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INTERPOLATION BY LOOP'S SUBDIVISION FUNCTIONS *[1)]

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Abstract

For the problem of constructing smooth functions over arbitrary surfaces from discrete data, we propose to use Loop's subdivision functions as the interpolants. Results on the existence, uniqueness and error bound of the interpolants are established. An efficient progressive computation algorithm for the interpolants is also presented.

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Key words: Interpolation, Loop's subdivision function.

1. Introduction

The problem of constructing interpolants on surfaces arises in some application areas such as characterizing the rain fall on the earth, the pressure on the wing of an airplane and the temperature on a human body. The problem was first proposed as an open question by Barnhill [5] in 1985. After that, a considerable number methods have been developed for dealing with it (for surveys see [7], [12]). Most of these methods interpolate the scattered data over planar or spherical domain surfaces. In [8] and [11], the domains are generalized to convex surface and topological genus zero surface, respectively. Pottmann [17] presented a method which does not possess similar restrictions on the domain surface but requires it to be of C^2. In [6] this restriction was left and the function on surface is constructed by transfinite interpolation. It seems that, the currently known approaches possess restrictions either on domain surfaces or functions on surfaces. The domain surfaces are usually assumed to be spherical, convex or genus zero. The functions on surfaces are not always polynomial [6], [15] or rather higher order polynomial [18]. The aim of this paper is to design a low order piecewise polynomial interpolation scheme over triangulated surfaces.

In several recent developments in computer graphics and numerical analysis (see [2, 3, 4, 9, 10]), Loop's subdivision (see [14]) surfaces and functions on surfaces have played a key role. In these developments, Loop's subdivision surfaces and function on surfaces are used to construct the finite element function space in a discretization process of a partial differential equation. However, the convergence analysis or error estimation in these discretization process require the interpolation error estimation by the function in the finite element function space. Such a result currently is not available. In this paper, we estimate the interpolation error bound and further provide an efficient method for constructing smooth multi-resolution functions over a surface. Precisely, we consider the following problem:

Given a discretized triangular surface mesh $T \subset \mathbb{R}^3$ and a discretized function $D \subset \mathbb{R}^\kappa$. Each of the function values is attached to one vertex of the surface mesh. Our primary goal is to construct smooth (non-discretized) representations for the surface functions that interpolate the discretized data. Our secondary goal is to estimate the error of the interpolation. Our tertiary goal is to establish a progressive computational method for the interpolation functions.

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We propose to use Loop's subdivision functions as the interpolants. Results on the existence, uniqueness and error bound of the interpolants are established. An efficient progressive computation algorithm for the interpolants is also presented.

The rest of the paper is organized as follows. In Section 2 we review some basic aspects on Loop's subdivision. In Section 3, we formulate the interpolation problem and then establish the result on the solvability of the interpolation problem. Section 4 is devoted to the interpolation error and convergence and Section 5 is for the efficient computation of the interpolation functions. Numerical examples are given in Section 6.

2. Loop's Subdivision Surfaces and Functions

Let us introduce some notations used in this paper:
S : domain surface of the interpolation, the limit surface of Loop's subdivision.
T : a triangulation of S.
$T^{(k)}$: a sequence of triangulation of S.
M : control mesh of T.
$M^{(k)}$: control mesh of $T^{(k)}$.
In Loop's subdivision scheme, the initial control mesh $M^{(0)}$ and the subsequent refined meshes $M^{(k)}$ consist of triangles only. In the refinement, each triangle is subdivided into 4 sub-triangles. Then the vertex position of the refined mesh is computed as the weighted average of the vertex position of the unrefined mesh. Consider a vertex x_0^k at level k with neighbor vertices x_i^k for $i = 1, \cdots, n$, where n is the valence of vertex x_0^k. The positions of the newly generated vertices x_i^{k+1} on the edges of the previous mesh are computed as

$$x_i^{k+1} = \frac{3x_0^k + 3x_i^k + x_{i-1}^k + x_{i+1}^k}{8}, \quad i = 1, \cdots, n, \tag{2.1}$$

where index i is to be understood modulo n. The old vertices get new positions according to

$$x_0^{k+1} = (1 - na)x_0^k + a\left(x_1^k + x_2^k + \cdots + x_n^k\right), \tag{2.2}$$

where $a = \frac{1}{n}\left[\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}cos\frac{2\pi}{n}\right)^2\right]$. Note that all newly generated vertices have a valence of 6, while the vertices inherited from the original mesh at level zero may have a valence other than 6. We will refer to the former case as *ordinary* and to the later case as *extraordinary*. The limit surface S of Loop's subdivision is C^2 everywhere except at the extraordinary points where it is C^1.

2.1. The Limit Surface Corresponding to Vertices

Lemma 2.1. *Let x_0^0 be a vertex with x_i^0, $i = 1, \cdots, n$, being the 1-ring neighbor vertices of the initial control mesh $M^{(0)}$. Then all these vertices converge to a single position*

$$v_0^T := (1 - nl)x_0^0 + l\sum_{i=1}^{n} x_i^0, \quad l = 1/[n + 3/(8a)] \tag{2.3}$$

as the subdivision step goes to infinity (see [4] for the proof of the Lemma).

Let x_0^1, x_i^1, $i = 1, \cdots, n$ be the control vertices generated by subdivision once around x_0^0 of the initial control mesh $M^{(0)}$. Then

$$v_0^T = (1 - nl)x_0^1 + l\sum_{i=1}^{n} x_i^1. \tag{2.4}$$

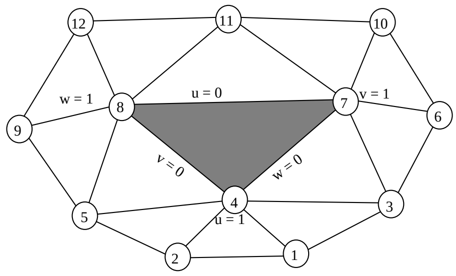


Fig 2.1: The vertex numbering of a regular patch with 12 control points. A regular patch is defined over the shaded triangle. Here $(u, v, w) = (\xi_0, \xi_1, \xi_2)$

This can be verified easily by substituting (2.1) and (2.2) to the right-handed side of (2.4). Lemma 2.1 and (2.4) mean that we can evaluate the limit position of the surface at any finite subdivision level and at any vertex by averaging the vertex and its neighbors. The surface tangents corresponding to the edges $[x_0^0 x_j^0]$ around x_0^0 are given by the following formula

$$t_{j+1} = cos\left(\frac{2\pi j}{n}\right) a_1^0 + sin\left(\frac{2\pi j}{n}\right) a_{n-1}^0, \quad j = 0, \cdots, n-1,$$

where $a_1^0 = \frac{2}{n}\sum_{i=0}^{n-1} cos\left(\frac{2\pi i}{n}\right) x_{i+1}^0, \quad a_{n-1}^0 = \frac{2}{n}\sum_{i=0}^{n-1} sin\left(\frac{2\pi i}{n}\right) x_{i+1}^0.$

2.2. Evaluation of Regular Surface Patches

To obtain a local parameterization of the limit surface S for each of the triangles in the initial control mesh, we choose (ξ_1, ξ_2) as two of the barycentric coordinates (ξ_0, ξ_1, ξ_2) and define $\mathcal{T}$ as

$$\mathcal{T} = \{(\xi_1, \xi_2) \in I\!R^2 : \xi_1 \geq 0, \xi_2 \geq 0, \xi_1 + \xi_2 \leq 1\}. \tag{2.5}$$

Consider a generic triangle in the mesh and introduce a local numbering of vertices lying in its immediate 1-ring neighborhood (see Fig 2.1). If all its vertices have a valence of 6, the resulting patch of the limit surface is exactly described by a single quartic box-spline patch, for which an explicit form exists. We refer to such a patch as *regular*. A regular patch is expressed by the linear combination of 12 basis functions:

$$x(\xi_1, \xi_2) = \sum_{i=1}^{12} N_i(\xi_1, \xi_2) x_i, \tag{2.6}$$

where the label i refers to the local numbering of the vertices that is shown in Fig 2.1. The basis N_i are given as follows (see [19]):

$$\begin{aligned}
N_1 &= \tfrac{1}{12}(\xi_0^4 + 2\xi_0^3\xi_1), \\
N_2 &= \tfrac{1}{12}(\xi_0^4 + 2\xi_0^3\xi_2), \\
N_3 &= \tfrac{1}{12}\left[\xi_0^4 + \xi_1^4 + 6\xi_0^3\xi_1 + 6\xi_0\xi_1^3 + 12\xi_0^2\xi_1^2 + (2\xi_0^3 + 2\xi_1^3 + 6\xi_0^2\xi_1 + 6\xi_0\xi_1^2)\xi_2\right], \\
N_4 &= \tfrac{1}{12}[6\xi_0^4 + 24\xi_0^3(\xi_1 + \xi_2) + \xi_0^2(24\xi_1^2 + 60\xi_1\xi_2 + 24\xi_2^2) \\
&\quad + \xi_0(8\xi_1^3 + 36\xi_1^2\xi_2 + 36\xi_1\xi_2^2 + 8\xi_2^3) + (\xi_1^4 + 6\xi_1^3\xi_2 + 12\xi_1^2\xi_2^2 + 6\xi_1\xi_2^3 + \xi_2^4)],
\end{aligned} \tag{2.7}$$

where (ξ_0, ξ_1, ξ_2) are barycentric coordinates of the triangle with vertices numbered as $4, 7, 8$, and $\xi_0 = 1 - \xi_1 - \xi_2$. Other basis functions are similarly defined. For example, replacing

(ξ_0, ξ_1, ξ_2) by (ξ_1, ξ_2, ξ_0) in N_1, N_2, N_3, N_4, we get N_{10}, N_6, N_{11}, N_7. Replacing (ξ_0, ξ_1, ξ_2) by (ξ_2, ξ_0, ξ_1) we get N_9, N_{12}, N_5, N_8.

2.3. Evaluation of Irregular Surface Patches

If a triangle is irregular, i.e., at least one of its vertices has a valence other than 6, the resulting patch is not a quartic box spline. We assume extraordinary vertices are isolated, i.e., there is no edge in the control mesh such that both its vertices are extraordinary. This assumption can be fulfilled by subdividing the mesh once. Under this assumption, any irregular patch has only one extraordinary vertex. For evaluation of irregular patches, we use the scheme proposed by Stam [19]. In this scheme the mesh needs to be subdivided repeatedly until the parameter values of interest are interior to a regular patch. We now summarize briefly the central idea of Stam's scheme. First, it is easy to see that each subdivision of an irregular patch produces three regular patches and one irregular patch. Repeated subdivision of the irregular patch will produce a sequence of regular patches. The surface patch is piecewise parameterized. The subdomains $\mathcal{T}_j^k$ are given as follows:

$$\begin{aligned}
\mathcal{T}_1^k &= \{(\xi_1, \xi_2) : \xi_1 \in [2^{-k}, 2^{-k+1}], &&\xi_2 \in [0, 2^{-k+1} - \xi_1]\},\\
\mathcal{T}_2^k &= \{(\xi_1, \xi_2) : \xi_1 \in [0, 2^{-k}], &&\xi_2 \in [2^{-k} - \xi_1, 2^{-k}]\},\\
\mathcal{T}_3^k &= \{(\xi_1, \xi_2) : \xi_1 \in [0, 2^{-k}], &&\xi_2 \in [2^{-k}, 2^{-k+1} - \xi_1]\}.
\end{aligned} \tag{2.8}$$

These subdomains are mapped onto $\mathcal{T}$ by the following transforms:

$$\begin{aligned}
t_{k,1}(\xi_1, \xi_2) &= (2^k\xi_1 - 1, 2^n\xi_2), && (\xi_1, \xi_2) \in \mathcal{T}_1^k,\\
t_{k,2}(\xi_1, \xi_2) &= (1 - 2^k\xi_1, 1 - 2^k\xi_2), && (\xi_1, \xi_2) \in \mathcal{T}_2^k,\\
t_{k,3}(\xi_1, \xi_2) &= (2^k\xi_1, 2^k\xi_2 - 1), && (\xi_1, \xi_2) \in \mathcal{T}_3^k.
\end{aligned}$$

Hence $\mathcal{T}_j^k$ form a tiling of $\mathcal{T}$ except for the point $(\xi_1, \xi_2) = (0, 0)$. The surface patch is then defined by its restriction to each triangle

$$x(\xi_1, \xi_2)|_{\mathcal{T}_j^k} = \sum_{i=1}^{12} x_i^{k,j} N_i(t_{k,j}(\xi_1, \xi_2)), \quad j = 1, 2, 3; \quad k = 1, 2, \cdots, \tag{2.9}$$

where $x_i^{k,j}$ are the properly chosen 12 control vertices around the irregular patch at the level k that define a regular surface patch. Using the vertex numbering and local coordinate system shown in Fig 2.1, it is easy to see that these three set control vertices are

$$\begin{aligned}
\{x_i^{k,1}\}_{i=1}^{12} &= [x_3^k, x_1^k, x_{n+4}^k, x_2^k, x_{n+1}^k, x_{n+9}^k, x_{n+3}^k, x_{n+2}^k, x_{n+5}^k, x_{n+8}^k, x_{n+7}^k, x_{n+10}^k],\\
\{x_i^{k,2}\}_{i=1}^{12} &= [x_{n+7}^k, x_{n+10}^k, x_{n+3}^k, x_{n+2}^k, x_{n+5}^k, x_{n+4}^k, x_2^k, x_{n+1}^k, x_{n+6}^k, x_3^k, x_1^k, x_n^k],\\
\{x_i^{k,3}\}_{i=1}^{12} &= [x_1^k, x_n^k, x_2^k, x_{n+1}^k, x_{n+6}^k, x_{n+3}^k, x_{n+2}^k, x_{n+5}^k, x_{n+12}^k, x_{n+7}^k, x_{n+10}^k, x_{n+11}^k].
\end{aligned}$$

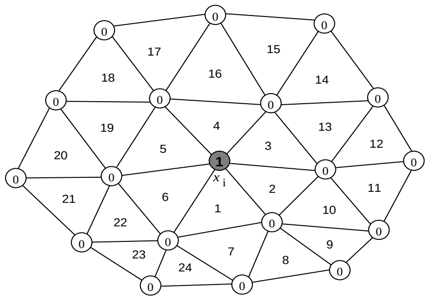
Hence, the main task is to compute these control vertices. As usual, the subdivision around an irregular patch is formulated as a linear transform from the level $k-1$ 1-ring vertices of the irregular patch to the related level k vertices, i.e.,

$$X^k = AX^{k-1} = \cdots = A^k X^0, \quad \tilde{X}^{k+1} = \tilde{A}X^k = \tilde{A}A^k X^0,$$

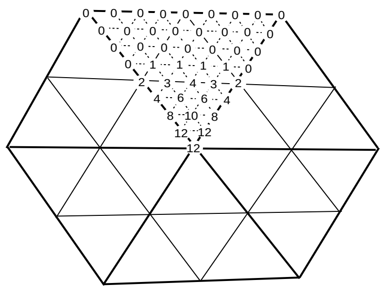
where $X^k = [x_1^k, \cdots, x_{n+6}^k]^T$, $\tilde{X}^k = [x_1^k, \cdots, x_{n+6}^k, x_{n+7}^k, \cdots, x_{n+12}^k]^T$, and A and $\tilde{A}$ are defined by the subdivision masks. Hence, $k+1$ subdivisions lead to the computation of A^k. A novel idea proposed by Stam is to use the Jordan canonical form $A = UJU^{-1}$. The computation of the A^k amount to computing J^k, which makes the cost of the computation nearly independent of k and hence very efficient. The beauty of the scheme is that explicit forms of U and J exist. We refer to [19] for details.

2.4. Basis Functions

For each vertex x_i of a control mesh M, we associate with a basis function ϕ_i, where ϕ_i is defined by the limit of the Loop's subdivision for the zero control values everywhere except at x_i where it is one (see Fig. 2.2.a). Hence the support of ϕ_i is local and it covers the 2-ring neighborhood of vertex x_i. Let e_j, $j = 1, \cdots, m_i$ be the 2-ring neighborhood elements. Then if e_j is regular, the explicit box-spline expression as in (2.6) exists for ϕ_i on e_j. Using (2.7), we can derive the BB-form coefficients for basis ϕ_i (see Fig. 2.2.b). These expressions can be used to evaluate ϕ_i. If e_i is irregular, local subdivision, as described in §, is needed around e_i until the parameter values of interest are interior to a regular patch. Using the basis $\{\phi_i\}$, the limit surface of Loop's subdivision is expressed as $S = \sum x_i \phi_i(x)$. It is known that S is C^2 everywhere, except at extraordinary vertices where it is C^1 (see [14]).



(a)



(b)

Fig 2.2: The quartic Bézier coefficients (each has a factor 1/24) of basis function. The coefficients on the other five macro-triangles are obtained by rotating the top macro-triangle around the center to the other five positions.

3. Solvability of the Interpolation Problem

Now we introduce the following notations: $x_i \in M$ is the i-th control vertex; $v_i \in T$ is the i-th vertex on the Loop's subdivision surface S; $f_i = f(v_i)$ is the i-th interpolation function value; g_i is the i-th control function value; ϕ_i is the i-th basis function, where $i = 1, \cdots, \mu$. Using these notations, we can formulate the interpolation problem as follows: For the given function values $\{f_i\}_1^\mu$, find the control function values $\{g_i\}_1^\mu$ such that

$$\sum_{j=1}^{\mu} g_j \phi_j(v_i) = f_i, \quad i = 1, \cdots, \mu. \tag{3.1}$$

Theorem 3.1. *The interpolation problem (3.1) always has a unique solution.*

It follows from (2.3) that equation (3.1) is equivalent to

$$(1 - n_i l_i) g_i + l_i \sum_{j=1}^{n_i} g_{k_j} = f_i, \quad i = 1, \cdots, \mu. \tag{3.2}$$

Hence we need to show that the system of equations (3.2) is always solvable uniquely. To this end, we introduce a simple lemma.

Lemma 3.1. *Let l_i be defined as in (2.3). Then*

$$(2n_i - 1)l_i = 1 \quad if \quad n_i = 3; \tag{3.3}$$
$$(2n_i - 1)l_i < 1 \quad if \quad n_i = 4, 5; \tag{3.4}$$
$$2n_i l_i = 1 \quad if \quad n_i = 6; \tag{3.5}$$
$$2n_i l_i < 1 \quad if \quad n_i \geq 7. \tag{3.6}$$

Using the definition of l_i, these relations can be easily verified. Now we start to prove Theorem 3.1. Suppose $f_i = 0$, we show that the corresponding homogeneous equation to (3.2) has only zero solution. On the contrary, we assume $\{g_i\}$ be a non-zero solution of it. Let ξ be an index such that

$$g_\xi = \max_j |g_j|$$

Without loss of generality, we may assume $g_\xi > 0$, otherwise we can multiply (-1) on both sides of the equation. Then if $n_\xi \geq 7$, we have, from (3.2) and (3.6),

$$\begin{aligned} 0 &= (1 - n_\xi l_\xi)g_\xi + l_\xi \textstyle\sum_{j=1}^{n_\xi} g_{k_j} \\ &\geq (1 - n_\xi l_\xi)g_\xi - l_\xi \textstyle\sum_{j=1}^{n_\xi} |g_{k_j}| \\ &\geq (1 - n_\xi l_\xi)g_\xi - n_\xi l_\xi g_\xi \\ &= (1 - 2n_\xi l_\xi)g_\xi \\ &> 0, \end{aligned}$$

a contradiction. Hence, we assume $n_\xi \leq 6$ in the following, and we show that a contradiction will be yielded again. First from the inequalities

$$\begin{aligned} 0 &= (1 - n_\xi l_\xi)g_\xi + l_\xi \textstyle\sum_{j=1, j\neq l}^{n_\xi} g_{k_j} + l_\xi g_{k_l} \\ &\geq (1 - n_\xi l_\xi)g_\xi - l_\xi \textstyle\sum_{j=1, j\neq l}^{n_\xi} |g_{k_j}| + l_\xi g_{k_l} \\ &= (1 - (2n_\xi - 1)l_\xi)g_\xi + l_\xi g_{k_l} \\ &\geq l_\xi g_{k_l}, \end{aligned}$$

we have $g_{k_l} \leq 0$ for any $l = 1, \cdots, n_\xi$. Now let m be an index, such that

$$|g_{k_m}| = \max_{1\leq j\leq n_\xi} |g_{k_j}|.$$

Then from $(1 - n_\xi l_\xi)g_\xi + l_\xi \sum_{j=1}^{n_\xi} g_{k_j} = 0$, it is easy to see that

$$g_{k_m} \leq \alpha(n_\xi)g_\xi \quad \text{with} \quad \alpha(n_\xi) = -\frac{1 - n_\xi l_\xi}{n_\xi l_\xi}.$$

Furthermore, we can derive that

$$g_{k_{m-1}} + g_{k_{m+1}} \leq \beta(n_\xi)g_\xi \quad \text{with} \quad \beta(n_\xi) = -\frac{1 - (2n_\xi - 2)l_\xi}{l_\xi}.$$

Now consider equation (3.2) for $i = k_m$. Using the inequalities obtained above, we have

$$\begin{aligned} 0 &= (1 - n_{k_m} l_{k_m})g_{k_m} + l_{k_m} \textstyle\sum_{j=1}^{n_{k_m}} g_{k_j} \\ &= (1 - n_{k_m} l_{k_m})g_{k_m} + l_{k_m} \textstyle\sum_{j\neq m-1, m+1} g_{k_j} + l_{k_m}(g_{k_{m-1}} + g_{k_{m+1}}) \\ &\leq \alpha(n_\xi)(1 - n_{k_m} l_{k_m})g_\xi + (n_{k_m} - 2)l_{k_m} g_\xi + \beta(n_\xi)l_{k_m} g_\xi \\ &= h(n_\xi, n_{k_m})g_\xi, \end{aligned}$$

where

$$h(n_\xi, n_{k_m}) = \alpha(n_\xi)(1 - n_{k_m} l_{k_m}) + (n_{k_m} - 2)l_{k_m} + \beta(n_\xi)l_{k_m}.$$

For each fixed n_ξ ($n_\xi = 3, 4, 5, 6$), $h(n_\xi, n_{k_m})$ is an increasing function with respect to n_{k_m}, and $h(n_\xi, \infty) < 0$. Therefore, $h(n_\xi, n_{k_m})g_\xi < 0$. This is a contradiction. Hence, the homogeneous system (3.2) has only zero solution and the theorem is proved.

Iterative Computation. The coefficient matrix of system (3.2) is very sparse. An iterative approach for solving the system is desirable. We even do not need to store the matrix since its elements can be easily computed during the iteration. The special structure of the matrix makes the following Jacobi-like iteration converge.

$$g_i^{k+1} = f_i + n_i l_i g_i^k - l_i \sum_{j=1}^{n_i} g_{k_j}^k, \quad i = 1, \cdots, \mu$$

In matrix form, it can be written as

$$Y^{k+1} = BY^k + C.$$

A good initial value of g_i^0 for the iteration is f_i. Note that B is a Metzler matrix.

An interesting fact is that the classical Jacobi or Gauss-Siedel iteration does not converge. As a very simple example to illustrate this, we choose a mesh consists of the faces of a tetrahedron that has four triangular faces and four vertices. In this case, $l_i = \frac{1}{5}$, and it is easy to derive that -1.5 is an eigenvalue of the iterative matrix of the Jacobi iteration. Hence the spectral radius of the iterative matrix is greater than one. Therefore, the Jacobi iteration is divergent. However, the spectral radius of the iterative matrix B of our Jacobi-like iteration is $\frac{4}{5}$, that is $\rho(B) < 1$. Another example, for that the spectral radius of the iterative matrix is easy to compute exactly, is the regular triangulation of a ring. In this case, every vertex has valence 6 and $l_i = \frac{1}{12}$. It is easy to see that 1 is an eigenvalue of the Jacobi iterative matrix (which is a stochastic matrix, see [13], pages 547-550). It follows from Gerschgorin Theorem, all the eigenvalues of B is in the disc $\{p : \|p - (0.5, 0)^T\| \leq 0.5\}$. Futhermore, using a similar approach of the proof Theorem 3.1, we can show that 1 is not the eigenvalue of B. Hence $\rho(B) < 1$. For the general case, we can derive that $B^k \to 0$ as $k \to \infty$.

4. Interpolation Error and Convergence

Theorem 4.1. *If the interpolated data $\{f_i\}$ comes from a linear function in $I\!R^3$, then the interpolant recovers this linear function.*

Proof. Let $f(x) = a^T x + b$ be a linear function, where a is a vector in $I\!R^3$. Suppose $f_i = f(v_i)$. Then by the uniqueness of the solution of the interpolation problem (3.1), we have the control function value $g_i = f(x_i)$, because the limit function of the subdivision, from the control function value g_i, is given by

$$\begin{aligned}(1 - n_i l_i)g_i + l_i \textstyle\sum_{j=1}^{n_i} g_{k_j} &= (1 - n_i l_i)(a^T x_i + b) + l_i \textstyle\sum_{j=1}^{n_i} (a^T x_{k_j} + b) \\ &= a^T[(1 - n_i l_i)x_i + l_i \textstyle\sum_{j=1}^{n_i} x_{k_j}] + b \\ &= a^T v_i + b \\ &= f_i\ .\end{aligned}$$

Let $[x_i x_j x_k]$ be a regular triangle, the surface patch corresponding to this triangle is defined by

$$S_{ijk}(\xi_1, \xi_2) = \sum_{l=1}^{12} x_{k_l} N_l(\xi_1, \xi_2).$$

The interpolant over the surface patch, which is the limit function of the same subdivision procedure, is given by

$$\begin{aligned} F_{ijk}(\xi_1,\xi_2) &= \sum_{l=1}^{12}(a^T x_{k_l}+b)N_l(\xi_1,\xi_2) \\ &= a^T\sum_{l=1}^{12} x_{k_l}N_l(\xi_1,\xi_2)+b \\ &= a^T S_{ijk}(\xi_1,\xi_2)+b, \end{aligned} \tag{4.1}$$

where the equality $\sum_{l=1}^{12} N_l(\xi_1,\xi_2)\equiv 1$ is employed.

If the triangle $[x_ix_jx_k]$ is not regular, the 1 to 4 subdivision is needed. For the newly generated vertices, say $\tilde{p}_i$, the control function is given by $a^T\tilde{p}_i+b$, since the subdivision rules are weighted averaging. Hence, for the newly produced regular sub-triangles, the interpolants are given by (4.1) as well. Repeat this procedure, we can see that at any point $v\in S_{ijk}$, the function value of the interpolant is given by a^Tv+b. This concludes the proof of the theorem.

Theorem 4.2. *Let I_S be the interpolation operator, defined by (3.1), on the surface S, then I_S is of linear and $\|I_S\|$ is uniformly bounded above for any S, where $\|I_S\|=\sup_{\|f\|_S=1}\|I_Sf\|_S$, $\|f\|_S=\max_{x\in S}|f(x)|$.*

Proof. It is obvious that I_S is linear. Now we show that I_S is bounded above. First note that

$$\|I_S\|=\sup_{\|f\|_S=1}\|I_Sf\|_S=\sup_{\|f\|_S=1}\|\sum_{i=1}^{\mu}g_i\phi_i\|_S,$$

where g_i is the control value corresponding to $f(v_i)$. Since $\phi_i\ge 0$, $g_i\equiv C$ makes the supremum $\sup_{\|f\|_S=1}\|\sum_{i=1}^{\mu}g_i\phi_i\|_S$ being achieved. Hence we have $f(v_i)=1$. Therefore

$$\|I_S\|=1 \tag{4.2}$$

Note that $\|I_S\|$ does not depend upon S.

Theorem 4.3. *Let f be a sufficiently smooth function on the surface S. Let S_{ijk} be the surface patch corresponding to the triangle $[x_ix_ix_k]$. Then*

$$\|f-I_Sf\|_{S_{ijk}}<Ch^2$$

where h is the size of the triangle $[x_ix_ix_k]$, C is a constant depending on f but not S, $\|f\|_{S_{ijk}}=\max_{x\in S_{ijk}}|f(x)|$.

Proof. Let f_L be a linear approximation of f on the triangle $[x_ix_ix_k]$. For example, f_L be the linear interpolation of f, then we know that $\|f-f_L\|_{S_{ijk}}<Ch^2$ for a constant C. Hence,

$$\begin{aligned} \|f-I_Sf\|_{S_{ijk}} &\le \|f-f_L\|_{S_{ijk}}+\|f_L-I_Sf\|_{S_{ijk}} \\ &= \|f-f_L\|_{S_{ijk}}+\|I_S(f_L-f)\|_{S_{ijk}} \\ &\le \|f-f_L\|_{S_{ijk}}+\|I_S\|\|f_L-f\|_{S_{ijk}} \\ &< C(1+\|I_S\|)h^2. \end{aligned}$$

This completes the proof.

Let I_S^k be the interpolation operator over the surface S and control mesh $M^{(k)}$. Then Theorem 4.3 implies that

$$\lim_{k\to\infty} I_S^kf=f$$

uniformly, for any smooth function f on S.

5. Progressive Computations

Suppose we are given a coarse control mesh $M^{(0)}$. Using Loop's subdivision repeatedly, we can generate a sequence of control meshes $M^{(j)}$, $j = 1, 2, \cdots$. It is known that all these mesh defined the same limit surface S. Let $V^{(j)} := \{v_i^{(j)}\} \subset T^{(j)}$ be the set of limit vertices of $M^{(j)}$. Then

$$V^{(0)} \subset V^{(1)} \subset V^{(2)} \cdots$$

The vertices in $V^{(j)} \setminus V^{(j-1)}$ are corresponding to the edges of $M^{(j-1)}$. Now suppose we are given function values $f(v_i^{(j)})$, we want to compute efficiently a sequence of interpolant $I_S^j f$. Our method is first to compute $I_S^0 f$, and then $I_S^1 f$ and so on. Now suppose $I_S^0 f$, which is assumed to be a small problem, has been computed, we go further to compute $I_S^1 f$. Let

$$V^{(0)} := \{v_i\}_{i=1}^{\mu}, \quad V^{(1)} := \{v_i\}_{i=1}^{\nu}.$$

The problem we want to solve is to find g_i, such that

$$\begin{cases} g_i + \dfrac{l_i}{1-n_i l_i} \displaystyle\sum_{j=1}^{n_i} g_{k_j} = \dfrac{f_i}{1-n_i l_i}, & i = 1, \cdots, \mu, \\ (1-n_i l_i) g_i + l_i \displaystyle\sum_{j=1}^{n_i} g_{k_j} = f_i, & i = \mu+1, \cdots, \nu. \end{cases} \tag{5.1}$$

Note that the newly added vertices always have valence 6, and the vertices corresponding to the previous mesh are separated each other by the newly added vertices. The matrix form of (5.1) can be written as

$$\begin{bmatrix} I & LC \\ \frac{1}{12}C^T & U \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{5.2}$$

where

$$\begin{aligned} G_1 &= [g_1 \ \cdots \ g_\mu]^T, & G_2 &= [g_{\mu+1} \ \cdots \ g_\nu]^T, \\ F_1 &= \left[\tfrac{f_1}{1-n_1 l_1} \ \cdots \ \tfrac{f_\mu}{1-n_\mu l_\mu}\right]^T, & F_2 &= [f_{\mu+1} \ \cdots \ f_\nu]^T, \\ L &= \mathrm{diag}\left(\tfrac{l_1}{1-n_1 l_1} \ \cdots \ \tfrac{l_\mu}{1-n_\mu l_\mu}\right) \end{aligned}$$

and $I \in \mathbb{R}^{\mu\times\mu}$ is a unit matrix. Let

$$C = (c_{ij})_{i=1,j=\mu+1}^{\mu,\nu}, \quad U = (u_{ij})_{i=\mu+1,j=\mu+1}^{\nu,\nu}.$$

Then we have

$$c_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

$$u_{ij} = \begin{cases} \frac{1}{2} & \text{if } i = j, \\ \frac{1}{12} & \text{if } i \neq j \text{ and } v_i,\ v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$
\begin{bmatrix} I & LC \\ \frac{1}{12}C^T & U \end{bmatrix} =
$$

$$
\begin{array}{c}
 \\ \vdots \\ j \\ \vdots \\ k \\ \vdots \\ \mu \\ \vdots \\ i \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc}
\cdots & j & \cdots & k & \cdots & \mu & \cdots & j_1 & \cdots & j_2 & \cdots & i & \cdots & k_1 & \cdots & k_2 & \cdots
\end{array} \\
\left[\begin{array}{cccccc|ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & a_j & \cdots & a_j & \cdots & a_j & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_k & \cdots & a_k & \cdots & a_k & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & \frac{1}{12} & \cdots & \frac{1}{12} & \cdots & \cdots & \cdots & \frac{1}{12} & \cdots & \frac{1}{12} & \cdots & \frac{1}{2} & \cdots & \frac{1}{12} & \cdots & \frac{1}{12} & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}\right]
\end{array}
\qquad (5.3)
$$

where $a_i = \frac{l_i}{1-n_i l_i}$. Since each v_i $(i > \mu)$ has four neighbors v_j $(j > \mu)$, U is a symmetric and positive definite matrix. Let

$$R = Y - \tilde{Y},$$

where $R = [R_1^T, R_2^T]^T$, $Y = [G_1^T, G_2^T]^T$, $\tilde{Y} = [\tilde{G}_1^T, \tilde{G}_2^T]^T$, and $\tilde{Y}$ is produced by subdivision once from the previous solution using Loop's subdivision rules. Then we have

$$
\begin{bmatrix} I & LC \\ \frac{1}{12}C^T & U \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \qquad (5.4)
$$

where

$$G_2 = F_2 - \frac{1}{12}C^T\tilde{G}_1 - S\tilde{G}_2.$$

From (5.4) we have

$$
\begin{cases} R_1 = -LCR_2, \\ \left(U - \frac{1}{12}C^T LC\right) R_2 = G_2. \end{cases} \qquad (5.5)
$$

Hence, we need to solve the second equation of (5.5) first for the unknown R_2, and then compute R_1 from the first equation. Note that the coefficient matrix $\tilde{S} = U - \frac{1}{12}C^T LC$ of the second equation of (5.5) is symmetric. Now we show that it is positive definite. To this end, we need the following two inequalities

$$
\frac{3}{8n_i} < a_i \le \frac{1}{2}, \quad (n_i - 1)a_i \le 1 \ \text{ with } \ a_i = \frac{l_i}{1 - n_i l_i}. \qquad (5.6)
$$

These are valid from Lemma 3.1. We will show that the diagonal elements of $\tilde{S}$ is strictly dominated. First we can see that the sum of the i-th row of $\frac{1}{12}C^T LC$ (see the matrix (5.3)) is $\frac{1}{12}\left(n_j a_j + n_k a_k\right)$, where j and k are the indices of vertices v_j and v_k with vertex v_i corresponding to the edge $[v_j v_k]$ (see Fig. 5.1.)

Next we show that each element of $C^T LC$ is not larger than one. From matrix (5.3), we can see that the off-diagonal and the non-zero element on the i-th row of $C^T LC$ is a_j or a_k. The diagonal element is $a_j + a_k$. It follows from (5.6) that each element of $C^T LC$ is not larger than one. Hence the elements of the i-th row of $\tilde{S}$ corresponding to the vertex indices j_1, j_2, k_1, k_2 (see Fig 5.1) are either $\frac{1}{12} - \frac{1}{12}a_j$ or $\frac{1}{12} - \frac{1}{12}a_k$. If we denote the elements of $\tilde{S}$ as $\tilde{s}_{uv}$.

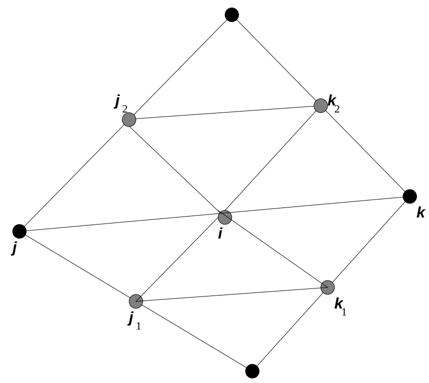


Fig 5.1: Indices of the vertices around the i-th vertex.

Then we have

$$\tilde{s}_{ii} = \frac{1}{2} - \frac{1}{12}(a_j + a_k), \quad \tilde{s}_{ij_1} + \tilde{s}_{ij_2} + \tilde{s}_{ik_1} + \tilde{s}_{ik_2} = \frac{1}{3} - \frac{1}{6}(a_j + a_k).$$

The sum of the remaining elements in absolute value is $[(n_j - 2)a_j + (n_k - 2)a_k]/12$. Hence

$$\begin{aligned} \tilde{s}_{ii} - \sum_{v \neq i} |\tilde{s}_{iv}| &\geq \frac{1}{6} - \frac{(n_j - 3)a_j + (n_k - 3)a_k}{12} \\ &= \frac{1}{6} - \frac{(n_j - 1)a_j + (n_k - 1)n_k}{12} + \frac{a_j + a_k}{6} \\ &\geq \frac{a_j + a_k}{6} \\ &> \frac{5}{48}\left(\frac{1}{n_j} + \frac{1}{n_k}\right). \end{aligned} \tag{5.7}$$

Therefore, the diagonal elements of $\tilde{S}$ is strictly dominated and hence $\tilde{S}$ is a positive definite matrix. Using the analysis above, we can further prove that

$$\tilde{s}_{ii} + \sum_{v \neq i} |\tilde{s}_{iv}| \leq 1 - \frac{1}{8}\left(\frac{1}{n_j} + \frac{1}{n_k}\right). \tag{5.8}$$

Note that the subdivision process does not produce vertices with valences other than 6, hence by the Gerschgorin theorem on eigenvalues, the upper and lower bounds of the eigenvalues of $\tilde{S}$ are given by the right-handed sides of (5.7) and (5.8), respectively. These bounds are not changed by the repeatedly subdivision. After one subdivision, there is at least one of the n_j and n_k is 6. Hence the eigenvalues are in the range $(5/288, 47/48)$, hence the condition number $\kappa_2(\tilde{S}) := \lambda_{max}/\lambda_{min} \leq 56.4$.

The development above transforms a bigger problem into a smaller one. More importantly, the smaller problem is better behaved since it has much smaller off-diagonal elements. Since the coefficient matrix is positive definite, the conjugate gradient method with diagonal preconditioner is very efficient.

6. Examples

Since the dimension of the interpolation function space V_M is ν, which is the same as that of linear element function space, and since the interpolant has linear precision, we compare the numerical behaviors of our approach with the linear interpolation approach. We use a benchmark of examples from the literatures [16].

Example 6.1. The aim of this example is to show the convergence rate of the interpolants, where the domain surfaces are defined by the limits of Loop's subdivision. The control meshes are defined such that the limit surfaces interpolate the regular triangulations $T^{(1)}, \cdots, T^{(6)}$ of an octahedron. $T^{(1)}, \cdots, T^{(6)}$ have 2^7, 2^9, 2^{11}, $\cdots$, 2^{17} triangles, respectively. Fig 6.1 shows the triangulation $T^{(4)}$, the control mesh $M^{(4)}$ and the limit surface S. The test functions are

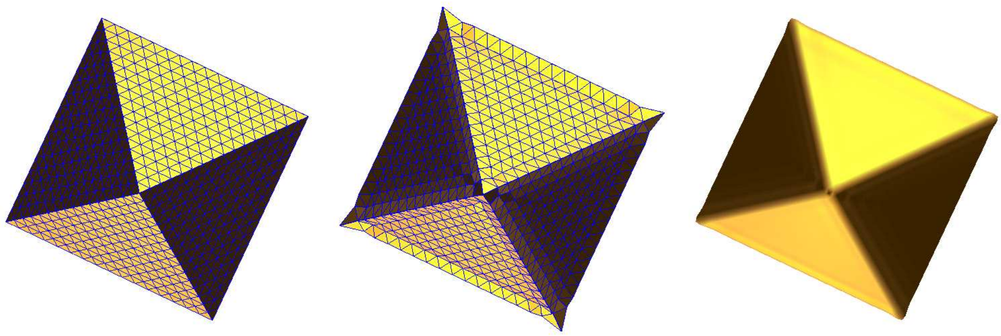


Fig 6.1: Left: Triangulation $T^{(4)}$; Middle: Control mesh $M^{(4)}$; Right: Limit surface S.

chosen from [16]:

$$\begin{aligned}
F_1(x,y,z) &= 0.75\exp\left\{-[(9x-2)^2+(9y-2)^2+(9z-2)^2]/4\right\}\\
&\quad + 0.75\exp\left\{-(9x+1)^2/49-(9y+1)/10-(9z+1)/10\right\}\\
&\quad + 0.5\exp\left\{-[(9x-7)^2+(9y-3)^2+(9z-5)^2]/4\right\}\\
&\quad - 0.2\exp\{-(9x-4)^2-(9y-7)^2-(9z-5)^2\};\\
F_2(x,y,z) &= [\tanh(9z-9x-9y)+1]/9;\\
F_3(x,y,z) &= [1.25+\cos(5.4y)]\cos(6z)/[6+6(3x-1)^2];\\
F_4(x,y,z) &= \exp\left\{-81\left[(x-0.5)^2+(y-0.5)^2+(z-0.5)^2\right]/16\right\}/3;\\
F_5(x,y,z) &= \exp\left\{-81\left[(x-0.5)^2+(y-0.5)^2+(z-0.5)^2\right]/4\right\}/3;\\
F_6(x,y,z) &= \sqrt{64-81[(x-0.5)^2+(y-0.5)^2+(z-0.5)^2]}/9-0.5.
\end{aligned}$$

Table 6.1. Maximal Errors of Loop's Interpolation

	F_1	F_2	F_3	F_4	F_5	F_6
ELP_1	0.363E-1	0.285E-1	0.103E-1	0.460E-3	0.507E-2	0.230E-2
ELP_2	0.756E-2	0.629E-2	0.246E-2	0.164E-3	0.567E-3	0.643E-3
ELP_3	0.149E-2	0.837E-3	0.616E-3	0.517E-4	0.143E-3	0.166E-3
ELP_4	0.346E-3	0.139E-3	0.154E-3	0.142E-4	0.358E-4	0.421E-4
ELP_5	0.855E-4	0.200E-4	0.387E-4	0.373E-5	0.894E-5	0.108E-4
ELP_6	0.213E-4	0.267E-5	0.969E-5	0.970E-6	0.225E-5	0.275E-5

Table 6.1 shows the maximal errors ELP of Loop's interpolants for the 6 test functions over the domain surfaces $T^{(i)}$. Comparing with the errors ELN of linear interpolations, that are shown in Table 6.2, the errors of Loop's interpolants are usually much smaller than that of linear interpolants. Table 6.3 gives the ratios $\mathrm{ELP}_{i+1}/\mathrm{ELP}_i$ of the maximal errors of Loop's interpolants. These ratios are near to $\frac{1}{4}$ as the surface meshes are subdivided.

Table 6.2. Maximal Errors of Linear Interpolation

	F_1	F_2	F_3	F_4	F_5	F_6
ELN_1	0.232E-1	0.166E-1	0.139E-1	0.477E-2	0.509E-2	0.586E-2
ELN_2	0.878E-2	0.874E-2	0.518E-2	0.245E-2	0.122E-2	0.323E-2
ELN_3	0.340E-2	0.443E-2	0.237E-2	0.123E-2	0.619E-3	0.169E-2
ELN_4	0.190E-2	0.222E-2	0.114E-2	0.618E-3	0.310E-3	0.865E-3
ELN_5	0.925E-3	0.111E-2	0.569E-3	0.309E-3	0.155E-3	0.438E-3
ELN_6	0.456E-3	0.555E-3	0.285E-3	0.155E-3	0.776E-4	0.220E-3

Since the size of the triangles of $T^{(i)}$ is twice of that of the triangles of $T^{(i+1)}$, the ratios (that is around $\frac{1}{4}$ for some function) show that the order of interpolation error can not be bigger than 2. Here we should mention the result of Arden [1]. In his thesis, he proved that Loop's function space has approximation order 3, where the error is measured by projecting the function on the tangent plane.

Table 6.3. The Ratios of Maximal Errors of Loop's Interpolation

	F_1	F_2	F_3	F_4	F_5	F_6
ELP_2/ELP_1	0.208	0.220	0.238	0.357	0.112	0.280
ELP_3/ELP_2	0.197	0.134	0.251	0.314	0.253	0.258
ELP_4/ELP_3	0.232	0.166	0.251	0.274	0.250	0.254
ELP_5/ELP_4	0.247	0.143	0.251	0.263	0.250	0.256
ELP_6/ELP_5	0.249	0.134	0.250	0.260	0.251	0.256

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