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QUANTUM COMPLEXITY OF THE INTEGRATION PROBLEM FOR ANISOTROPIC CLASSES *1)

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Abstract

We obtain the optimal order of high-dimensional integration complexity in the quantum computation model in anisotropic Sobolev classes $W^{\mathbf{r}}_{\infty}([0,1]^d)$ and Hölder Nikolskii classes $H^{\mathbf{r}}_{\infty}([0,1]^d)$. It is proved that for these classes of functions there is a speed-up of quantum algorithms over deterministic classical algorithms due to factor n^{-1} and over randomized classical methods due to factor $n^{-1/2}$. Moreover, we give an estimation for optimal query complexity in the class $H^{\Lambda}_{\infty}(D)$ whose smoothness index is the boundary of some complete set in $\mathbb{Z}^d_+$.

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1. Introduction

Quantum computers, whose basic operators are based on the theory of quantum mechanics, equip with the amazing computational speed which is much faster than that of classical computers. The questions arisen by the powerful conceptual machines are studied in computer science but seldom done in numerical analysis, see [4, 24, 14]. The pioneering work about the quantum complexity for numerical problem was done by Novak, [19]. After that, a series of papers about summation of sequences and multivariate integration of functions by Novak and Heinrich were published, see [12, 10, 11]. In [25], Traub initially discussed the quantum complexity of path integration.

In this paper, we continue the study of the problem of high-dimensional integration. Usually, the need to understand the complexity of the problems in the deterministic and randomized settings will help to judge the possible gains by quantum computation. In information-based complexity theory, the complexity of integration problems is well known for classical function classes. Recently, Fang and Ye [7] obtained the exact order of integration problem for anisotropic Sobolev classes and Holder-Nikolskii classes in the classical deterministic and randomized settings. Our goal is to study the complexity in the quantum computation model. Compared to the known results of complexities for some anisotropic classes, we hope that there exists an essential speed-ups under quantum computation similar to what happens for the classical Sobolev classes.

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We obtain the optimal order of high-dimensional integration complexity in the quantum computation model for anisotropic Sobolev classes $W^{\mathbf{r}}_{\infty}([0,1]^d)$ and Hölder Nikolskii classes $H^{\mathbf{r}}_{\infty}([0,1]^d)$. Our method is based on the discrete skill which is used in [11]. But we develop some new skills to overcome the difficulties of anisotropy and weaker smoothness which arise from the the study of our classes. For more details on the quantum setting for numerical problems we refer to [10]. For general background on quantum computing we refer to the surveys [8, 21] and to the monographs [16, 22].

We organize this paper as follows. In section two, we review the quantum computation model. In section three, the integration problems in anisotropic classes are briefly introduced. Moreover, we present the main results of our paper. Section four reviews some known results which is used in the proof of theorems. Finally, the proof of the new results are presented in section five.

2. Quantum Computation Model

In this section we introduce the quantum computation model. We start with adopting some notations following [11, 19]. For nonempty sets Ω and K we denote the set of all function from Ω to K by $\mathcal{F}(\Omega, K)$. Let G be a normed space with scalar field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$, and let S be any mapping from F to G, where $F \subset \mathcal{F}(\Omega, \mathbb{R})$. we want to approximate $S(f)$ for $f \in F$ by quantum computations. Denote

$$\mathbb{Z}[0, N) := \{0, \dots, N-1\}$$

for $N \in \mathbb{N}$. Let H_m be m-fold tensor product of H_1, two-dimensional Hilbert space over $\mathbb{C}$, and let $\{e_0, e_1\}$ be two orthonormal basis of H_1. An orthonormal basis of H_m, denoted by $\mathcal{C}_m$, consist of the vectors $|l >:= e_{i_0} \otimes \dots \otimes e_{i_m}$ $(l \in \mathbb{Z}[0, 2^{m-1}))$, where $\otimes$ is the tensor product, $i_j \in \{0,1\}$ and $l = \sum_{j=0}^{2^m-1} i_j 2^{m-1-j}$. Let $\mathcal{U}(H_m)$ stand for the set of unitary operator on H_m.

Two mappings are defined respectively by

$$\tau : Z \to \Omega \qquad \text{and} \qquad \beta : K \to \mathbb{Z}[0, 2^{m''}).$$

where for m, m', $m'' \in \mathbb{N}$, $m' + m'' \leq m$ and Z is the nonempty subset of $\mathbb{Z}[0, 2^{m'})$. A quantum query on F is give by a tuple

$$Q = (m, m', m'', Z, \tau, \beta),$$

and the number of quits $m(Q) := m$. We define the unitary operator Q_f for a given query Q by setting for each $f \in F$

$$Q_f |i > |x > |y > := \begin{cases} |i > |x \oplus \beta(f(\tau(i))) > |y > & \text{if } i \in Z, \\ |i > |x > |y > & \text{otherwise,} \end{cases}$$

where set $|i > |x > |y > \in \mathcal{C}_m := \mathcal{C}_{m'} \otimes \mathcal{C}_{m''} \otimes \mathcal{C}_{m-m'-m''}$ and denote addition modulo $2^{m''}$ by $\oplus$.

Let tuple $A = (Q, (U_j)_{j=0}^n)$ denote a quantum algorithm on F with no measurement, where Q is a quantum query on F, $n \in \mathbb{N}_0$ $(\mathbb{N} = \mathbb{N} \bigcup \{0\})$ and $U_j \in \mathcal{U}(H_m)$, with $m = m(Q)$. For each $f \in F$, we have $A_f \in \mathcal{U}(H_m)$ with the following form

$$A_f = U_n Q_f U_{n-1} \dots U_1 Q_f U_0.$$

The number of queries is denoted by $n_q(A) := n$. A quantum algorithm $A : F \to G$ with k measurements is defined for $k \in \mathbb{N}$ by

$$A = ((A_l)_{l=0}^{k-1}, (b_l)_{l=0}^{k-1}, \phi),$$

where $A_l (l \in \mathbb{Z}[0, k))$ are quantum algorithms on F with no measurements, $b_0 \in \mathbb{Z}[0, 2^{m_0})$ is a fixed basis state with which A starts. for $1 \le l \le k-1$, apply the quantum operations to b_{l-1} and get a random state ζ_{l-1}. The resulting state ζ_{l-1} is memorized and transformed into a new basis state b_l,

$$b_l : \prod_{i=0}^{l-1} \mathbb{Z}[0, 2^{m_i}) \to \mathbb{Z}[0, 2^{m_l}),$$

where we denote $m_l := m(A_l)$ and ϕ is a function

$$\phi : \prod_{i=0}^{k-1} \mathbb{Z}[0, 2^{m_l}) \to G.$$

Let $n_q := \sum_{l=0}^{k-1} n_q(A_l)$ denote the number of queries used by A and $(A_f(x, y))_{x,y \in \mathcal{C}_m}$ the matrix of the transformation A_f in the canonical basis $\mathcal{C}_m$, $A_f(x, y) =< x|A_f|y >$. The output of A at input $f \in F$ will be a probability measure $A(f)$ on G, defined as follows:

$$\begin{aligned} p_{A,f}(x_0, \ldots, x_{k-1}) = \quad & |A_{0,f}(x_0, b_0)|^2 |A_{1,f}(x_1, b_1(x_0))|^2 \\ & \ldots |A_{k-1,f}(x_{k-1}, b_{k-1}(x_0, \ldots, x_{k-2}))|^2. \end{aligned}$$

Define $A(f)$ by

$$A(f)(C) = \sum_{\phi(x_0,\ldots,x_{k-1}) \in C} p_{A,f}(x_0, \ldots, x_{k-1}), \qquad \forall C \subset G.$$

The probabilistic error of A is defined as follows: Let $0 \le \theta < 1$, $f \in F$, let ξ be any random variable with distribution $A(f)$, and let

$$e(S, A, f, \theta) = \inf\{\epsilon \ge 0 : P\{\|s(f) - \xi\| > \epsilon\} \le \theta\},$$

$$e(S, A, F, \theta) = \sup_{f \in F} e(S, A, f, \theta),$$

$$e(S, A, f) = e(S, A, f, 1/4),$$

and

$$e(S, A, F) = e(S, A, F, 1/4).$$

The nth minimal query error is defined by

$$e_n^q(S, F, \theta) = \inf\{e(s, A, F, \theta)\} : A \text{ is any quantum algorithm with } \quad n_q(A) \le n, n \in \mathbb{N}_0\}$$

and the query complexity is defined for $\epsilon > 0$ by

$$comp_\epsilon^q(S, F) = \min\{n_q(A) : A \text{ is any quantum algorithm with } \quad e(S, A, F) \le \epsilon\}.$$

3. Integration

The main results are stated in this section. First, we recall some notation of function classes on $D = [0,1]^d$. For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{N}^d$ and $1 \le p \le \infty$, the anisotropic Sobolev classes $W_p^{\mathbf{r}}(D)$ are defined as follows

$$W_p^{\mathbf{r}}(D) := \Big\{ f \in L_p(D) : \|f\|_{L_p(D)} + \sum_{j=1}^{d} \|f\|_{L_p^{r_j}} \le 1 \Big\}, \tag{3.1}$$

where

$$\|f\|_{L_p^{r_j}} = \Big\| \frac{\partial^{r_j} f}{\partial x_j^{r_j}} \Big\|_{L_p(D)}, \qquad j = 1, \ldots, d.$$

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$, $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ and $1 \le p \le \infty$, the Nikolskii classes $H_p^{\mathbf{r}}(D)$ is defined by

$$H_p^{\mathbf{r}}(D) := \Big\{ f \in L_p(D) : \omega_{a_j}(f, \sigma_j, D)_p \le \sigma_j^{r_j}, a_j = [r_j] + 1, j = 1, \ldots, d \Big\}, \tag{3.2}$$

where

$$\omega_{a_j}(f, \sigma_j, D)_p = \sup_{0 \le h_j \le \sigma_j} \|\triangle_{h_j}^{a_j}(f, \mathbf{t})\|_{L_p(D)}$$

is the p-th modulus of smoothness of f at the j-th coordinate, $\triangle_{h_j}^{a_j}$ is the usual a_jth forward difference of step length h_j with respect to t_j, and $[r_j]$ denoted by the integer part of r_j. For more details about anisotropic classes we refer to [13, 17, 23]. We introduce the notation

$$g(\mathbf{r}) = (\sum_{j=1}^{d} \frac{1}{r_j})^{-1}, \tag{3.3}$$

which is a good measurement of average smoothness of anisotropy and plays an important role in our error estimation. We define the integration operator $I_d : \mathbf{S}(D) \to \mathbb{R}$ by

$$I_d(f) = \int_D f(\mathbf{t}) d\mathbf{t}, \tag{3.4}$$

where $\mathbf{S}$ denote $W_\infty^{\mathbf{r}}$ or $H_\infty^{\mathbf{r}}$ or $H_\infty^{\mathbf{\Lambda}}$. In the following, let $\mathbf{e} \in \mathbb{N}^d = (1, 1, \ldots, 1)$ and let $\mathbf{e}_j = (\delta_{i,j})$, the Kronecker notation. Denote the weak equivalence of two functions $a(n)$ and $b(n)$, by $a(n) \asymp b(n)$, which means that for sufficiently large n there exist two positive constant c_1 and c_2 such that $c_1 b(n) \le a(n) \le c_2 b(n)$. The results of Norvak and Heinrich, see [19, 11], imply the following theorem:

Theorem A. *Let r, $d \in \mathbb{N}$, $1 \le p < \infty$ and assume $rp > d$, Then for all $n \in \mathbb{N}$ with $n > 4$*

$$e_n^q(I_d, W_p^{r \cdot \mathbf{e}}) \asymp n^{-\frac{r}{d} - 1} \cdot \alpha,$$

where

$$\alpha := \begin{cases} 1 & \textit{if } 2 < p \le \infty, \\ (\log\log n)^{3/2} \log\log\log n & \textit{if } p = 2, \\ (\log n)^{2/p - 1} & \textit{if } 1 \le p < 2. \end{cases}$$

We partially extend above results. Our main result is

Theorem 3.1. *Let F be one of the classes $W^{\mathbf{r}}_\infty(D)$ or $H^{\mathbf{r}}_\infty(D)$. Then*

$$e^q_n(I_d, F) \asymp n^{-g(\mathbf{r})-1}.$$

Comparing this with the known recent results of Fang and Ye about the classes W and H, where they proved the optimal rate of convergence is $n^{-g(\mathbf{r})}$ while that of randomized algorithms is $n^{-g(\mathbf{r})-1/2}$, we find that there is a speed-up of quantum algorithms over deterministic classical algorithm due to factor n^{-1} and a speed-up over randomized classical methods due to factor $n^{-1/2}$. Notice that the case of uniform norm is particularly interesting, since the integration problem in this case is intractable in the worst case deterministic setting, see [1-3,20,7].

Moreover we state the result of integration problem for the generalized Hölder-Nikolskii class as follow. Let us recall some definitions which we need from [6]. We consider a bounded set $\Lambda \in \mathbb{Z}^d_+$ to be complete, if $\alpha' \in \Lambda$ when $\alpha \in \Lambda$ and $\alpha' \le \alpha$. The boundary of Λ is defined by

$$\partial\Lambda := \{\, \alpha : \ \alpha \notin \Lambda, \ \text{and if } \alpha' < \alpha, \ \text{then } \alpha' \in \Lambda \,\}, \tag{3.5}$$

where $\alpha' < \alpha$ means that there exists at least one coordinate direction j such that $\alpha'_j < \alpha_j$. For $\alpha \in \mathbb{Z}^d_+$, and $\sigma = (\sigma_1, \ldots, \sigma_d) > \mathbf{0}$, we define

$$\Delta^\alpha_\sigma(f, \mathbf{t}) := \Delta^{\alpha_1}_{\sigma_1} \cdots \Delta^{\alpha_d}_{\sigma_d}(f, \mathbf{t}),$$

and multivariate modulus of smoothness

$$\omega_\alpha(f, \sigma, D)_p := \sup_{\mathbf{0}<\mathbf{h}<\sigma} \|\Delta^\alpha_\sigma(f, \mathbf{t})\|_p(D(\alpha, \sigma)),$$

where

$$D(\alpha, \sigma) := \{\, \mathbf{t} : (t_j + s_j\alpha_j)^d_1 \in D \text{ for all } \mathbf{s} \le \sigma, \mathbf{s} \in \mathbb{R}^d_+ \,\}.$$

For Λ is a complete set, the generalized Hölder-Nikolskii class $H^\Lambda_p(D)$, which is a generalization of the classical and anisotropic Hölder-Nikolskii class, is defined by

$$H^\Lambda_p(D) := \Big\{\, f \in L_p(D) : \ \omega_\Lambda(f, \sigma, D)_p \le \sum_{\alpha\in\partial\Lambda} \sigma^\alpha, \ \alpha = (\alpha_1, \ldots \alpha_d) \in \partial\Lambda \,\Big\},$$

where $\sigma^\alpha = \prod_{j=1}^d \sigma_j^{\alpha_j}$ and $\omega_\Lambda(f, \sigma, D)_p = \sum_{\alpha\in\partial\Lambda} \omega_\alpha(f, \sigma, D)_p$.

Theorem 3.2. *Let F be the class $H^\Lambda_\infty(D)$. Then there is a vector $\mathbf{r}_\Lambda \in \partial\Lambda$ such that*

$$e^q_n(I_d, F) \le cn^{-g(\mathbf{r}_\Lambda)-1}.$$

4. Some Auxiliary Results

In this section we cite some known results [12, 11] which will serve as building blocks in our proof of theorems. We introduce a mapping $\Gamma : F \to \tilde{F}$ which defined below. For κ, $m^* \in \mathbb{N}$ there are mappings

$$\begin{aligned} \eta_j &: \ \tilde{\Omega} \to \Omega, \\ \beta &: \ k \to \mathbb{Z}[0, 2^{m^*}), \\ \varrho &: \ \tilde{\Omega} \times \mathbb{Z}[0, 2^{m^*})^\kappa \to \tilde{K} \end{aligned}$$

such that for $f \in F$ and $s \in \tilde{\Omega}$

$$(\Gamma(f))(s) = \varrho(s, \beta \circ f \circ \eta_0(s), \ldots, \beta \circ f \circ \eta_{\kappa-1}(s))\,. \tag{4.1}$$

Therefore, we can receive a function $\tilde{f} = \Gamma(f) \in \tilde{F}$ for $f \in F$. Using an already known and reduced algorithm $\tilde{A}$ on $\tilde{F}$, we construct an algorithm A on F.

Lemma A. *Given a mapping $\Gamma : F \to \tilde{F}$ as in (4.6), a normed space G and a algorithm $\tilde{A}$ from $\tilde{F}$ to G, there is a quantum algorithm A from F to G with*

$$n_q(A) = 2\kappa n_q(\tilde{A})$$

and for all $f \in F$

$$A(f) = \tilde{A}(\Gamma(f)).$$

Consequently, if $\tilde{S} : \tilde{F} \to G$ is any mapping and $S = \tilde{S} \circ \Gamma$, then for each $n \in \mathbb{N}_0$

$$e^q_{2\kappa n}(S, F) \leq e^q_n(\tilde{S}, \tilde{F})\,.$$

Lemma B. *Let Ω, K and $F \subseteq \mathcal{F}(\Omega, K)$ be nonempty sets, let $k \in \mathbb{N}_0$ and let $S_l : F \to \mathbb{R}$ $(l = 0, \ldots, k)$ be mappings. Define $S : F \to \mathbb{R}$ by $S(f) = \sum_{l=0}^{k} S_l(f)$ $(f \in F)$. Let $n_0, \ldots, n_k \in \mathbb{N}_0$.*

(i) *Assume $\theta_0, \ldots, \theta_k \geqslant 0$ and put $n = \sum_{l=0}^{k} n_l$. Then*

$$e^q_n(S, F, \sum_{l=0}^{k} \theta_l) \leq \sum_{l=0}^{k} e^q_{n_l}(S_l, F, \theta_l).$$

(ii) *Assume $v_0, \ldots, v_k \in \mathbb{N}$ satisfy $\sum_{l=0}^{k} e^{-v_l/8} \leq 1/4$. Put $n = \sum_{l=0}^{k} v_l n_l$. Then*

$$e^q_n(S, F) \leq \sum_{l=0}^{k} e^q_{n_l}(S_l, F).$$

Lemma C. *Let S, $T : F \to G$ be any mappings, $n \in N_0$ and assume that $e^q_n(S, F)$ is finite. Then the following hold:*

(i) $e^q_n(T, F) \leq e^q_n(S, F) + \sup_{f \in F} |T(f) - S(f)|$.

(ii) *If $K = \mathbb{K}$ and S is a linear operator from $\mathcal{F}(D, K)$ to G, then for all $\lambda \in \mathbb{K}$*

$$e^q_n(S, \lambda F) = |\lambda| \cdot e^q_n(S, F)\,.$$

Due to the need of next section, we introduce a known results about summation from [10]. Let the space L^N_∞ consists of all functions $f : \mathbb{Z}[0, N) \to \mathbb{R}$, equipped with the norm

$$\|f\|_{L^N_\infty} = \max_{0 \leq i \leq N-1} |f(i)|\,.$$

A mapping S_N from L^N_∞ to $\mathbb{R}$ is defined by

$$S_N f = \frac{1}{N} \sum_{i=0}^{N-1} f(i), \tag{4.2}$$

and $\mathcal{B}(L_\infty^N)$ denote the unit ball of L_∞^N, which is the set of all functions $f \in L_\infty^N$ whose norm is no more than one.

Theorem B. *There is constant c such that for all n, $N \in \mathbb{N}$ with $2 < n \le cN$,*

$$e_n^q(S_N, \mathcal{B}(L_\infty^N)) \asymp n^{-1}.$$

5. The Proof of Results

Our presentation is essentially based on ideas and methods developed for isotropic Sobolev spaces. However, because of the anisotropy, several basic properties applied in the isotropic setting so not hold or are different. Consequently, a series of arguments in the subsequent analysis have to be adapted, replaced, and generalized.

Lemma D. *Let Q be a rectangle with side length vector $\delta = (\delta_1, \ldots, \delta_d)$. Then for each $f \in L_\infty(Q)$ and $\mathbf{a} \in \mathbb{N}^d$ there exists a polynomial $g \in P_{\mathbf{a}}$ with*

$$\|f - g\|_{L_\infty(Q)} \le c\omega_{\mathbf{a}}(f, \delta, Q)_\infty,$$

where $\omega_{\mathbf{a}}(f, \delta, Q)_\infty = \sum_{j=1}^d \omega_{a_j}(f, \delta, Q)_\infty$ and $P_{\mathbf{a}} = \operatorname{span}\{x^{\mathbf{k}} : k_j < a_j, 1 \le j \le d\}$.

Lemma 5.1. *For $\mathbf{a} \in \mathbb{N}^d$, let J be any quadrature rule on $C(D)$ with bounded operator norm, which is exact on $P_{\mathbf{a}}(D)$, i.e.,*

$$Jg = I_d g \qquad \forall g \in P_{\mathbf{a}}(D). \tag{5.1}$$

Then for $f \in L_\infty(D)$ we have

$$|I_d f - Jf| \le c \cdot \omega_{\mathbf{a}}(f, \mathbf{e}, D)_\infty. \tag{5.2}$$

Proof. Using Lemma D, we have

$$\begin{aligned} |I_d f - Jf| &\le \inf_{g \in P_{\mathbf{a}}(D)} |I_d(f - g) - J(f - g)| \\ &\le \inf_{g \in P_{\mathbf{a}}(D)} \max(\|I_d\|, \|J\|)\|f - g\|_{L_\infty(D)} \\ &\le c \cdot \omega_{\mathbf{a}}(f, \mathbf{e}, D)_\infty \,. \end{aligned} \tag{5.3}$$

The proof of Theorem 3.1. By means of imbedding relationship that $W_\infty^{\mathbf{r}}(D)$ is imbedded into $H_\infty^{\mathbf{r}}(D)$, denoted by $W_\infty^{\mathbf{r}}(D) \hookrightarrow H_\infty^{\mathbf{r}}(D)$, see [17], it is only required to prove the upper estimate bound for the class $H_\infty^{\mathbf{r}}(D)$ and the lower bound for the class $W_\infty^{\mathbf{r}}(D)$.

First, we utilize the smoothness in different coordination direction to slip the cube D. For $P_0 \in \mathbb{N}_0$, let $m_j(k) = [2^{P_0 \frac{g(\mathbf{r})}{r_j}}], j = 1, \ldots, d$, where P_0 is sufficiently large such that satisfy $m_j(k) > 1$. According to the equation (3.4) we can find

$$\prod_{j=1}^d m_j(k) \asymp [2^{P_0 g(\mathbf{r}) \dot{\sum}_{j=1}^d 1/r_j}] \asymp 2^{P_0} \,. \tag{5.4}$$

The cube D is divided into $2^{P_0 l}$ congruent rectangle T_{li} of disjoint interior, i.e.

$$D = \bigcup_{i=0}^{2^{P_0 l}-1} T_{li}.$$

Let $\mathbf{s}_{li}$ denote the point in T_{li} with the smallest Euclidean norm. We introduce the following extension operator

$$E_{li}:\mathcal{F}(D,\mathbb{R})\to\mathcal{F}(D,\mathbb{R})$$

by setting

$$(E_{li}f)(\mathbf{s})=f(\mathbf{s}_{li}+(m_1^{-l},\ldots,m_d^{-l})^T\mathbf{s}), \tag{5.5}$$

for $f\in\mathcal{F}(D,\mathbb{R})$ and $\mathbf{s}\in D$. Now define

$$Jf=\sum_{j=0}^{\kappa-1}b_jf(\mathbf{t}_j)\qquad (f\in C(D)),$$

where $b_j\in\mathbb{R}$ and $\mathbf{t}_j\in D$. For $l\in\mathbb{N}_0$, let

$$\begin{aligned} J_lf &= 2^{-P_0l}\sum_{i=0}^{2^{P_0l-1}}J(E_{li}f)\\ &= 2^{-P_0l}\sum_{i=0}^{2^{P_0l-1}}\sum_{j=0}^{\kappa-1}b_jf(\mathbf{s}_{li}+(m_1^{-l},\ldots,m_d^{-l})\mathbf{t}_j). \end{aligned} \tag{5.6}$$

By Lemma 5.1 and the imbedding relationship $H_\infty^{\mathbf{r}}\hookrightarrow C(D)$, we have for $f\in H_\infty^{\mathbf{r}}(D)$

$$\begin{aligned} |I_df-J_lf| &= |I_df-2^{-P_0l}\sum_{i=0}^{2^{P_0l}-1}J(E_{li}f)|\\ &\le 2^{-P_0l}\sum_{i=0}^{2^{P_0l}-1}|I_d(E_{li}f)-J(E_{li}f)|\\ &\le c\cdot 2^{-P_0l}\sum_{i=0}^{2^{P_0l}-1}\omega_{\mathbf{a}}(E_{li}f,\mathbf{e},D)_\infty. \end{aligned} \tag{5.7}$$

Note that

$$\begin{aligned} \omega_{a_j}(E_{li}f,\mathbf{e}_j,D)_\infty &= \sup_{0\le h_j\le\mathbf{e}_j}\ \operatorname*{ess\,sup}_{t\in I^d}|\triangle^{a_j}_{h_j\cdot m_j^{-l}}(f,s_{li}+(m_1^{-l},\ldots,m_d^{-l})^T\mathbf{t})|\\ &= \sup_{0\le h_j\le m_j^{-l}\mathbf{e}_j}\ \operatorname*{ess\,sup}_{t\in T_{li}}|\triangle^{a_j}_{h_j}(f,t)|\\ &\le c\omega_{a_j}(f,m_j^{-l}\mathbf{e}_j,D)_\infty\\ &\le |m_j^{-l}|^{r_j}\\ &\le c2^{-P_0g(\mathbf{r})l},. \end{aligned} \tag{5.8}$$

Hence, we conclude that

$$|I_df-J_lf|\le 2^{-P_0g(\mathbf{r})l}\,. \tag{5.9}$$

Let $J'f:=(J_1-J_0)f$. Then

$$\begin{aligned} J'f &= 2^{-P_0}\sum_{i=0}^{2^{P_0-1}}\sum_{j=0}^{\kappa-1}b_jf(\mathbf{s}_{1i}+(m_1^{-1},\ldots,m_d^{-1})^T\mathbf{t}_j)-\sum_{j=0}^{\kappa-1}b_jf(\mathbf{t}_j)\\ &= \sum_{j=0}^{\kappa'-1}b'_jf(\mathbf{t'}_j) \end{aligned} \tag{5.10}$$

where

$$\kappa' \leq \kappa(2^{P_0+1}). \tag{5.11}$$

For $l \in \mathbb{N}_0$, set

$$J'_{li}f = J'(E_{li}f) = \sum_{j=0}^{\kappa'-1} b'_j f(\mathbf{s}_{li} + (m_1^{-l}, \ldots, m_d^{-l})^T \mathbf{t}_j) \tag{5.12}$$

$$J'_l = 2^{-P_0 l} \sum_{i=0}^{2^{P_0 l}-1} J'_{li}. \tag{5.13}$$

Let us now recall the discretization techniques that has been developed in Heinrich [11]. We approximate $I_d f$ by $J_k f$ with the error bound $n^{-g(\mathbf{r})-1}$, however the node number required by J_k might be much larger than n. We split J_k into J_{k_0} and J'_l ($l = k_0, \ldots, k-1$). In order to reach the aim of further reduction, the computation of $J'_l f$ is replaced by that of the mean of uniformly bounded sequences for proper N_l. We then estimate the error order by using the known results.

From (5.13) and (5.12), we have that

$$\begin{aligned}
J_1(E_{li}f) &= 2^{-P_0} \sum_{i_0=0}^{2^{P_0}-1} \sum_{j=0}^{\kappa-1} J(E_{1i_0}(E_{li}f)) \\
E_{1i_0}(E_{li}f) &= E_{1i_0}(f(\mathbf{s}_{li} + (m_1^{-l}, \ldots, m_d^{-l})^T \mathbf{t}_j)) \\
&= f(\mathbf{s}_{li} + (m_1^{-l}, \ldots, m_d^{-l})^T (\mathbf{s}_{1i_0} + (m_1^{-1}, \ldots, m_d^{-1})^T \mathbf{t}_j)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&2^{-P_0 l} \sum_{i=0}^{2^{P_0 l}-1} J_1(E_{li}f) \\
&= 2^{-P_0(l+1)} \sum_{i=0}^{2^{P_0 l}-1} \sum_{i_0=0}^{2^{P_0}-1} \sum_{j=0}^{\kappa-1} f(\mathbf{s}_{li} + (m_1^{-l}, \ldots, m_d^{-l})^T \mathbf{s}_{1i_0} + (m_1^{-(l+1)}, \ldots, m_d^{-(l+1)})^T \mathbf{t}_j).
\end{aligned}$$

Obviously, $\mathbf{s}_{li} + (m_1^{-l}, \ldots, m_d^{-l})^T \mathbf{s}_{1i_0}$ is the point in T_{l+1i} with the smallest Euclidean norm. We conclude that:

$$J_{l+1}f = 2^{-P_0 l} \sum_{i=0}^{2^{P_0 l}-1} J_1(E_{li}f)$$

and hence

$$J_{l+1}f - J_l f = 2^{-P_0 l} \sum_{i=0}^{2^{P_0} l-1} J'_{li}f = J'_l f. \tag{5.14}$$

We can conclude that for k_0, $k \in \mathbb{N}$ and $k_0 < k$

$$J_k = J_{k_0} + \sum_{l=k_0}^{k-1} J'_l. \tag{5.15}$$

Using the technique similar to that of (5.15), we have that

$$\begin{aligned}|I_d(E_{li}f)-J_1(E_{li}f)| &\le c\cdot 2^{-P_0 l}\sum_{i=0}^{2^{P_0}l-1}\omega_{\mathbf{a}}(E_{1i_0}E_{li}f,\mathbf{e},D)_\infty\\ &= c\cdot 2^{-P_0 l}\sum_{i=0}^{2^{P_0}l-1}\sum_{i_0=0}^{2^{P_0}-1}\sum_{j=1}^{d}\omega_{a_j}(E_{l_i}f,m_j^{-1}\mathbf{e}_j,T_{1,i_0})\\ &\le c\cdot 2^{-P_0 g(\mathbf{r})(l+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\max_{0\le i\le N_l-1}|J'_{li}f| &= \max_{0\le i\le N_l-1}|J'(E_{li}f)| = \max_{0\le i\le N_l-1}|J_1(E_{li}f)-J_0(E_{li}f)|\\ &\le \max_{0\le i\le N_l-1}(|(I_d-J_1)(E_{li}f)|+|(I_d-J_0)(E_{li}f)|)\\ &\le c\cdot 2^{-P_0 g(\mathbf{r})(l+1)}+2^{-P_0 g(\mathbf{r})l}\\ &\le c\cdot 2^{-P_0 g(\mathbf{r})l}.\end{aligned} \tag{5.16}$$

For $n\ge\max(\kappa,5)$, let

$$k_0=\lfloor\log(n/\kappa)/P_0\rfloor, \tag{5.17}$$

and

$$k=\lceil(1+1/g(\mathbf{r}))k_0\rceil. \tag{5.18}$$

Therefore, we have $0\le k_0<k$ and put $N_l=2^{P_0 l}$ for $k_0\le l<k$. In order to define the mapping $\Gamma_l: H^{\mathbf{r}}_\infty(D)\to L_p^{N_l}(D)$, we will introduce some required mappings. First of all, let us fix an $m^*\in N$ with

$$2^{-m^*/2}\le k^{-1}2^{-g(\mathbf{r})kP_0} \tag{5.19}$$

and

$$2^{m^*/2-1}\ge c, \tag{5.20}$$

where the constant c comes from the inequality $\|f\|_{C(D)}\le c\|f\|_{H^{\mathbf{r}}_\infty}$ (according to the imbedding relationship $H^{\mathbf{r}}_\infty(D)\hookrightarrow C(D)$). Therefore for $f\in H^{\mathbf{r}}_\infty(D)$

$$\|f\|_{C(D)}\le 2^{m^*/2-1}. \tag{5.21}$$

Let the mapping $\eta_{l_j}(i):\mathbb{Z}[0,N_l)\to D$ $(j=1,\dots,\kappa'-1)$ be

$$\eta_{l_j}(i)=\mathbf{s}_{l_i}+(m_1^{-l},\dots,m_d^{-l})^T\mathbf{t}'_j, \tag{5.22}$$

Define $\beta:R\to\mathbb{Z}[0,2^{m*})$ by

$$\beta(z):=\begin{cases}0 & \text{if } z<-2^{m^*/2-1},\\ \lfloor 2^{m^*/2}(z+2^{m^*/2-1})\rfloor & \text{if } -2^{m^*/2-1}\le z<2^{m^*/2-1},\\ 2^{m^*}-1 & \text{if } z\ge 2^{m^*/2-1},\end{cases} \tag{5.23}$$

and $\gamma: Z[0,2^{m^*})\to\mathbb{R}$ by

$$\gamma(y)=2^{-m^*/2}y-2^{m^*/2-1}. \tag{5.24}$$

It is obvious that for $-2^{m^*/2-1} \le z \le 2^{m^*/2-1}$,

$$\gamma(\beta(z)) \le z \le \gamma(\beta(z)) + 2^{-m^*/2}. \tag{5.25}$$

The mapping $\varrho : Z[0, 2^{m^*})^{\kappa'} \to \mathbb{R}$ is defined by

$$\varrho(y_0, \ldots, y_{\kappa'-1}) = \sum_{j=0}^{\kappa'-1} b'_j \gamma(y_j). \tag{5.26}$$

Since the needed tools are already provided, we give the expression of the compound mapping Γ_l for $f \in H^{\mathbf{r}}_\infty(D)$, i.e.,

$$\Gamma_l(f)(i) = \varrho((\beta \circ f \circ \eta_{l_j}(i))_{j=0}^{\kappa'-1}). \tag{5.27}$$

Combining (5.19), (5.29), (5.33) and (5.34), we let $\mathbf{x} = \mathbf{s}_{l_i} + (m_1^{-l}, \ldots, m_d^{-l})^T \mathbf{t}'_j$ and easily obtain

$$|J'_{l_i} f - \Gamma_l(f)(i)| \le \sum_{j=0}^{\kappa'-1} |b'_j||f(\mathbf{x}) - \gamma(\beta(f(\mathbf{x})))|.$$

By (5.32), (5.28)and (5.26),

$$|J'_{l_i} f - \Gamma_l(f)(i)| \le 2^{-m^*/2} \sum_{j=0}^{\kappa'-1} |b'_j| \le ck^{-1} 2^{-g(\mathbf{r})kP_0}. \tag{5.28}$$

By (5.20) and (4.7), it is obvious that

$$|J'_l f - S_{N_l} \Gamma_l(f)| \le ck^{-1} 2^{-g(\mathbf{r})kP_0}, \tag{5.29}$$

for all $f \in H^{\mathbf{r}}_\infty(D)$. Using (5.23) and (5.35), we have

$$\begin{aligned} \|\Gamma_l(f)\|_{L^{N_l}_\infty} &\le \|(J'_{l_i} f)_{i=0}^{N_l-1}\|_{L^{N_l}_\infty} + \|\Gamma_l(f) - (J'_{l_i} f)_{i=0}^{N_l-1}\|_{L^{N_l}_\infty} \\ &\le c \cdot 2^{-g(\mathbf{r})P_0 l}. \end{aligned} \tag{5.30}$$

We conclude that

$$\Gamma_l(H^{\mathbf{r}}_\infty(D)) \subseteq c 2^{-g(\mathbf{r})P_0 l} \mathcal{B}(L^{N_l}_p). \tag{5.31}$$

Choose μ to satisfy the following condition

$$0 < \mu < g(\mathbf{r})P_0, \tag{5.32}$$

and let

$$n_l = \lceil 2^{P_0 k_0 - \mu(l-k_0)} \rceil, \tag{5.33}$$

$$v_l = \lceil 8(2\ln(l - k_0 + 1) + \ln 8) \rceil, \tag{5.34}$$

for $l = k_0, \ldots, k-1$. A simple computation leads to

$$\sum_{l=k_0}^{k-1} e^{-v_l/8} < 1/4\,. \tag{5.35}$$

For convenience, we set

$$\tilde{n} = n + 2\kappa' \sum_{l=k_0}^{k-1} v_l n_l. \tag{5.36}$$

By (5.40) (5.41) (5.18) and (5.25),

$$\tilde{n} \leq n + 2\kappa(2^{P_0}+1)2^{P_0 k_0} \sum_{l=0}^{k-k_0-1} \lceil 8(2\ln(l+1)+\ln 8)\rceil \lceil 2^{-\mu l}\rceil \leq c2^{P_0 k_0}.$$

Therefore, by (5.24),

$$\tilde{n} \leq cn\,. \tag{5.37}$$

Using Lemma C (i) and (5.16), we have

$$e^q_{\tilde{n}}(I_d, H^{\mathbf{r}}_\infty(D)) \leq c2^{-g(\mathbf{r})P_0 k} + e^q_{\tilde{n}}(J_k, H^{\mathbf{r}}_\infty(D)). \tag{5.38}$$

Since $\kappa 2^{P_0 k_0} \leq n$,

$$e^q_n(J_{k_0}, H^{\mathbf{r}}_\infty(D), 0) = 0. \tag{5.39}$$

According to the Lemma B, (5.46), (5.43), (5.22) and (5.42), it is suffices to prove that

$$\begin{aligned} e^q_{\tilde{n}}(J_k, H^{\mathbf{r}}_\infty(D)) &\leq e^q_n(J_{k_0}, H^{\mathbf{r}}_\infty(D)), 0) + e^q_{\tilde{n}-n}(J_k - J_{k_0}, H^{\mathbf{r}}_\infty(D)) \\ &\leq \sum_{l=k_0}^{k-1} e^q_{2\kappa' n_l}(J'_l, H^{\mathbf{r}}_\infty(D)). \end{aligned} \tag{5.40}$$

From (5.36), Lemma C, (4.38), and Lemma A, we have

$$\begin{aligned} e^q_{2\kappa' n_l}(J'_l, H^{\mathbf{r}}_\infty(D)) &\leq ck^{-1}2^{-g(\mathbf{r})P_0 k} + e^q_{2\kappa' n_l}(S_{N_l}\Gamma_l, H^{\mathbf{r}}_\infty(D)) \\ &\leq ck^{-1}2^{-g(\mathbf{r})P_0 k} + c2^{-g(\mathbf{r})P_0 l} e^q_{n_l}(S_{N_l}, \mathcal{B}(L^{N_l}_\infty)). \end{aligned} \tag{5.41}$$

Combining (5.45)-(5.48), we conclude

$$e^q_{\tilde{n}}(I_d, H^{\mathbf{r}}_\infty(D)) \leq c2^{-g(\mathbf{r})P_0 k} + c\sum_{l=k_0}^{k-1} c2^{-g(\mathbf{r})P_0 l} e^q_{n_l}(S_{N_l}, \mathcal{B}(L^{N_l}_\infty)). \tag{5.42}$$

According to (5.49), Theorem B, (5.40), (5.39) and (5.25), we estimate the error order that

$$\begin{aligned} e^q_{\tilde{n}}(I_d, H^{\mathbf{r}}_\infty(D)) &\leq c2^{-g(\mathbf{r})P_0 k} + \sum_{l=k_0}^{k-1} c\cdot 2^{-g(\mathbf{r})P_0 l}\cdot n_l^{-1} \\ &\leq c2^{-(g(\mathbf{r})+1)P_0 k_0} + c2^{-(g(\mathbf{r})+1)P_0 k_0} \sum_{l=0}^{k-k_0-1} 2^{-(g(\mathbf{r})P_0-\mu)l} \\ &\leq c\cdot 2^{-(g(\mathbf{r})+1)\log(\frac{n}{\kappa})} \\ &\leq c\cdot n^{-g(\mathbf{r})-1}. \end{aligned} \tag{5.43}$$

Now we prove lower bound, as mentioned above we only need to prove the class $W^{\mathbf{r}}_{\infty}(D)$. Combine the method of Heinrich in [11] and the skill of treating anisotropy we have

$$cn^{-1} \le e^q_{2n}(S_N, \mathcal{B}(L^N_{\infty})) \le cn^{g(\mathbf{r})} e^q_n(I_d, W^{\mathbf{r}}_{\infty}(D)) + \varsigma,$$

here $\varsigma \in \mathbb{R}_+$ can be made arbitrary small. Thus the lower bound immediately comes out. We omit the details.

The proof of Theorem 3.2. The conclusion of Theorem 3.2 easily follows from Theorem 3.1. Actually, it follows from the definition of complete set Λ that for any $\mathbf{l} = (l_1, \ldots, l_d) \in \Lambda$ implies $l_j \mathbf{e}_j \in \Lambda$, $j = 1, \ldots, d$, and then it follows from the definition of the boundary $\partial\Lambda$, there is a unique d-dimensional vector $\mathbf{r}_{\Lambda} = (r_1, \ldots, r_d)$ satisfying $r_j \mathbf{e}_j \in \partial\Lambda$. This combines with the definition of the Hölder-Nikolskii class implies the imbedding relationship

$$H^{\Lambda}_{\infty}(D) \hookrightarrow H^{\mathbf{r}_{\Lambda}}_{\infty}(D).$$

It follows from a result of Theorem 3.1 that

$$e^q_n(I_d, H^{\mathbf{k}}_{\infty}(D)) \le cn^{-g(\mathbf{r}_{\Lambda})-1}.$$

Thus we have the following estimate for the upper bound

$$e^q_n(I_d, H^{\Lambda}_{\infty}(D)) \le e^q_n(I_d, H^{\mathbf{r}_{\Lambda}}_{\infty}(D)) \le n^{-g(\mathbf{r}_{\Lambda})-1}.$$

The proof of Theorem 3.2 is complete.

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