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NON C^0 NONCONFORMING ELEMENTS FOR ELLIPTIC FOURTH ORDER SINGULAR PERTURBATION PROBLEM [*1)]

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Abstract

In this paper we give a convergence theorem for non C^0 nonconforming finite element to solve the elliptic fourth order singular perturbation problem. Two such kind of elements, a nine parameter triangular element and a twelve parameter rectangular element both with double set parameters, are presented. The convergence and numerical results of the two elements are given.

Mathematics subject classification: 65N12, 65N30.
Key words: Singular perturbation problem, Nonconforming element, Double set parameter method.

1. Introduction

We consider the following elliptic singular perturbation problem [1]:

$$\begin{cases} \varepsilon^2\Delta^2 u - \Delta u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f \in L^2(\Omega), \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator, $\Delta^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^2$, $\Omega \subset R^2$ is a bounded polygonal domain, $\partial\Omega$ is the boundary of Ω, $\frac{\partial}{\partial n}$ denotes the outer normal derivative on $\partial\Omega$, and ε is a real parameter such that $0 < \varepsilon \leq 1$. When ε tends to zero, (1) formally degenerates to Poisson's equation. Hence, (1) is a plate model which may degenerate toward an elastic membrane problem.

A conforming plate element should have C^1 continuity which makes the element complicated, so nonconforming plate elements are widely used. For convergence criterion there are Patch-Test[10] which is convenient to use for engineers, and Generalized Patch-Test[9] which is a sufficient and necessary condition. According to Generalized Patch-Test, Professor Shi presented F-E-M-Test[11] which is easier to use. Many successful nonconforming plate elements [5,7,3,12,13,14] have been presented, but not all of them are convergent for (1) uniformly respect to ε.

It is proved[1] that the non-C^0 nonconforming plate element— Morley's element [2],—is not convergent for (1) when $\varepsilon \to 0$. In [1] a C^0 nonconforming plate element is presented, which is convergent for (1) uniformly in ε. In this paper we study the convergence of non-C^0 nonconforming plate elements for (1). In section 2 we give a general convergence theorem for non-C^0 nonconforming plate elements solving (1). In section 3 the double set parameter method to construct nonconforming finite element is presented. In section 4 a triangular and a rectangular non-C^0 nonconforming plate elements [3][4] are presented and their convergence for (1) uniformly in ε is proved. In section 5 some numerical results are given.

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2. A Convergence Theorem

The inner product on $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$, $H^m(\Omega)$ is the usual Sobolev space of functions with partial derivatives of order less than or equal to m in $L^2(\Omega)$, and the corresponding norm by $\|\cdot\|_{m,\Omega}$. The seminorm derived from the partial derivatives of order equal to m is denoted by $|\cdot|_{m,\Omega}$. The space $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Alternatively, we have

$$H_0^1(\Omega)=\{v\in H^1(\Omega); v|_{\partial\Omega}=0\}, H_0^2(\Omega)=\{v\in H^2(\Omega); v=\frac{\partial v}{\partial n}=0, on\partial\Omega\}$$

Let Du be the gradient of u and $D^2u=(\frac{\partial^2 u}{\partial x_i\partial x_j})_{2\times 2}$ be the 2×2 tensor of the second order partial derivatives.

The weak form of (1) is : find $u\in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v)+b(u,v)=(f,v) \qquad \forall v\in H_0^2(\Omega) \tag{2}$$

where

$$a(u,v)=\int_\Omega D^2u:D^2v dx, \qquad b(u,v)=\int_\Omega Du\cdot Dv dx. \tag{3}$$

From Green's formula[5], it is easy to see that

$$\int_\Omega D^2u:D^2v dx=\int_\Omega \triangle u\triangle v dx \qquad \forall u,v\in H_0^2(\Omega) \tag{4}$$

However this identity does not hold on the noncomforming finite element spaces. We use the form (3) like in [1].

Assume that $\{T_h\}$ is a quasi-uniform [5] and shape-regular[5] family of triangulations of Ω, here the discretization parameter h is a characteristic diameter of the elements in T_h. We use V_h to denote the finite element space which is piecewise polynomial space and satisfies the boundary conditions of (1) in some way. Then the finite element approximation of (2) is: find $u_h\in V_h$ such that

$$\varepsilon^2 a_h(u_h,v_h)+b_h(u_h,v_h)=(f,v_h) \qquad \forall v_h\in V_h \tag{5}$$

where

$$a_h(u,v)=\sum_{K\in T_h}\int_K D^2u:D^2v dx, \qquad b_h(u,v)=\sum_{K\in T_h}\int_K Du\cdot Dv dx.$$

We define a seminorm $|||\cdot|||_{\varepsilon,h}$ by [1]

$$|||w|||_{\varepsilon,h}^2=\varepsilon^2 a_h(w,w)+b_h(w,w)=\varepsilon^2|w|_{2,h}^2+|w|_{1,h}^2 \tag{6}$$

where $|\cdot|_{i,h}^2=\sum_K|\cdot|_{i,K}^2, i=1,2$.

The interpolation operator derived by V_h is denoted by Π_h. Let $\Pi_K=\Pi_h|_K$ for $K\in\mathcal{T}_h$. $P_m(K)$ is the polynomial space of degree less than or equal to m on K. Let F denote any edge of an element.

Theorem 1. *Let u and u_h be solutions of (2)and (5) respectively. If V_h satisfies the following conditions:*
(c1) $|||\cdot|||_{\varepsilon,h}$ is a norm on V_h.
(c2) $\forall K\in T_h, \forall v\in P_2(K), \Pi_K v=v$.
(c3) $\forall v_h\in V_h, v_h$ is continuous at the vertics of elements and is zero at the vertics on $\partial\Omega$.
(c4) $\forall v_h\in V_h, \int_F v_h ds$ is continuous across the element edge F and is zero on $F\subset\partial\Omega$.
(c5) $\forall v_h\in V_h, \int_F \frac{\partial v_h}{\partial n}ds$ is continuous across the element edge F and is zero on $F\subset\partial\Omega$.
Then

$$|||u-u_h|||_{\varepsilon,h}\le ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega}+\|f\|_{0,\Omega}) \tag{7}$$

where c is independent of ε, h and u.

Proof. That the second Strang Lemma [2][5] is used to problem (2) and (5) results[1],

$$|||u-u_h|||_{\varepsilon,h} \le c\left(\inf_{v_h\in V_h}|||u-v_h|||_{\varepsilon,h} + \sup_{w_h\in V_h}\frac{|E_{\varepsilon,h}(u,w_h)|}{|||w_h|||_{\varepsilon,h}}\right) \tag{8}$$

where

$$E_{\varepsilon,h}(u,w_h) = \varepsilon^2 a_h(u,w_h) + b_h(u,w_h) - (f,w_h) \tag{9}$$

Obviously the discrete problem (5) has a unique solution from condition c1) by Lax-Milgram Lemma. By condition c2) and interpolation theory [5] we have

$$\inf_{v_h\in V_h}|||u-v_h|||_{\varepsilon,h} \le |||u-\Pi_h u|||_{\varepsilon,h} = (\varepsilon^2|u-\Pi_h u|^2_{2,h} + |u-\Pi_h u|^2_{1,h})^{\frac{1}{2}}$$
$$\le ch(\varepsilon|u|_{3,\Omega} + |u|_{2,\Omega}). \tag{10}$$

Since

$$D^2u : D^2w_h = \Delta u\Delta w_h + (2\partial_{12}u\partial_{12}w_h - \partial_{11}u\partial_{22}w_h - \partial_{22}u\partial_{11}w_h).$$

From Green's formula [2][5]

$$\int_K \triangle u\triangle w_h dx = \int_{\partial K}\triangle u\frac{\partial w_h}{\partial n}ds - \int_K \nabla\triangle u\nabla w_h dx.$$

$$\int_K 2\partial_{12}u\partial_{12}w_h - \partial_{11}u\partial_{22}w_h - \partial_{22}u\partial_{11}w_h)dx = \int_{\partial K}(\frac{\partial^2 u}{\partial n\partial s}\frac{\partial w_h}{\partial s} - \frac{\partial^2 u}{\partial s^2}\frac{\partial w_h}{\partial n})ds.$$

Then

$$a_h(u,w_h) = \sum_{K\in T_h}\int_K D^2u : D^2w_h dx$$
$$= \sum_{K\in T_h}\left\{\int_{\partial K}\left[(\triangle u - \frac{\partial^2 u}{\partial s^2})\frac{\partial w_h}{\partial n} + \frac{\partial^2 u}{\partial s\partial n}\frac{\partial w_h}{\partial s}\right]dx - \int_K \nabla\triangle u\nabla w_h dx\right\}. \tag{11}$$

where $\frac{\partial}{\partial s}$ is the tangent derivative on the edges of elements.

Let w_h^I be the piecewise interpolation polynomial of w_h such that:

For triangular elements, $\forall K\in\mathcal{T}_h, w_h^I|_K\in P_2(K), w_h^I|_K(a_i) = w_h(a_i), \int_{F_i} w_h^I ds = \int_{F_i} w_h ds, F_i\in \partial K, i=1,2,3.$

For rectangular elements, $w_h^I|_K \in P_2(K)\cup\left\{x^2y, xy^2\right\}, w_h^I|_K(a_i) = w_h(a_i), \int_{F_i} w_h^I ds = \int_{F_i} w_h ds, F_i\in\partial K, i=1,2,3,4.$

Then from conditions c3) and c4) we have

$$w_h^I \in H_0^1(\Omega), \int_F (w_h - w_h^I)ds = 0, \forall F\subset\partial K, \forall K\in\mathcal{T}_h \tag{12}$$

$$(f,w_h^I) = \sum_{K\in T_h}\int_K(\varepsilon^2\triangle^2 u - \triangle u)w_h^I dx$$
$$= -\sum_{K\in T_h}\int_K(\varepsilon^2\nabla\triangle u - \nabla u)\cdot\nabla w_h^I dx \tag{13}$$

Substituting (11) and (13) into (9) results:

$$E_{\varepsilon,h}(u,w_h) = \sum_{K\in\mathcal{T}_h}\left\{\int_{\partial K}\varepsilon^2\left[(\Delta u - \frac{\partial^2 u}{\partial s^2})\frac{\partial w_h}{\partial n} + \frac{\partial^2 u}{\partial s\partial n}\frac{\partial w_h}{\partial s}\right]ds\right.$$

$$-\int_K \varepsilon^2\nabla\triangle u\cdot\nabla(w_h-w_h^I)dx+\int_K\nabla u\cdot\nabla(w_h-w_h^I)dx-(f,w_h-w_h^I)\Big\}. \quad (14)$$

Now we estimate every term in (14).

From conditions c3) and c5) we have

$$\int_F\left[\frac{\partial w_h}{\partial n}\right]ds=\int_F\left[\frac{\partial w_h}{\partial s}\right]ds=0,\forall F\subset\partial K,\forall K\in\mathcal{T}_h$$

where $[v]$ is the jump of v across F. Using the formal skill for nonconforming elements of plate bending problem [3][5] we get

$$\left|\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left[(\Delta u-\frac{\partial^2 u}{\partial s^2})\frac{\partial w_h}{\partial n}+\frac{\partial^2 u}{\partial s\partial n}\frac{\partial w_h}{\partial s}\right]ds\right|$$

$$\le ch|u|_{3,\Omega}|w_h|_{2,h}\le ch\varepsilon^{-1}|u|_{3,\Omega}|||w_h|||_{\varepsilon,h}. \quad (15)$$

From interpolation theory[5] we have

$$\left|\sum_{K\in\mathcal{T}_h}\int_K\nabla\triangle u\cdot\nabla(w_h-w_h^I)dx\right|$$

$$\le ch|u|_{3,\Omega}|w_h|_{2,h}\le ch\varepsilon^{-1}|u|_{3,\Omega}|||w_h|||_{\varepsilon,h}. \quad (16)$$

$$|(f,w_h-w_h^I)|\le c\|f\|_{0,\Omega}\|w_h-w_h^I\|_{0,\Omega}$$

$$\le ch\|f\|_{0,\Omega}|w_h|_{1,h}\le ch\|f\|_{0,\Omega}|||w_h|||_{\varepsilon,h}. \quad (17)$$

Let $\Pi_0 v=\frac{1}{K}\int_K vdx$. From (12) $\int_K\nabla(w_h-w_h^I)dx=\int_{\partial K}(w_h-w_h^I)nds=0$, then

$$\left|\sum_{K\in\mathcal{T}_h}\int_K\nabla u\cdot\nabla(w_h-w_h^I)dx\right|=\left|\sum_{K\in\mathcal{T}_h}\int_K(\nabla u-\Pi_0\nabla u)\nabla(w_h-w_h^I)dx\right|$$

$$\le\sum_{K\in\mathcal{T}_h}\|\nabla u-\Pi_0\nabla u\|_{0,K}|w_h-w_h^I|_{1,K}\le ch|u|_{2,\Omega}|w_h|_{1,h}$$

$$\le ch|u|_{2,\Omega}|||w_h|||_{\varepsilon,h}. \quad (18)$$

Substituting (15)-(18) into (14) we get

$$|E_{\varepsilon,h}(u,w_h)|\le ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega}+\|f\|_{0,\Omega})|||w_h|||_{\varepsilon,h} \quad (19)$$

Then (7) follows from (8) (10) (19).

Remark 2.1. Morley's element does not satisfy (c4) and has been proved[1] not convergent for (2).

3. Two Non-C^0 Nonconforming Elements with Double Set Parameters

1. A Nine Parameter Triangular Element[3].

Given a triangle K with vertices $a_i=(x_i,y_i),1\le i\le 3$, we denote by F_i,n_i,s_i, respectively , the side opposite to a_i, the unit outward normal and the tangential vectors on F_i. Let λ_i be the area coordinates for the triangle K, Δ be the area of K, v_i,v_{ix},v_{iy} be the function value of v and its first derivatives at a_i, and a_{12},a_{23},a_{31} be the midpoints of F_3,F_1,F_2 respectively. Put

$$b_i=y_{i+1}-y_{i-1},c_i=x_{i-1}-x_{i+1},r_i=(b_{i+1}b_{i-1}+c_{i+1}c_{i-1})/\Delta,$$

$$t_i = F_i^2/\Delta, i = 1,2,3(\text{mod } 3).$$

The shape function space is

$$P(K) = P_3(K) = span\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1,$$
$$\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2, \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2, \lambda_3^2\lambda_1 - \lambda_3\lambda_1^2, \lambda_1\lambda_2\lambda_3\} \tag{20}$$

Degrees of freedom are

$$D(v) = (d_1(v), \cdots, d_{10}(v))^\top \tag{21}$$

where $d_i(v) = v_i, i = 1,2,3, d_4(v) = v(a_{12}), d_5(v) = v(a_{23}), d_6(v) = v(a_{31})$, $d_7(v) = -2\int_{F_1} \frac{\partial v}{\partial n} ds, d_8(v) = -2\int_{F_2} \frac{\partial v}{\partial n} ds, d_9(v) = -2\int_{F_3} \frac{\partial v}{\partial n} ds$, $d_{10}(v) = -4\int_{F_1} \lambda_2 \frac{\partial v}{\partial n} ds$.

$\forall v \in P_3(K)$, suppose that

$$v = \beta_1\lambda_1 + \beta_2\lambda_2 + \beta_3\lambda_3 + \beta_4\lambda_1\lambda_2 + \beta_5\lambda_2\lambda_3 + \beta_6\lambda_3\lambda_1$$
$$+\beta_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + \beta_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + \beta_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2) + \beta_{10}\lambda_1\lambda_2\lambda_3 \tag{22}$$

Substituting (22) into (21) results

$$Cb = D(v) \tag{23}$$

where $b = (\beta_1, \cdots, \beta_{10})^\top$, the interpolation matrix is

$$C = \begin{pmatrix} 1 & 0 & 0 & & & & & & & \\ 0 & 1 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & & & & \\ t_1 & r_3 & r_2 & \frac{t_1}{2} & -\frac{t_1}{2} & \frac{t_1}{2} & -\frac{t_1}{3} & 0 & \frac{t_1}{3} & \frac{t_1}{6} \\ r_3 & t_2 & r_1 & \frac{t_2}{2} & \frac{t_2}{2} & -\frac{t_2}{2} & \frac{t_2}{3} & -\frac{t_2}{3} & 0 & \frac{t_2}{6} \\ r_2 & r_1 & t_3 & -\frac{t_3}{2} & \frac{t_3}{2} & \frac{t_3}{2} & 0 & \frac{t_3}{3} & -\frac{t_3}{3} & \frac{t_3}{6} \\ t_1 & r_3 & r_2 & \frac{2t_1}{3} & \frac{r_2-t_1}{3} & \frac{t_1}{3} & -\frac{t_1}{2} & -\frac{t_1}{6} & \frac{t_1}{6} & \frac{t_1}{6} \end{pmatrix}_{10\times 10}$$

It is easy to see that

$$\det C = \frac{t_1^2 t_2 t_3}{36} \neq 0. \tag{24}$$

Nodal parameters are defined by

$$Q(v) = (v_1, v_{1x}, v_{1y}, v_2, v_{2x}, v_{2y}, v_3, v_{3x}, v_{3y})^\top. \tag{25}$$

We approximate the degrees of freedom (21) in terms of the nodal parameters (25) as follows

$$d_i(v) = v_i, i = 1,2,3. \tag{26}$$

$d_4 = H_3(a_{12}), d_5 = H_1(a_{23}), d_6(v) = H_2(a_{31})$,here $H_i(x,y)$ is the Hermite interpolation polynomial of order 3 of v on $F_i, i = 1,2,3$, resulting

$$\left\{\begin{array}{l} d_4 = \frac{1}{2}\left\{v_1 + v_2 + \frac{1}{4}[(v_{1x} - v_{2x})c_3 - (v_{1y} - v_{2y})b_3]\right\} \\ d_5 = \frac{1}{2}\left\{v_2 + v_3 + \frac{1}{4}[(v_{2x} - v_{3x})c_1 - (v_{2y} - v_{3y})b_1]\right\} \\ d_6 = \frac{1}{2}\left\{v_3 + v_1 + \frac{1}{4}[(v_{3x} - v_{1x})c_2 - (v_{3y} - v_{1y})b_2]\right\} \end{array}\right. \tag{27}$$

Let $|F_i|$ be the measure of F_i. For the degrees of freedom $d_7(v), d_8(v), d_9(v)$ we use the trapezoidal rule, giving

$$\left\{\begin{array}{l} d_7 = b_1(v_{2x}+v_{3x}) + c_1(v_{2y}+v_{3y}) + O(|F_1|^3|v|_{3,K,\infty}) \\ d_8 = b_2(v_{3x}+v_{1x}) + c_2(v_{3y}+v_{1y}) + O(|F_2|^3|v|_{3,K,\infty}) \\ d_9 = b_3(v_{1x}+v_{2x}) + c_3(v_{1y}+v_{2y}) + O(|F_3|^3|v|_{3,K,\infty}) \end{array}\right. \tag{28}$$

and $d_{10}(v) = -4\int_{F_1}\lambda_2 I_1(\frac{\partial v}{\partial n})ds$, here I_1 is the linear interpolation operator on F_1, we get

$$d_{10}(v) = \frac{2}{3}[(2v_{2x}+v_{3x})b_1 + (2v_{2y}+v_{3y})c_1] + O(|F_1|^3|v|_{3,K,\infty})$$

Then we have

$$D(v) = GQ(v) + \delta(v) \tag{29}$$

where $\delta(v) = (0,0,0,0,0,0,\varepsilon(v),\varepsilon(v),\varepsilon(v),\varepsilon(v))^\top, \varepsilon(v) = O(h^3|v|_{3,K,\infty})$

$$G = \begin{pmatrix} 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{c_3}{8} & -\frac{b_3}{8} & \frac{1}{2} & -\frac{c_3}{8} & \frac{b_3}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{c_1}{8} & -\frac{b_1}{8} & \frac{1}{2} & -\frac{c_1}{8} & \frac{b_1}{8} \\ \frac{1}{2} & -\frac{c_2}{8} & \frac{b_2}{8} & 0 & 0 & 0 & \frac{1}{2} & \frac{c_2}{8} & -\frac{b_2}{8} \\ 0 & 0 & 0 & 0 & b_1 & c_1 & 0 & b_1 & c_1 \\ 0 & b_2 & c_2 & 0 & 0 & 0 & 0 & b_2 & c_2 \\ 0 & b_3 & c_3 & 0 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4b_1}{3} & \frac{4c_1}{3} & 0 & \frac{2b_1}{3} & \frac{2c_1}{3} \end{pmatrix}_{10\times 9}.$$

Neglecting the term $\delta(v)$ and combining (23) we get

$$b = C^{-1}GQ(v). \tag{30}$$

(22) and (30) are the real expression of the shape function v, $Q(v)$ is the real nodal parameters.

2. A Twelve Parameter Rectangular Element [4]

Suppose the rectangular element K is on the (x,y) plane with the center (x_0, y_0), its sides are parallel to axes of coordinates and the side lengths are $2a$ and $2b$ respectively, $a(x_i, y_i), F_i = [a_i, a_{i+1}], 1 \le i \le 4$ are the vertices and the sides of K. The reference element $\widehat{K}$ is a square on the plane (ξ, η) with center $(0,0)$, $\widehat{a}_1(-1,-1), \widehat{a}_2(1,-1), \widehat{a}_3(1,1), \widehat{a}_4(-1,1)$ are 4 nodes of $\widehat{K}$. Under the affine mapping $\xi = (x-x_0)/a, \eta = (y-y_0)/b, K \to \widehat{K}$, and $v(x,y) = \widehat{v}(\xi,\eta)$.

We chose degrees of freedom as

$$D(v) = (d_1(v), \cdots, d_{12}(v))^\top \tag{31}$$

where

$d_i(v) = v_i, 1 \le i \le 4.$
$d_5(v) = \frac{1}{a}\int_{F_1} v ds = \int_{-1}^1 \widehat{v}(\xi,-1)d\xi, d_6(v) = \frac{1}{b}\int_{F_2} v ds = \int_{-1}^1 \widehat{v}(1,\eta)d\eta,$
$d_7(v) = -\frac{1}{a}\int_{F_3} v ds = \int_{-1}^1 \widehat{v}(\xi,1)d\xi, d_8(v) = -\frac{1}{b}\int_{F_4} v ds = \int_{-1}^1 \widehat{v}(-1,\eta)d\eta,$
$d_9(v) = -\frac{b}{a}\int_{F_1} \frac{\partial v}{\partial n}ds = \int_{-1}^1 \frac{\partial \widehat{v}}{\partial \eta}(\xi,-1)d\xi,$
$d_{10}(v) = \frac{a}{b}\int_{F_2} \frac{\partial v}{\partial n}ds = \int_{-1}^1 \frac{\partial \widehat{v}}{\partial \xi}(1,\eta)d\eta,$
$d_{11}(v) = -\frac{b}{a}\int_{F_3} \frac{\partial v}{\partial n}ds = \int_{-1}^1 \frac{\partial \widehat{v}}{\partial \eta}(\xi,1)d\xi,$
$d_{12}(v) = \frac{a}{b}\int_{F_4} \frac{\partial v}{\partial n}ds = \int_{-1}^1 \frac{\partial \widehat{v}}{\partial \xi}(-1,\eta)d\eta.$

This means that degrees of freedom are the function values of v at 4 nodes, the mean values of v and the integrations of out normal derivatives of v along 4 sides.

The shape function space is

$$P(K) = P_3(K) \cup \{x^4, y^4\} = Span\{p_1, \cdots, p_{12}\} \tag{32}$$

where $p_1 = \frac{1}{4}(1-\xi)(1-\eta), p_2 = \frac{1}{4}(1+\xi)(1-\eta)$
$p_3 = \frac{1}{4}(1+\xi)(1+\eta), p_4 = \frac{1}{4}(1-\xi)(1+\eta),$
$p_5 = (1-\xi^2), p_6 = (1-\eta^2), p_7 = (1-\xi^2)\eta, p_8 = (1-\eta^2)\xi,$
$p_9 = (1-\xi^2)\xi, p_{10} = (1-\eta^2)\eta, p_{11} = (1-\xi^2)\xi^2, p_{12} = (1-\eta^2)\eta^2.$

Let

$$\forall v \in P(K), \qquad v = \beta_1 p_1 + \cdots + \beta_{12} p_{12} \tag{33}$$

Substituting (33) into (31), resulting

$$D(v) = Cb \tag{34}$$

where $b = (\beta_1, \cdots, \beta_{12})^\top$

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 & & & & & & & & \\
0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & \\
1 & 1 & 0 & 0 & \frac{4}{3} & 0 & -\frac{4}{3} & 0 & 0 & 0 & \frac{4}{15} & 0 \\
0 & 1 & 1 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & \frac{4}{15} \\
0 & 0 & 1 & 1 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & \frac{4}{15} & 0 \\
1 & 0 & 0 & 1 & 0 & \frac{4}{3} & 0 & -\frac{4}{3} & 0 & 0 & 0 & \frac{4}{15} \\
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 4 & \frac{4}{3} & 0 & 0 & -4 & 0 & 4 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -4 & 0 & 0 & \frac{4}{3} & -4 & 0 & -4 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -4 & \frac{4}{3} & 0 & 0 & -4 & 0 & -4 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 4 & 0 & 0 & \frac{4}{3} & -4 & 0 & 4 & 0
\end{pmatrix}$$

It is easy to see that

$$\det C = \frac{2^{24}}{3^4 \cdot 5^2} \neq 0 \tag{35}$$

The nodal parameters are taken as

$$Q(v) = (v_1, v_{1x}, v_{1y}, \cdots, v_4, v_{4x}, v_{4y})^\top. \tag{36}$$

The following methods are used to approximate the degrees of freedom $D(v)$ into the linear combinations of nodal parameters $Q(v)$:

$$d_i(v) = v_i, i = 1, 2, 3, 4. \tag{37}$$

$d_{i+4} = \frac{2}{|F_i|}\int_{F_i} H_i(v)ds, 1 \le i \le 4$, here $H_i(v)$ is the Hermite interpolation polynomial of order 3 of v on $F_i, 1 \le i \le 4$, giving

$$\begin{cases}
d_5(v) = v_1 + v_2 + \frac{a}{3}(v_{1x} - v_{2x}) + O(h^4|v|_{4,K,\infty}) \\
d_6(v) = v_2 + v_3 + \frac{b}{3}(v_{2y} - v_{3y}) + O(h^4|v|_{4,K,\infty}) \\
d_7(v) = v_3 + v_4 + \frac{a}{3}(-v_{3x} + v_{4x}) + O(h^4|v|_{4,K,\infty}) \\
d_8(v) = v_1 + v_4 + \frac{b}{3}(v_{1y} - v_{4y}) + O(h^4|v|_{4,K,\infty})
\end{cases} \tag{38}$$

For $d_9(v)-d_{12}(v)$ trapezoidal rule of numerical integration is used, resulting

$$\begin{cases} d_9(v)=b(v_{1y}+v_{2y})+O(h^3|v|_{3,K,\infty}) \\ d_{10}(v)=a(v_{2x}+v_{3x})+O(h^3|v|_{3,K,\infty}) \\ d_{11}(v)=b(v_{3y}+v_{4y})+O(h^3|v|_{3,K,\infty}) \\ d_{12}(v)=a(v_{1x}+v_{4x})+O(h^3|v|_{3,K,\infty}) \end{cases} \tag{39}$$

The above discretizing can be expressed as

$$D(v)=GQ(v)+\delta(v) \tag{40}$$

where $\delta(v)=(0,0,0,0,\varepsilon_1(v),\varepsilon_1(v),\varepsilon_1(v),\varepsilon_1(v),\varepsilon_2(v),\varepsilon_2(v),\varepsilon_2(v),\varepsilon_2(v))^\top,\varepsilon_1(v)=O(h^4|v|_{4,K,\infty})$, $\varepsilon_2(v)=O(h^3|v|_{3,K,\infty})$,

$$G=\begin{pmatrix} 1 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{a}{3} & 0 & 1 & -\frac{a}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{b}{3} & 1 & 0 & -\frac{b}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{a}{3} & 0 & 1 & \frac{a}{3} & 0 \\ 1 & 0 & \frac{b}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{b}{3} \\ 0 & 0 & b & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \end{pmatrix}.$$

Similarly, neglecting the term $\delta(v)$, we get the real shape function v which is still as (33), and $b=C^{-1}GQ(v)$.

Theorem 2. *The above nine parameter triangle element and twelve parameter rectangle element are convergent for (2) uniformly in ε and*

$$|||u-u_h|||_{\varepsilon,h}\le ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega}+\|f\|_{0,\Omega}) \tag{41}$$

where c is independent of ε,h and u.

Proof. It is only needed to check the conditions c1)-c5) of Theorem 1 for both elements.

Suppose $v_h\in V_h$ and $|||v_h|||_{\varepsilon,h}=0$, this means $v_h|_K=$const, $\forall K\in\mathcal{T}_h$, then $v_h\equiv 0$ in Ω follows from $v_h|_{\partial\Omega}=0$ and v_h is continuous at the nodes of $\mathcal{T}_h$. So $|||\cdot|||_{\varepsilon,h}$ is a norm on V_h and c1) is satisfied for both elements.

$\forall v\in P_2(v)$, from (29) and (40) we have $\delta(v)=0$, then $D(v)=GQ(v)$ is hold exactly, and $D(\Pi_K v)=GQ(v)=D(v)$, thus $\Pi v=v$. This means c2) is satisfied for both elements.

From $d_1(v)-d_3(v)$ of (21) and $d_1(v)-d_4(v)$ of (31), as well as (26) and (37) we know v is continuous at nodes of $\mathcal{T}_h$, so c3) is satisfied for both elements.

For the triangle element, $\forall F\subset\partial K,\forall K\in\mathcal{T}_h$, from $d_1(v)-d_6(v)$ of (21), $\forall v\in V_h, v$ is continuous at two ends and midpoint of F, so $\int_F vds$ is continuous across F by Simpson formula of numerical integration. For the rectangle element $\int_F vds$ is continuous across F from $d_5(v)-d_8(v)$ of (31). Thus c4) is satisfied for both elements.

Obviously c5) is satisfied for both elements from $d_7(v)-d_9(v)$ of (21) and $d_9(v)-d_{12}(v)$ of (31).

4. Numerical Experiments

Consider problem (1) with $\Omega=[0,1]^2\subset R^2$ and $f=\varepsilon^2\triangle^2u-\triangle u$, where $u=(\sin(\pi x_1)\sin(\pi x_2))^2$[1].

For a comparison with Example 4.1 of [1], we compute the relative error in the energy norm, $|||u_h^I - u_h|||_{\varepsilon,h}/|||u_h^I|||_{\varepsilon,h}$, for different values of ε and h. Here u_h^I denote the interpolant of u on a finite element space V_h . We also consider the case $\varepsilon = 0$, the poisson problem with Dirichlet boundary conditions, and the biharmonic problem $\triangle^2 u = f$.

In the figures we show errors in the norm $|u - u_h|_{l,h}, l = 0, 1$ for each mesh respectively and for different values of ε and h. The norm $|\cdot|_{l,h}$ is defined as

$$|g|_{l,h} = \max_{|\alpha|=l, a\in M(T_h)} |D^\alpha g(a)|, \forall g \in V$$

where $M(T_h)$ is the set of vertices of all $K \in T_h$.

Experiment 1. *To solve the problem (1) with the twelve parameter rectangular element in Section 4, we use two rectangular meshes which are shown in Figure 1 (case n=8) . The relative errors measured by the energy norm for mesh 1 and mesh 2 are given in Table 1 and Table 2 respectively. The errors measured by the norm* $|\cdot|_{l,h}, l = 0, 1$ *for mesh 1 and mesh 2 are shown in Figure 2 and Figure 3 respectively.*

Experiment 2. *To solve the same problem as Experiment 1 with the nine parameter triangular element in Section 4, we use four triangular meshes which are shown in Figure 4 (case n=8). The relative errors measured by the energy norm for mesh 3 to mesh 6 are given in Table 3 to Table 6 respectively. The errors measured by the norm* $|\cdot|_{l,h}, l = 0, 1$ *for mesh 3 to mesh 6 are shown in Figure 5 to Figure 8 respectively.*

From the above numerical experiments it can be seen that these numerical results are consistent with the theoretical analysis.

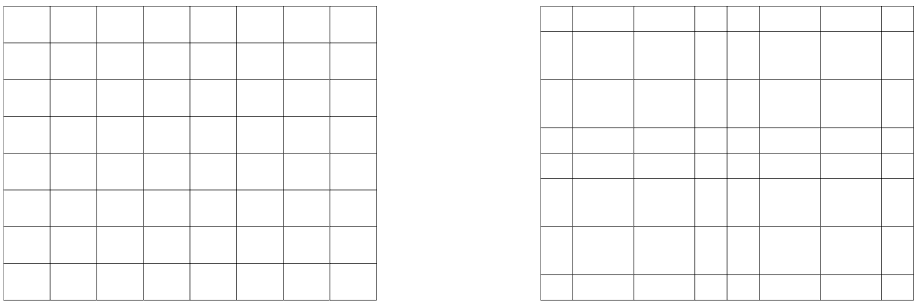

Figure 1: Two subdivisions: mesh 1 (the left) and mesh 2 (the right)

Table 1. The relative error measured by the energy norm for mesh 1

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	8.16e-003	2.85e-003	7.69e-004	1.96e-004
2^{-2}	6.37e-003	2.23e-003	6.01e-004	1.53e-004
2^{-4}	1.37e-003	4.89e-004	1.33e-004	3.40e-005
2^{-6}	7.95e-005	2.97e-005	9.34e-006	2.49e-006
2^{-8}	1.40e-005	6.91e-007	3.12e-007	1.27e-007
2^{-10}	1.14e-005	7.15e-008	3.46e-009	1.96e-009
Poisson	1.12e-005	5.33e-008	2.19e-010	8.67e-013
Biharmonic	8.31e-003	2.90e-003	7.84e-004	2.00e-004

Table 2. The relative error measured by the energy norm for mesh 2

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	3.11e-003	1.92e-003	6.04e-004	1.62e-004
2^{-2}	2.40e-003	1.50e-003	4.72e-004	1.27e-004
2^{-4}	4.57e-004	3.17e-004	1.04e-004	2.81e-005
2^{-6}	5.72e-005	1.48e-005	6.36e-006	1.98e-006
2^{-8}	9.23e-005	8.65e-006	5.46e-007	7.63e-008
2^{-10}	9.74e-005	1.05e-005	8.39e-007	5.22e-008
Poisson	9.78e-005	1.06e-005	8.82e-007	6.24e-008
Biharmonic	3.17e-003	1.96e-003	6.16e-004	1.66e-004

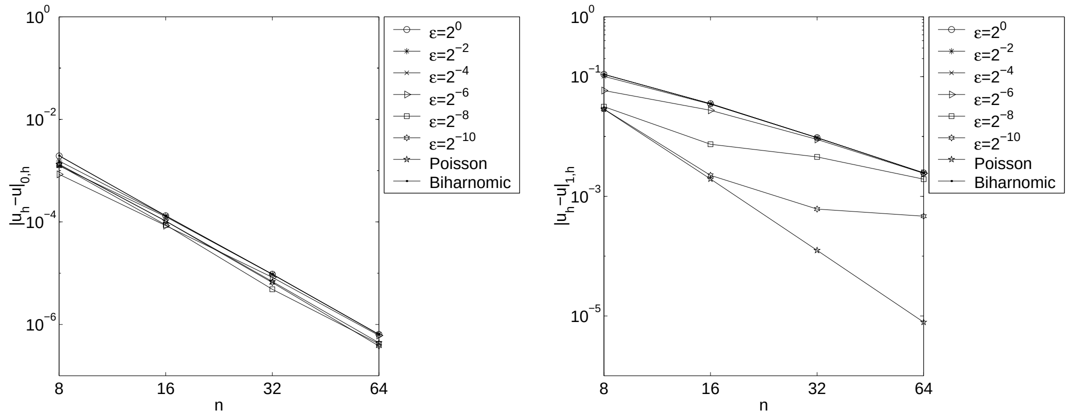

Figure 2: The error of u_h measured by the norms $|\cdot|_{l,h}, l = 0, 1$ for mesh 1

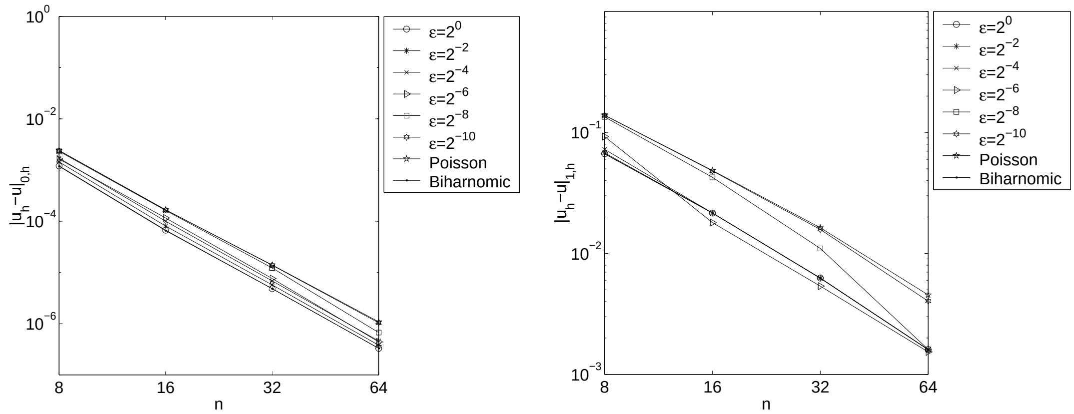

Figure 3: The error of u_h measured by the norms $|\cdot|_{l,h}, l = 0, 1$ for mesh 2

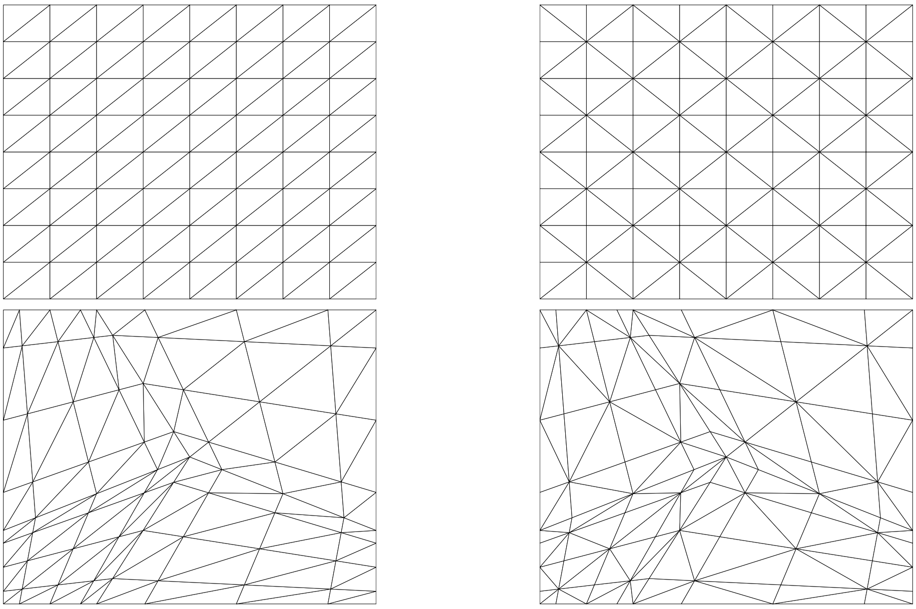

Figure 4: Four triangulations: mesh 3 (the top left), mesh 4 (the top right), mesh5 (the bottom left) and mesh6 (the bottom right)

Table 3. The relative error measured by the energy norm for mesh 3

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	3.89e-002	9.89e-003	2.46e-003	6.14e-004
2^{-2}	3.09e-002	7.76e-003	1.92e-003	4.80e-004
2^{-4}	8.16e-003	1.80e-003	4.32e-004	1.07e-004
2^{-6}	3.16e-003	2.75e-004	3.90e-005	8.34e-006
2^{-8}	3.18e-003	2.29e-004	1.50e-005	1.14e-006
2^{-10}	3.19e-003	2.32e-004	1.54e-005	9.73e-007
Poisson	3.20e-003	2.32e-004	1.55e-005	9.90e-007
Biharmonic	3.96e-002	1.01e-002	2.50e-003	6.26e-004

Table 4. The relative error measured by the energy norm for mesh 4

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	4.39e-002	1.24e-002	3.21e-003	8.12e-004
2^{-2}	3.50e-002	9.74e-003	2.51e-003	6.35e-004
2^{-4}	9.19e-003	2.25e-003	5.64e-004	1.41e-004
2^{-6}	3.11e-003	2.93e-004	4.77e-005	1.08e-005
2^{-8}	2.98e-003	2.02e-004	1.36e-005	1.19e-006
2^{-10}	2.98e-003	2.02e-004	1.30e-005	8.19e-007
Poisson	2.98e-003	2.02e-004	1.30e-005	8.19e-007
Biharmonic	4.47e-002	1.26e-002	3.27e-003	8.27e-004

Table 5. The relative error measured by the energy norm for mesh 5

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	3.68e-001	2.61e-001	1.96e-001	1.65e-001
2^0	3.16e-001	2.17e-001	1.60e-001	1.34e-001
2^0	9.84e-002	5.88e-002	4.08e-002	3.33e-002
2^0	1.63e-002	5.23e-003	3.19e-003	2.56e-003
2^0	1.08e-002	1.18e-003	2.62e-004	1.66e-004
2^0	1.06e-002	9.58e-004	8.37e-005	1.49e-005
Poisson	1.06e-002	9.47e-004	7.28e-005	5.11e-006
Biharmonic	3.73e-001	2.64e-001	1.99e-001	1.68e-001

Table 6. The relative error measured by the energy norm for mesh 6

$\varepsilon \setminus n$	2^3	2^4	2^5	2^6
2^0	3.35e-001	2.64e-001	2.21e-001	2.04e-001
2^{-2}	2.86e-001	2.20e-001	1.82e-001	1.67e-001
2^{-4}	8.82e-002	6.03e-002	4.73e-002	4.27e-002
2^{-6}	1.67e-002	5.57e-003	3.73e-003	3.31e-003
2^{-8}	1.23e-002	1.37e-003	3.06e-004	2.14e-004
2^{-10}	1.21e-002	1.14e-003	9.60e-005	1.86e-005
Poisson	1.21e-002	1.13e-003	8.31e-005	5.80e-006
Biharmonic	3.39e-001	2.68e-001	2.24e-001	2.07e-001

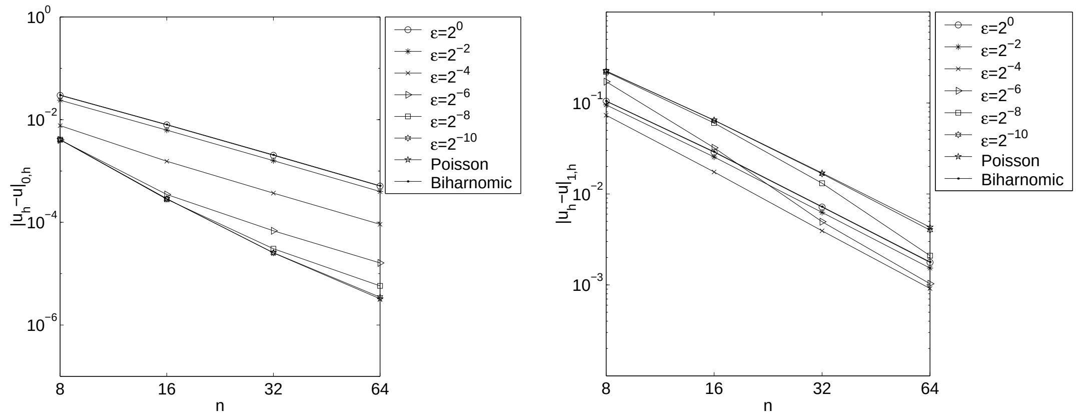

Figure 5: The error of u_h measured by the norms $|\cdot|_{l,h}, l=0,1$ for mesh 3

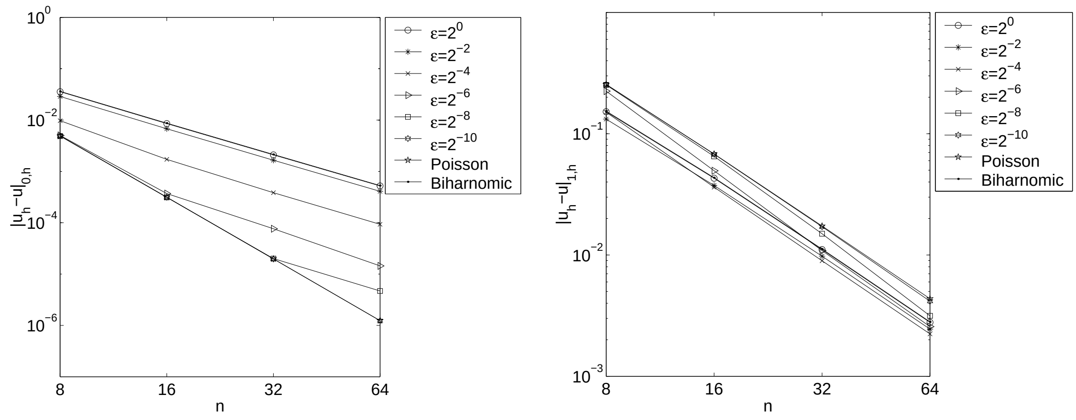

Figure 6: The error of u_h measured by the norms $|\cdot|_{l,h}, l=0,1$ for mesh 4

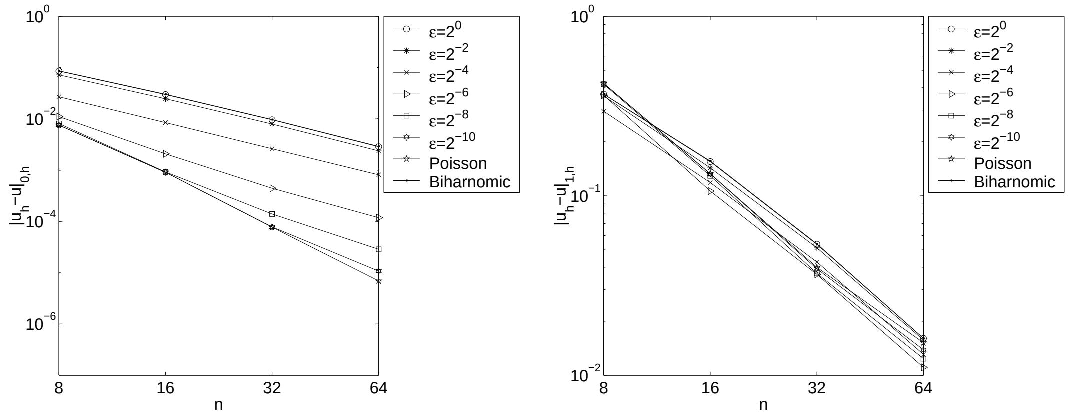

Figure 7: The error of u_h measured by the norms $|\cdot|_{l,h}, l=0,1$ for mesh 5

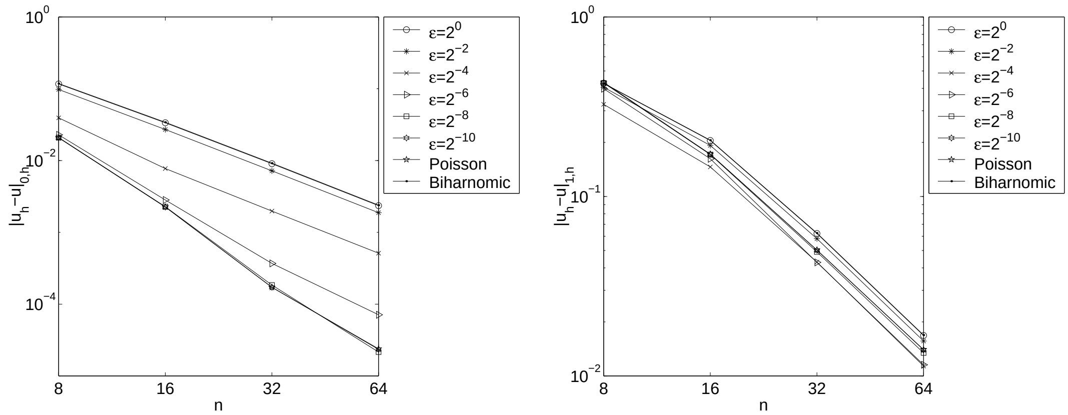

Figure 8: The error of u_h measured by the norms $|\cdot|_{l,h}, l=0,1$ for mesh 6

References

[1] T.K. Nilssen, X.C. Tai and R. Winther, A robust nonconforming H^2- element, *Math. Comp.*, **70**:234 (2001), 489-505.

[2] P. Lascaux and P. Lesaint, Some noncomforming finite element for the plate bending problem, *RAIRO, Anal. Numer.*, **R-1** (1975), 9-53

[3] S.C. Chen and Z.C. Shi, Double set parameter method of constructing a stiffness matrix, *Chinese J. Numer. Math. Appl.*, **4** (1991), 55-69.

[4] S.C. Chen and D.Y. shi, 12-parameter rectangular plate elements with geometric sysmmetry, *Numer. Math. J. Chinese Uni.,* (in Chinese). **3** (1996), 233-238.

[5] P.G. Ciarlet, The finite element method for elliptic problems, North-Holland Publishing Company, 1978.

[6] Z.C. Shi, Error Estimates of Morley element, *Math. Numer. Sinica*, **12** (1990), 113-118.

[7] Z.C. Shi, The generalized patch test for Zienkiewicz's triangle, *J. Comput. Math.*, **2** (1984), 276-286

[8] B. Specht, Modified shape functions for the three node plate bending element passing the patch test, *I. J. Numer. Maths. Eng.*, **26** (1988), 705-715.

[9] Z.C. Shi and S.C. Chen, Analysis of a nine degree plate bending element of Specht, *Chinese J. Numer. Math. Appl.*, **4** (1989), 73-79.

[10] F. Stummel, The generalized patch test, *SIAM. J. Numer. Anal.*, **16** (1979), 449-471.

[11] G. Strang and G. J. Fix, An analysis of the finite element method, Prentice-Hall, 1973.

[12] Z. C. Shi, The F-E-M-Test for convergence of nonconforming finite elements, *Math. Comp.*, *49* (1987), 391-405.

[13] D.G. Bergan, M.K. Nygard, Finite elements with increased freedom in choosing shape functions, *Int. J. Numer. Math. Eng.*, **20** (1984), 643-664.

[14] L. M. Tang et. al., Quasi-conforming elements in finite element analysis, *J. Dalian Institute of Technology*, **2** (1980), 19-35 (in Chinese).

[15] Y.Q. Long et. al., Generalized conforming elements, *J. Civil. Eng.*, **1** (1987), 1-14 (in Chinese).