) du \geq -\ln a - 1$$

and

$$E(\ln \xi) \geq -2\ln a - 1.$$

Since $C_n^m \leq 2^n$ and $\ln C_n^m \leq n$, therefore we have

$$\begin{aligned} E(\ln \xi) &\geq -2\ln nC_n^m - 1 \\ &= -2\ln n - 2\ln C_n^m - 1 \\ &\geq -2\ln n - 2n - 1. \end{aligned}$$

Combining theorem 3.2 and lemma 4.3 we have the following theorem.

Theorem 4.4. *The expected number of iterations to solve the probabilistic LP model, using primal-dual interior point algorithms satisfying (6), is bounded above by* $O(n^{1.5})$.

5. High Probability Analysis

In this section we will give a "high probability" analysis on the number of iterations required for termination of some interior point algorithms. An event is said to be *anticipated* if the probability that it occurs approaching one as $n \to \infty$. If an event is likely to happen for large n, then it is called *high probability* event. The high probability analysis of this section complements the expected result obtained in section 4.

Theorem 5.1. *The anticipated number of iterations required to terminate the interior point algorithms satisfying* (6) *is bounded above by* $O(n^{1.5})$.

Proof. Since $f_\xi(u) \leq nC_n^m$ $(0 < u < \frac{1}{2})$ by lemma 4.2, therefore

$$\begin{aligned} P(0 \leq \xi \leq \frac{1}{n^2 C_n^m}) &= \int_0^{\frac{1}{n^2 C_n^m}} f_\xi(u) du \\ &\leq \int_0^{\frac{1}{n^2 C_n^m}} nC_n^m du \\ &= \frac{1}{n}. \end{aligned}$$

Hence

$$P(\xi > \frac{1}{n^2 C_n^m}) > 1 - \frac{1}{n},$$

and

$$P(\ln \xi > -\ln n^2 C_n^m) > 1 - \frac{1}{n},$$

or

$$P(\ln \xi > -O(n)) > 1 - \frac{1}{n}.$$

From theorem 3.2 we know that the anticipated number of iterations required to terminate the interior point algorithms satisfying (6) is bounded above by $O(n^{1.5})$.

6. Further Remarks

In this paper we have shown that the expected number of iterations of some $O(\sqrt{n}L)$ interior-point algorithms is bounded above by $O(n^{1.5})$. The random LP problem that we dealt with in this paper is an extension of the probabilistic LP model introduced by Todd [9]. We also showed that the anticipated number of iterations needed to terminate these algorithms is bounded above by $O(n^{1.5})$. We expect that the result in the paper can be extended to more general random LP models. For example, if the entries of A are independently and identically distributed as stable distribution with characteristic function $e^{-c|t|^\alpha}$ $(0 < \alpha \leq 2)$, then the result in this paper corresponding to $\alpha = 1$ and the result in [3] corresponding to $\alpha = 2$. It would be interesting to see the result in this paper hold for $0 < \alpha \leq 2$. It will be a more difficult task.

The probabilistic analysis in this paper has been concentrated on improve the factor L in the worst-case complexity to $O(n)$ in expected complexity. Another important issue of probabilistic analysis of interior point algorithms is the average improvement per iteration compared to the worst-case complexity analysis. This could lead to improve the factor of $\sqrt{n}$ in the worst-case complexity. It remains an interesting and open problem.

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