Journal of Computational Mathematics, Vol.23, No.2, 2005, 199–204.

NEW ESTIMATES FOR SINGULAR VALUES OF A MATRIX *

Ming-xian Pang
(*Normal Faculty of Science, Jilin Beihua University, Jilin 132013, China*)

Abstract

New estimates are provided for singular values of a matrix in this paper. These results generalize and improve corresponding estimates for singular values in [4]-[6].

Mathematics subject classification: 15A18, 65F20.
Key words: Singular value, Directed graph, Circuit.

1. Introduction and Denotations

In terms of matrix entries, Gerschgorin's theorem, Brauer's theorem and Brualdi's theorem provide useful estimates for eigenvalues of a matrix ([9],[10]). Using these theorems, some researchers made many corresponding estimates for singular values of a matrix (see [1]-[8]). In this paper, several new estimates for singular values of a matrix are presented. These results generalize and improve corresponding estimates in [4]-[6].

The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$. Let $A=(a_{ij}) \in C^{n\times n}$, $\sigma(A)$ be the set of all singular values of A, and

$$r_i(A)=\sum_{j\neq i}|a_{ij}|, \qquad c_i(A)=\sum_{j\neq i}|a_{ji}|, \qquad a_i=|a_{ii}|, \qquad i\in\langle n\rangle=\{1,2,...,n\}.$$

Suppose the partition $N_j \subseteq \langle n\rangle,\ j\in\langle m\rangle$. Then it satisfies that $\bigcup\limits_{j\in\langle m\rangle} N_j=\langle n\rangle$, and for $\forall\ i\neq j$, $N_i\cap N_j=\phi$. For all $i\in\langle n\rangle$,denote $i\in N_{\sigma_i},\ \sigma_i\in\langle m\rangle$, and let $(\sigma_1,...,\sigma_m)$ be a permutation of $(1,...,m)$. For the sake of convenient, we also use the following denotations:

$$r^{(i)}_{N_{\sigma_i}}(A)=\sum_{j\in N_{\sigma_i}\backslash\{i\}}|a_{ij}|, \qquad c^{(i)}_{N_{\sigma_i}}(A)=\sum_{j\in N_{\sigma_i}\backslash\{i\}}|a_{ji}|;$$
$$\bar r^{(i)}_{N_{\sigma_i(A)}}=r_i(A)-r^{(i)}_{N_{\sigma_i}}(A), \qquad \bar c^{(i)}_{N_{\sigma_i}}(A)=c_i(A)-c^{(i)}_{N_{\sigma_i}}(A);$$
$$S^{(i)}_{N_{\sigma_i}}(A)=\max\{r^{(i)}_{N_{\sigma_i}}(A),\, c^{(i)}_{N_{\sigma_i}}(A)\}, \qquad \bar S^{(i)}_{N_{\sigma_i}}(A)=\max\{\bar r^{(i)}_{N_{\sigma_i}}(A),\, \bar c^{(i)}_{N_{\sigma_i}}(A)\}.$$

Let $\Gamma(A)$ be the directed graph of A with vertex set $V=\langle n\rangle$ and $E=\{(i,j): a_{ij}\neq 0\}$. The sets of out-neighbors and in-neighbors of i in $\Gamma(A)$ are denoted by $\Gamma_i^+(A)$ and $\Gamma_i^-(A)$, respectively, namely,

$$\Gamma_i^+(A)=\{j\in V\backslash\{i\}: (i,j)\in E\}, \quad \Gamma_i^-(A)=\{j\in V\backslash\{i\}: (j,i)\in E\}.$$

For a given $A=(a_{ij})\in C^{n\times n}$, we define the undirected graph $G(A)=(\tilde V,\tilde E)$ with vertex set $\tilde V=\langle n\rangle$ and edge set $\tilde E=\{\{i,j\}: a_{ij}\neq 0 \text{ or } a_{ji}\neq 0,\ 1\le i\neq j\le n\}$, and denote $G_i(A)=\Gamma_i^+(A)\cup\Gamma_i^-(A),\ E_\sigma=\{\{i,j\}\in\tilde E: i\in N_{\sigma_i},\, j\in N_{\sigma_j},\ \sigma_i\neq\sigma_j\}$.

2. Main Results

In this section we give an improved Brauer-type estimate for singular values .

* Received August 25, 2003; final revised November 6, 2003.

Lemma 2.1. *Let* $A=(a_{ij})\in C^{n\times n}$ *and give a partition* $\langle n\rangle=\bigcup_{j\in\langle m\rangle}N_j,\ N_i\cap N_j=\phi,\ i,j\in\langle m\rangle,\ i\neq j$. *If* $G_i(A)\neq\phi,\ \forall i\in\langle n\rangle$ *and* $G_i(A)\cap N_{\sigma_j}\supseteq G_j(A)\cap N_{\sigma_j},\ \forall j\in G_i(A)\backslash N_{\sigma_i},\ \forall i\in\langle n\rangle$. *Then*

$$\sigma(A)\subseteq(\bigcup_{i\in\langle n\rangle}D_i(A))\cup(\bigcup_{\substack{i\in N_{\sigma_i},\,j\in N_{\sigma_j}\\ \{i,j\}\in E_\sigma}}D_{ij}(A)),\tag{1}$$

where

$$D_i(A)=\{z\geq 0\,:\,|z-a_i|\leq S^{(i)}_{N_{\sigma_i}}(A)\},\qquad \forall i\in\langle n\rangle,$$

for all $i\neq j$,

$$D_{ij}(A)=\{z\geq 0\,:\,(|z-a_i|-S^{(i)}_{N_{\sigma_i}}(A))(|z-a_j|-S^{(j)}_{N_{\sigma_j}}(A))\leq\bar{S}^{(i)}_{N_{\sigma_i}}(A)\bar{S}^{(j)}_{N_{\sigma_j}}(A)\}.$$

Proof. For $\forall\sigma\in\sigma(A)$, there are two nonzero vectors $x=(x_1,\cdots,x_n)^T$ and $y=(y_1,\cdots,y_n)^T$ such that

$$\sigma x=Ay,\qquad \sigma y=A^*x.\tag{2}$$

We denote $z_i=\max\{|x_i|,\,|y_i|\},\forall i\in\langle n\rangle,\ z_p=\max_{j\in\langle n\rangle}\{z_j\},\,p\in N_{\sigma_p}$. Without loss of generality, we assume that $z_p=|y_p|\geq|x_p|$. Then the $p-$th equality in (2) implies

$$\sigma x_p-a_{pp}y_p=\sum_{j\in\Gamma_p^+(A)\cap N_{\sigma_p}}a_{pj}y_j+\sum_{j\in\Gamma_p^+(A)\backslash N_{\sigma_p}}a_{pj}y_j\tag{3}$$

$$\sigma y_p-\bar{a}_{pp}x_p=\sum_{j\in\Gamma_p^-(A)\cap N_{\sigma_p}}\bar{a}_{jp}x_j+\sum_{j\in\Gamma_p^-(A)\backslash N_{\sigma_p}}\bar{a}_{jp}x_j.\tag{4}$$

Write $\eta=x_p/y_p$. If $G_p(A)\subseteq N_{\sigma_p}$ or $z_j=0,\quad\forall j\in G_p(A)\backslash N_{\sigma_p}$, then (3) and (4) imply

$$|\sigma\eta-a_{pp}|\leq r^{(p)}_{N_{\sigma_p}}(A)\tag{5}$$

and

$$|\sigma-\eta\bar{a}_{pp}|\leq c^{(p)}_{N_{\sigma_p}}(A),\tag{6}$$

respectively . That $|\eta|\leq 1$. So, if $\sigma\leq a_p$, then $|\sigma-a_p|\leq|\eta||\sigma-a_p|\leq|\sigma\eta-a_{pp}|$, and if $\sigma\geq a_p$ then $|\sigma-a_p|\leq|\sigma-|\eta|a_p|\leq|\sigma-\eta\bar{a}_{pp}|$. Therefore, from (5) and (6) it can be deduced that

$$|\sigma-a_p|\leq S^{(p)}_{N_{\sigma_p}}(A),$$

i.e., $\sigma\in D_p(A)$.

If $\sigma\notin\bigcup_{i\in\langle n\rangle}D_i(A)$, by the above discussions we have $G_p(A)\backslash N_{\sigma_p}\neq\phi$ and $z_q=\max_{j\in G_p(A)\backslash N_{\sigma_p}}\{z_j\}>0$ (otherwise (5) and (6) imply $\sigma\in D_p(A)$). Thus equalities (3) and (4) imply

$$|\sigma-a_p|\leq S^{(p)}_{N_{\sigma_p}}(A)+\bar{S}^{(p)}_{N_{\sigma_p}}(A)\frac{z_q}{z_p}.\tag{7}$$

For $q\in N_{\sigma_q}\subset G_p(A)\backslash N_{\sigma_p}$, we have $G_q(A)\neq\phi$ (otherwise we can deduce $\sigma=a_q$, that is, $\sigma\in D_q(A)$). Similarly, if $z_q=|y_q|\geq|x_q|$,then it is easy to derive the following formula from the $q-$th equality in (2):

$$|\sigma-a_q|\leq S^{(q)}_{N_{\sigma_q}}(A)+\bar{S}^{(q)}_{N_{\sigma_q}}(A)\frac{z_p}{z_q}.\tag{8}$$

Note that $\sigma\notin\bigcup_{i\in\langle n\rangle}D_i(A)$, we have $|\sigma-a_p|>S^{(p)}_{N_{\sigma_p}}(A)$ and $|\sigma-a_q|>S^{(q)}_{N_{\sigma_q}}(A)$. Thus, from (7) and (8) we get

$$(|\sigma-a_p|-S^{(p)}_{N_{\sigma_p}}(A))(|\sigma-a_q|-S^{(q)}_{N_{\sigma_q}}(A))\leq\bar{S}^{(p)}_{N_{\sigma_p}}(A)\bar{S}^{(q)}_{N_{\sigma_q}}(A).\tag{9}$$

Since $q\in G_p(A)\backslash N_{\sigma_p}\neq\phi$, it holds that $\{p,q\}\in E_\sigma$.

Theorem 2.2. *Let $A = (a_{ij}) \in C^{n\times n}$ and give a partition $\langle n\rangle = \bigcup_{j\in\langle m\rangle} N_j,\ N_i \cap N_j = \phi,\ i, j \in \langle m\rangle,\ i \neq j$. Denote $\alpha_0 = \{i \in \langle n\rangle : G_i(A) = \phi\}$ and $G_i(A) \cap N_{\sigma_j} \supseteq G_j(A) \cap N_{\sigma_j}, \forall j \in G_i(A)\backslash N_{\sigma_i},\ \forall i \in \bar{\alpha}_0$.. Then*

$$\sigma(A) \subseteq \{a_i : i \in \alpha_0\} \cup (\bigcup_{i\in\bar{\alpha}_0} D_i(A)) \cup (\bigcup_{\substack{i\in N_{\sigma_i}\backslash\alpha_0\\ j\in N_{\sigma_j}\backslash\alpha_0\\ \{i,j\}\in E_\sigma}} D_{ij}(A)). \tag{10}$$

Proof. Without loss of generality, we assume that $\alpha_0 = \{1, \cdots, k\},\ k \in \langle n\rangle$. Then A has the form

$$A = \begin{pmatrix} a_{11} & & & \\ & \ddots & & 0 \\ & & a_{kk} & \\ & 0 & & A_{n-k} \end{pmatrix},$$

where $A_{n-k} \in C^{(n-k)\times(n-k)}$ is the principal submatrix of A with orders $n-k$ at least. Thus $\sigma(A) = \{a_i : i \in \alpha_0\} \cup \sigma(A_{n-k})$. It follows from Lemma 2.1 that

$$\sigma(A_{n-k}) \subseteq (\bigcup_{i\in\bar{\alpha}_0} D_i(A)) \cup (\bigcup_{\substack{i\in N_{\sigma_i}\backslash\alpha_0\\ j\in N_{\sigma_j}\backslash\alpha_0\\ \{i,j\}\in E_\sigma}} D_{ij}(A)).$$

This completes the proof.

Remark 1. Take $m = 2$. Then from Lemma 2.1, Theorem 1 of [5] can be deduced. Hence Theorem 2.2 of this paper is a generalization of the corresponding results in [4]-[6].

In the following, we will discuss the Brualdi-type estimate for singular values.

Theorem 2.3. *Let $A = (a_{ij}) \in C^{n\times n}$ and give a partition $\langle n\rangle = \bigcup_{j\in\langle m\rangle} N_j,\ N_i \cap N_j = \phi,\ i, j \in \langle m\rangle, i \neq j$. If $G_i(A) \neq \phi,\ \forall i \in \langle n\rangle$, and*

$$G_i(A) \cap N_{\sigma_j} \supseteq G_j(A) \cap N_{\sigma_j}, \quad \forall j \in G_i(A)\backslash N_{\sigma_i},\ \forall i \in \langle n\rangle, \tag{11}$$

then

$$\sigma(A) \subseteq (\bigcup_{i\in\langle n\rangle} D_i(A)) \cup (\bigcup_{\gamma\in C(A)} D_\gamma(A)), \tag{12}$$

where

$$D_\gamma(A) = \{z \geq 0 : \prod_{i\in\gamma}(|z - a_i| - S^{(i)}_{N_{\sigma_i}}(A)) \leq \prod_{i\in\gamma} \bar{S}^{(i)}_{N_{\sigma_i}}(A)\},\ \forall\gamma \in C(A)$$

and the set of nontrivial circuits, with length 2 at least, is denoted by $C(A)$ in $G(A)$.

Proof. Following the notations in Lemma 2.1, we let $z_p = \max_{j\in\langle n\rangle}\{z_j\} = |y_p| \geq |x_p|,\ z_j = \max\{x_j, y_j\},\ j \in \langle n\rangle,\ \eta = x_p/y_p$. Moreover, denote $z_{p_1} = \max_{j\in G_p(A)\backslash N_{\sigma_p}} \{z_j\}$. From equalities (3) and (4) we can deduce

$$|\sigma\eta - a_{pp}| \leq r^{(p)}_{N_{\sigma_p}}(A) + \bar{A}^{(p)}_{N_{\sigma_p}}(A)\frac{z_{p_1}}{z_p}, \tag{13}$$

$$|\sigma - \bar{a}_{pp}\eta| \leq c^{(p)}_{N_{\sigma_p}}(A) + \bar{c}^{(p)}_{N_{\sigma_p}}(A)\frac{z_{p_1}}{z_p}. \tag{14}$$

If $z_{p_1} = 0$ or $G_p(A)\backslash N_{\sigma_p} = \phi$, then (13) and (14) imply

$$|\sigma - a_p| \leq S^{(p)}_{N_{\sigma_p}}(A),$$

that is $\sigma \in D_p(A)$.

Let $\sigma \notin \bigcup_{i\in\langle n\rangle} D_i(A)$. Then $G_p(A)\backslash N_{\sigma_p} \neq \phi$ and $z_{p_1} > 0$. Let $z_{p_1} = |x_{p_1}| \geq |y_{p_1}|$ and $z_{p_2} = \max_{j\in G_{p_1}(A)\backslash N_{\sigma_{p_1}}} \{z_j\}$. Similarly, we can prove that $G_{p_1}(A)\backslash N_{\sigma_{p_1}} \neq \phi$ and $z_{p_2} > 0$ (otherwise $\sigma \in \bigcup_{i\in\langle n\rangle} D_i(A)$). Denote by $\xi = y_{p_1}/x_{p_1}$. Then, we have

$$|\sigma - \xi a_{p_1p_1}| \leq \sum_{j\in G_{p_1}(A)\cap N_{\sigma_{p_1}}} |a_{p_1j}||y_j|/|x_{p_1}| + \sum_{j\in G_{p_1}(A)\backslash N_{\sigma_{p_1}}} |a_{p_1j}||y_j|/|x_{p_1}|, \tag{15}$$

$$|\sigma\xi - \bar{a}_{p_1p_1}| \leq \sum_{j\in G_{p_1}(A)\cap N_{\sigma_{p_1}}} |\bar{a}_{jp_1}||x_j|/|x_{p_1}| + \sum_{j\in G_{p_1}(A)\backslash N_{\sigma_{p_1}}} |\bar{a}_{jp_1}||x_j|/|x_{p_1}|. \tag{16}$$

Note that $G_p(A)\backslash N_{\sigma_{p_1}} = \bigcup_{j\neq p} (G_p(A) \cap N_{\sigma_j})$, we see that (11) implies $z_{p_1} \geq \max_{j\in G_p(A)\cap N_{\sigma_{p_1}}} \{z_j\} \geq \max_{j\in G_{p_1}((A))\cap N_{\sigma_{p_1}}} \{z_j\}$. On the other hand, it is clear that $|\xi| \leq 1$. If $\sigma \leq a_{p_1}$ then $|\sigma - a_{p_1}| \leq |\sigma|\xi| - a_{p_1}| \leq |\sigma\xi - \bar{a}_{p_1p_1}|$, and if $\sigma \geq a_{p_1}$ then $|\sigma - a_{p_1}|| \leq |\sigma - a_{p_1}|\xi|| \leq |\sigma - \xi a_{p_1p_1}|$. Therefore, from (13)-(16), the following formulae are derived:

$$|\sigma - a_p| \leq S^{(p)}_{N_{\sigma_p}}(A) + \bar{S}^{(p)}_{N_{\sigma_p}}(A) z_{p_1}/z_p, \tag{17}$$

$$|\sigma - a_{p_1}| \leq S^{(p_1)}_{N_{\sigma_{p_1}}}(A) + \bar{S}^{(p_1)}_{N_{\sigma_{p_1}}}(A) z_{p_2}/z_{p_1}. \tag{18}$$

Moreover, since $\sigma \notin D_i(A)$ implies $|\sigma - a_i| > S^{(i)}_{N_{\sigma_i}}(A), \quad \forall i \in \langle n\rangle$, (17) and (18) are equivalent to

$$|\sigma - a_p| - S^{(p)}_{N_{\sigma_p}}(A) \leq \bar{S}^{(p)}_{N_{\sigma_p}}(A) z_{p_1}/z_p, \tag{19}$$

$$|\sigma - a_{p_1}| - S^{(p_1)}_{N_{\sigma_{p_1}}} \leq \bar{S}^{(p_1)}_{N_{\sigma_{p_1}}}(A) z_{p_2}/z_{p_1}. \tag{20}$$

Under similar discussions, by replacing p_1 by p_2 we have $z_{p_3} = \max_{j\in G_{p_2}(A)\backslash N_{\sigma_{p_2}}} \{z_j\} > 0$ and $G_{p_2}(A)\backslash N_{\sigma_{p_2}} \neq \phi$. Therefore,

$$|\sigma - a_{p_2}| - S^{(p_2)}_{N_{p_2}}(A) \leq \bar{S}^{(p_2)}_{N_{\sigma_{p_2}}} z_{p_3}/z_{p_2} \tag{21}$$

holds. Since $\sigma \notin D_i(A)$ and $G_i(p) \neq \phi, \quad \forall i \in \langle n\rangle$, the above process can be proceed continuously. Thus, in $G(A)$, the undirected edges $\{p, p_1\},\ \{p_1, p_2\},\ \{p_2, p_3\}, \cdots$ can be constituted. Because n is finite, there exists $s < t$ such that $p_s = p_t$, i.e., there exists a circuit γ_0 in $G(A) : \{q_1, , q_2\},\ \{q_2, q_3\},\ \{q_3, q_4\}, \cdots, \{q_t, q_{t+1}\} = \{q_t, q_1\}$ with $z_{q_{s+1}} = \max_{j\in G_{q_s}(A)\backslash N_{\sigma_{q_s}}} \{z_j\} > 0,\ z_{q_s} > 0,\ s \in \langle t\rangle$. From the above process we have

$$|\sigma - a_{q_s}| - S^{(q_s)}_{N_{\sigma_{q_s}}}(A) \leq \bar{S}^{(q_s)}_{N_{\sigma_{q_s}}}(A) z_{q_{s+1}}/z_{q_s}, \qquad \forall s \in \langle t\rangle. \tag{22}$$

Take product of the inequalities in (22) over all s, we obtain

$$\prod_{s=1}^{t} (|\sigma - a_{q_s}| - S^{q_s}_{N_{\sigma_{q_s}}}(A)) \leq \prod_{s=1}^{t} \bar{S}^{q_s}_{N_{\sigma_{q_s}}}(A),$$

that is

$$\prod_{j\in\gamma_0} (|\sigma - a_j| - S^{(j)}_{N_{\sigma_j}}(A)) \leq \prod_{j\in\gamma_0} \bar{S}^{(j)}_{N_{\sigma_j}}(A), \tag{23}$$

Thus, $\sigma \in D_{\gamma_0}(A)$.

Lemma 2.4. *Let $A = (a_{ij}) \in C^{n\times n}$ satisfy all assumptions in Theorem 2.3. Then*

$$\bigcup_{\gamma\in C(A)} D_\gamma(A) \subseteq \bigcup_{\substack{i\in N_{\sigma_i}\\ j\in N_{\sigma_j}\\ \{i,j\}\in E_\sigma}} D_{ij}(A). \tag{24}$$

Proof. For every $\gamma \in C(A)$, denote $\gamma: \{i_1, i_2\}, \{i_2, i_3\}, \cdots, \{i_k, i_{k+1}\} = \{i_k, i_1\}$ and the length of γ by $|\gamma| = k$. When $k = 2$, the result holds obviously.

When $k \geq 3$, if the result does not hold, then for any $z \in D_r(A)$ we have

$$(|z-a_i| - S^{(i)}_{N_{\sigma_i}}(A))(|z-a_j| - S^{(j)}_{N_{\sigma_j}}(A)) > \bar{S}^{(i)}_{N_{\sigma_i}}(A)\bar{S}^{(j)}_{N_{\sigma_j}}(A), \quad \forall \{i,j\} \in \gamma.$$

Moreover,

$$\begin{aligned}(\prod_{i\in\gamma}(|z-a_i| - S^{(i)}_{N_{\sigma_i}}(A)))^2 &= (|z-a_{i_1}| - S^{(i_1)}_{N_{\sigma_{i1}}}(A))(|z-a_{i_2}| - S^{(i_2)}_{N_{\sigma_{i_2}}}(A)) \\ &\quad (|z-a_{i_2}| - S^{(i_2)}_{N_{\sigma_{i_2}}}(A))\cdots(|z-a_{i_k}| - S^{(i_k)}_{N_{\sigma_{i_k}}}(A)) \\ &\quad (|z-a_{i_k}| - S^{(i_k)}_{N_{\sigma_{i_k}}}(A))(|z-a_{i_1}| - S^{(i_1)}_{N_{\sigma_{i_1}}}(A)) \\ &> \bar{S}^{(i_1)}_{N_{\sigma_{i_1}}}(A)\bar{S}^{(i_2)}_{N_{\sigma_{i_2}}}(A)\bar{S}^{(i_2)}_{N_{\sigma_{i_2}}}(A)\cdots\bar{S}^{(i_k)}_{N_{\sigma_{i_k}}}(A)\bar{S}^{(i_k)}_{N_{\sigma_{i_k}}}(A)\bar{S}^{(i_1)}_{N_{\sigma_{i_1}}}(A) \\ &= (\prod_{i\in\gamma}\bar{S}^{(i)}_{N_{\sigma_i}}(A))^2,\end{aligned}$$

that is, $\prod_{i\in\gamma}(|z-a_i| - S^{(i)}_{N_{\sigma_i}}(A)) > \prod_{i\in\gamma}\bar{S}^{(i)}_{N_{\sigma_i}}(A)$. This contradicts with $z \in D_\gamma(A)$.

From Theorem 2.3 and Lemma 2.4 we can obtain the following theorem immediately.

Theorem 2.5. *Let $A = (a_{ij}) \in C^{n\times n}$ satisfy all assumptions in Theorem 2.3. Then*

$$\sigma(A) \subseteq (\bigcup_{i\in\langle n\rangle} D_i(A)) \cup (\bigcup_{\gamma\in C(A)} D_\gamma(A)) \subseteq (\bigcup_{i\in\langle n\rangle} D_i(A)) \cup (\bigcup_{\substack{\{i,j\}\in\gamma \\ \gamma\in C(A)}} D_{ij}(A)). \tag{25}$$

In general, we have the following result.

Theorem 2.6. *Let $A = (a_{ij}) \in C^{n\times n}$ satisfy all assumptions in Theorem 2.2 and $\bar{\alpha}_0 = \langle n\rangle\backslash\alpha_0 \neq \phi$. If A satisfies $G_i(A) \cap (N_{\sigma_j}\backslash\alpha_0) \supseteq G_j(A) \cap (N_{\sigma_j}\backslash\alpha_0)$, $\forall j \in G_i(A)\backslash(N_{\sigma_j}\backslash\alpha_0)$, $\forall i \in \bar{\alpha}_0$, then*

$$\sigma(A) \subseteq \{a_i : i \in \alpha_0\} \cup (\bigcup_{i\in\bar{\alpha}_0} D_i(A)) \cup (\bigcup_{\gamma\in C(A)} D_\gamma(A)). \tag{26}$$

The proof is similar to Theorem 2.2 and is thus omitted.

Remark 2. In [8] the authors gave an interval of Brualdi-type for singular values under the assumptions $\Gamma_i^+(A) \neq \phi$, $\Gamma_i^+(A) \supseteq \Gamma_i^-(A)$, $\forall i \in \langle n\rangle$. But Theorem 2.3 is obtained under the assumptions $G_i(A) \neq \phi$, $G_i(A) \cap N_{\sigma_j} \supseteq G_j(A) \cap N_{\sigma_j}$, $\forall i \in \langle n\rangle$, $\forall j \in G_i(A)\backslash N_{\sigma_i}$ which are different from the assumptions in [8].

3. Example

By elementary calculations, [4] give the following result. Let $0 \leq a \leq b$ and $g \geq 0$. Set $c = (a+b)/2, d = (b-a)/2$. Then

$$\{z \geq 0 : |z-a||z-b| \leq g\} = [(c-(d^2+g)^{1/2})_+, c-((d^2-g)_+)^{1/2}] \cup [c+((d^2-g)_+)^{1/2}, c+(d^2+g)^{1/2}], \tag{29}$$

where $u_r = max\{0, u\}, u \in R$.

We shall arrange the singular values in decreasing order.

Example is the following: consider

$$A = \begin{pmatrix} 1 & 0.1 & 0.1 & 0 \\ 0 & 2 & 0.1 & 0.1 \\ 0 & 0 & 3 & 0.4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Apply Theorem 2.2 and (29). Let $N_1 = \{1\}, N_2 = \{2,3\}, N_3 = \{4\}$. It can be verified that

$$S_{N_1}{}^{(1)} = 0, \quad S_{N_2}{}^{(2)} = 0.1, \quad S_{N_2}{}^{(3)} = 0.1, \quad S_{N_3}{}^{(4)} = 0,$$
$$\overline{S}_{N_1}{}^{(1)} = 0.2, \quad \overline{S}_{N_2}{}^{(2)} = 0.1, \quad \overline{S}_{N_2}{}^{(3)} = 0.1, \quad \overline{S}_{N_3}{}^{(4)} = 0.2,$$

and

$$\begin{aligned}&D_1(A)=\{1\},\ D_2(A)=[1.9,\ 2.1],\ D_3(A)=[2.9,\ 3.1], D_4(A)=\{4\},\\&D_{12}(A)=[0.9783,\ 1.0228]\cup[1.8772,\ 2.1179],\\&D_{13}(A)=[0.9895,\ 1.0106]\cup[2.8894,\ 3],\\&D_{24}(A)=[2.0895,\ 2.1106]\cup[3.9894,\ 4.0105],\\&D_{34}(A)=[2.8821,\ 3.1228]\cup[3.9772,\ 4.0217].\end{aligned}$$

Thus by Theorem 6 in [4] the singular values of A satisfy

$$\begin{aligned}&\sigma_1\in[3.9772,\ 4.0217],\quad \sigma_2\in[2.8821,\ 3.1228],\\&\sigma_3\in[1.8772,\ 2.1179],\quad \sigma_4\in[0.9783,\ 1.0228].\end{aligned}\tag{30}$$

Applying the techniques in [4],we obtain

$$\begin{aligned}&\{z\geq 0:|z-1||z-2|\leq 0.04\}=[0.9615,\ 1.0417]\cup[1.9583,\ 2.0385],\\&\{z\geq 0:|z-1||z-3|\leq 0.04\}=[0.9802,\ 1.0202]\cup[2.9798,\ 3.0198],\\&\{z\geq 0:|z-2||z-3|\leq 0.04\}=[1.9615,\ 2.0417]\cup[2.9583,\ 3.0385],\\&\{z\geq 0:|z-2||z-4|\leq 0.04\}=[1.9802,\ 2.0202]\cup[3.9798,\ 4.0198],\\&\{z\geq 0:|z-3||z-4|\leq 0.04\}=[2.9615,\ 3.0417]\cup[3.9798,\ 4.0198].\end{aligned}$$

Therefore, the singular values of A satisfy

$$\begin{aligned}&\sigma_1\in[3.9583,\ 4.0385],\quad \sigma_2\in[2.9615,\ 3.0417],\\&\sigma_3\in[1.9583,\ 2.0385],\quad \sigma_4\in[0.9615,\ 1.0417].\end{aligned}\tag{31}$$

The bounds of σ_1 and σ_4 in (30) are batter than those in (31).

Clearly A satisfies all assumptions in Theorem 2.3 and there exist three circuits in $G(A)$: γ_1: 1-2-3-1, γ_2: 1-2-3-4-1, γ_3: 2-3-4-2. Moreover, we can obtain:

$$\begin{aligned}&D_{\gamma_1}(A)=\{z\geq 0:|z-1|(|z-2|-0.1)(|z-3|-0.1)\leq 0.002\},\\&D_{\gamma_2}(A)=\{z\geq 0:|z-1|(|z-2|-0.1)(|z-3|-0.1)|z-4|\leq 0.0004\},\\&D_{\gamma_3}(A)=\{z\geq 0:(|z-2|-0.1)(|z-3|-0.1)|z-4|\leq 0.002\}.\end{aligned}$$

Thus by Theorem 2.3 ,it can be verified that

$$\sigma(A)\subseteq(\bigcup_{i\in\langle 4\rangle}D_i(A))\cup(\bigcup_{j\in\langle 3\rangle}D_{\gamma_j}(A))\doteq[0.9988,\ 4.0012].\tag{32}$$

The upper bound of σ_1 and the lower bound of σ_4 in (32) are better than those in (30)and (31).

References

[1] Qi L, Some simple estimates for singular values of a matrix, *Linear Algebra Appl.*, **56** (1984), 105-119.

[2] Johnson, C.R., A Gerschgorin-type lower bound for the smallest singular value, *Linear Algebra Appl.*, **112** (1989), 1-7.

[3] Johnson, C.R., Szule.T., Futher lower bounds for the smallest singular value, *Linear Algebra Appl.*, **272** (1998), 169-179.

[4] Luoluo Li, The undirected graph and estimates of matrix singular values, *Linear Algebra Appl.*, **285** (1998), 181-188.

[5] Luoluo Li, New results on estimates for singular values, *Num. Math.J. of Chinese Universities*, **9**:1 (2000), 55-60.

[6] Li Saijian and Luoluo Li, Brauer-type theorems for estimating singular values of a matrix, *Act. Scien. Nature Univer. Sunyatseni*, **37**:4 (1998), 43-45.

[7] Zuxiang Li, Mingxian Pang, The estimations of lower bounds for singular value of matrix, *Math. Numer. Sinca,* **20**:4 (1998), 377-382.

[8] Jiongsheng Li, Kai Yang, Qingxue Wang, Digraphs and inclusion intervals of Brualdi-type for singular values, *Act. Math. Appl. Sinica*, **18**:3 (2002), 471-476.

[9] Horn.R.A., Johnson C.R., Matrix analysis, New York: Cambridge Uni. Press, 1985.

[10] Horn.R.A., Johnson C.R., Topics in matrix analysis, New York: Cambridge Uni. Press, 1991.