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AN ANISOTROPIC NONCONFORMING FINITE ELEMENT WITH SOME SUPERCONVERGENCE RESULTS *1)

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Abstract

The main aim of this paper is to study the error estimates of a nonconforming finite element with some superconvergence results under anisotropic meshes. The anisotropic interpolation error and consistency error estimates are obtained by using some novel approaches and techniques, respectively. Furthermore, the superclose and a superconvergence estimate on the central points of elements are also obtained without the regularity assumption and quasi-uniform assumption requirement on the meshes. Finally, a numerical test is carried out, which coincides with our theoretical analysis.

Mathematics subject classification: 65N30, 65N15.
Key words: Anisotropic meshes, Nonconforming finite element, Interpolation error and consistency error estimates, Superclose, Superconvergence.

1. Introduction

It is well-known that regular assumption or quasi-uniform assumption[1] of finite element meshes is a basic condition in analysis of finite element approximation both for conventional conforming and nonconforming elements. However, with the development of the finite element methods and its applications to more fields and more complex problems, the above regular assumption or quasi-uniform assumption are great deficient in the finite element methods. For example, the solution may have anisotropic behavior in parts of the domain. This means that the solution varies significantly only in certain directions. Such as the diffusion problems in domains with edges and singularly perturbed convection-diffusion-reaction problems where boundary or interior layers appear. In such cases, it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction.

Considering a bounded convex domain $\Omega \subset R^2$, we can describe the elements of anisotropic meshes mathematically. Let J_h be a family of meshes of Ω and denote the diameter of the finite element K and the supremum of the diameters of all circles contained in K by h_K and ρ_K respectively, $h = \max\limits_{K\in J_h} h_K$. It is assumed in the classical finite element theory that $\frac{h_K}{\rho_K} \leq C$, where C be a positive constant which is independent of K and the function considered. Such assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to 0$. Recently, Zenisek[2,3] and Apel[4,5] published a series of papers concentrating on the interpolation error estimates of some Lagrange Type elements(conforming elements), but nonconforming methods are hardly treated. As far as we know, it seems that there are few papers focused on the nonconforming elements under anisotropic meshes.

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On the other hand, the superconvergence study of the finite element methods is one of the most active research subject for a long time in theoretical analysis and practical computations. Many superconvergence results about conforming finite element methods have been obtained(see [6] [7]). Do these superconvergence results of conforming elements still hold for nonconforming ones? [8-10] studied the superconvergence of Wilson's element and obtained the superconvergence estimate of the gradient error on the centers of elements. Under square meshes, [11] recently obtained same superconvergentce results of rotated Q_1 element, too. However, to our knowledge, there are no papers published with respect to anisotropic meshes.

In our work, we firstly study the anisotropic interpolation property of a nonconforming finite element proposed by [12], which will play an important role in estimating the interpolation error. By employing some techniques different from the existing articles, we obtain the consistency error estimate. Then we get the superclose property and a superconvergence estimate on the centers of elements without the regularity assumption and quasi-uniform assumption requirements on the meshes. In the last section, some numerical examples are presented to illustrate the validity of our theoretical analysis.

2. Construction of the Finite Element Space with Anisotropic Interpolation Property

Assume $\hat{K} = [-1,1] \times [-1,1]$ to be the reference element, the four vertices are $\hat{d}_1 = (-1,-1), \hat{d}_2 = (1,-1), \hat{d}_3 = (1,1), \hat{d}_4 = (-1,1)$, let $\hat{l}_1 = \overline{\hat{d}_1\hat{d}_2}, \hat{l}_2 = \overline{\hat{d}_2\hat{d}_3}, \hat{l}_3 = \overline{\hat{d}_3\hat{d}_4}, \hat{l}_4 = \overline{\hat{d}_4\hat{d}_1}$.

We define the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ on $\hat{K}$ as follows

$$\hat{\Sigma} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5\}, \qquad \hat{P} = span\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\}, \tag{1}$$

where $\hat{v}_i = \frac{1}{|\hat{l}_i|}\int_{\hat{l}_i} \hat{v} d\hat{s}, \quad i = 1,2,3,4, \quad \hat{v}_5 = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{v} d\xi d\eta, \varphi(t) = \frac{1}{2}(3t^2-1)$.

It can be easily proved that the interpolation defined above is properly posed, the interpolation function is as follows

$$\hat{\Pi}\hat{v} = \hat{v}_5 + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta + \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi(\xi) + \frac{1}{2}(\hat{v}_3 + \hat{v}_1 - 2\hat{v}_5)\varphi(\eta) \tag{2}$$

For the sake of convenience, Let $\Omega \subset R^2$ to be a convex polygon composed by a family of rectangular meshes J_h which doesn't need to satisfy the regularity conditions. $\forall K \in J_h$, denote the barycenter of element K by (x_K, y_K), the length of edges parallel to x-axis and y-axis by $2h_x, 2h_y$ respectively, $h_K = \max\{h_x, h_y\}, h = \max\limits_{K \in J_h} h_K$.

$F_K : \hat{K} \longrightarrow K$ is defined as

$$\begin{cases} x = x_K + h_x\xi, \\ y = y_K + h_y\eta. \end{cases} \tag{3}$$

Define the finite element space as

$$V_h = \{v_h | \hat{v_h} = v_h|_K \circ F_K \in \hat{P},\ \forall K \in J_h, \int_F [v_h] ds = 0, F \subset \partial K\}, \tag{4}$$

where $[v_h]$ stands for the jump of v_h across the edge F if F is an internal edge, and it is equal to v_h itself if F is a boundary edge.

Let the general element K is a rectangle element in $x-y$ plane, the interpolate operator is defined as

$$\Pi_K : H^2(K) \to \hat{P} \circ F_K^{-1}, \Pi_K v = (\hat{\Pi}\hat{v}) \circ F_K^{-1}, \quad \Pi_h : H^2(\Omega) \to V_h, \Pi_h|_K = \Pi_K.$$

In order to obtain the anisotropic interpolation error estimate we should turn to the following lemma

Lemma 2.1. *The interpolation operator* $\hat{\Pi}$ *defined as (2) has the anisotropic interpolation properties, i.e., for* $|\alpha| = 1$*, such that*

$$\|\hat{D}^{\alpha}(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^{\alpha}\hat{v}|_{1,\hat{K}}. \tag{5}$$

Here and later, the positive constant C will be used as a generic constant, which is independent of h_K and of $\frac{h_K}{\rho_K}$.

Proof. When $\alpha = (1, 0)$,

$$\hat{D}^{\alpha}\hat{\Pi}\hat{v} = \frac{\partial \hat{\Pi}\hat{v}}{\partial \xi} = \frac{1}{2}(\hat{v}_2 - \hat{v}_4) + \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi^{'}(\xi). \tag{6}$$

Notice that $r = dim\hat{D}^{\alpha}\hat{P} = 2$. Obviously, $\{1, \varphi^{'}(\xi)\}$ is a basis of $\hat{D}^{\alpha}\hat{P}$, and denote

$$\hat{D}^{\alpha}\hat{\Pi}\hat{v} = \beta_1 + \beta_2\varphi^{'}(\xi),$$

where

$$\begin{aligned}
\beta_1 &= \frac{1}{2}(\hat{v}_2 - \hat{v}_4) = \frac{1}{4}(\int_{\hat{l}_2} \hat{v}(1, \eta)d\eta - \int_{\hat{l}_4} \hat{v}(-1, \eta)d\eta) = \frac{1}{|\hat{K}|}\int_{\hat{K}} \frac{\partial \hat{v}}{\partial \xi}d\xi d\eta, \\
\beta_2 &= \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5) = \frac{1}{4}(\int_{\hat{l}_2} \hat{v}(1, \eta)d\eta + \int_{\hat{l}_4} \hat{v}(-1, \eta)d\eta - 2\int_{\hat{K}} \hat{v}(\xi, \eta)d\xi d\eta) \\
&= \frac{1}{|\hat{K}|}\int_{\hat{K}} \xi\frac{\partial \hat{v}}{\partial \xi}d\xi d\eta.
\end{aligned}$$

$\forall \hat{w} \in H^1(\hat{K})$, let

$$F_1(\hat{w}) = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{w}d\xi d\eta,$$

$$F_2(\hat{w}) = \frac{1}{|\hat{K}|}\int_{\hat{K}} \xi\hat{w}d\xi d\eta.$$

Apparently $F_j \in (H^1(\hat{K}))^{'}, j = 1, 2$. Employing the basic anisotropic interpolation theorem[13] yields

$$\|\hat{D}^{\alpha}(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^{\alpha}\hat{v}|_{1,\hat{K}}.$$

Similarly, we can prove that (5) is valid for $\alpha = (0, 1)$. This completes the proof.

3. Anisotropic Error Estimates for the Second Order Elliptic Problem

Now, let us consider the following Poisson problem

$$\begin{cases} -\triangle u = f, & in \ \ \Omega, \\ u|_{\Gamma} = 0, & on \ \ \Gamma = \partial\Omega. \end{cases} \tag{7}$$

Let $V = H_0^1(\Omega)$, then the weak form of (7) is

$$\begin{cases} Find \ u \in V, & such \ \ that \\ a(u, v) = f(v), & \forall \ v \in V, \end{cases} \tag{8}$$

where

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx dy, \quad f(v) = \int_{\Omega} f v dx dy.$$

The approximation of (8) reads as follows

$$\begin{cases} Find\ u_h \in V_h, & such\ \ that \\ a_h(u_h, v_h) = f(v_h), & \forall\ v_h \in V_h. \end{cases} \tag{9}$$

We define

$$\|\cdot\|_h = (\sum_{K \in J_h} |\cdot|_{1,K}^2\ )^{\frac{1}{2}},$$

then it is easy to see that $\|\cdot\|_h$ is the norm over V_h.

Assume u and u_h to be the unique solution of (7) and (9) respectively, then by the second Strang lemma[1], we have

$$\|u - u_h\|_h \le C(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{v_h \in V_h \setminus \{0\}} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_h}). \tag{10}$$

Now we consider the first term on the right hand of (10), i.e., interpolation error.

By lemma 2.1, we have

$$\begin{aligned} \inf_{v_h \in V_h} \|u - v_h\|_h \quad & \le \|u - \Pi_h u\|_h = (\sum_{K \in J_h} |u - \Pi_K u|_{1,K}^2)^{\frac{1}{2}} \\ & = (\sum_{K \in J_h} \sum_{|\alpha|=1} \|D^\alpha (u - \Pi_K u)\|_{0,K}^2)^{\frac{1}{2}} \\ & = (\sum_{K \in J_h} \sum_{|\alpha|=1} h_K^{-2\alpha} (h_x h_y) \|\hat{D}^\alpha (\hat{u} - \hat{\Pi}_{\hat{K}} \hat{u})\|_{0,\hat{K}}^2)^{\frac{1}{2}} \\ & \le C(\sum_{K \in J_h} \sum_{|\alpha|=1} h_K^{-2\alpha} (h_x h_y) |\hat{D}^\alpha \hat{u}|_{1,\hat{K}}^2)^{\frac{1}{2}} \\ & \le C(\sum_{K \in J_h} \sum_{|\alpha|=1} \sum_{|\beta|=1} h_K^{2\beta} \|D^{\alpha+\beta} u\|_{0,K}^2)^{\frac{1}{2}} \\ & \le Ch|u|_{2,\Omega}. \end{aligned} \tag{11}$$

Then we turn to the second term on the right hand of (10), i.e., consistency error which will be very difficult to estimate without the usual regular assumption.

For $\forall K \in J_h, \forall v \in H^1(K)$, we define

$$\begin{aligned} P_{0i} v &= \frac{1}{2h_x} \int_{l_i} v dx, \quad i = 1, 3, \\ P_{0i} v &= \frac{1}{2h_y} \int_{l_i} v dy, \quad i = 2, 4, \\ P_0 v &= \frac{1}{|K|} \int_K v dx dy. \end{aligned}$$

It is easy to see that these projections are affine equivalent and the corresponding ones onto the reference element $\hat{K}$ denote by $\hat{P}_{0i}, i = 1, 2, 3, 4$ and $\hat{P}_0$.

Then by Green's formula we get

$$
\begin{aligned}
a_h(u,v_h)-(f,v_h) &= \sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds = \sum_K \sum_{l_i\subset\partial K} \int_{l_i} \frac{\partial u}{\partial n} v_h ds \\
&= \sum_{K\in J_h} [\int_{l_1} -(v_h - P_{01}v_h)(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y})dx \\
&+ \int_{l_3} (v_h - P_{03}v_h)(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y})dx \\
&+ \int_{l_2} (v_h - P_{02}v_h)(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x})dy \\
&- \int_{l_4} (v_h - P_{04}v_h)(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x})dy \\
&= \sum_K [I_1 + I_3 + I_2 + I_4], \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_{l_1} -(v_h - P_{01}v_h)(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y})dx, \\
I_2 &= \int_{l_2} (v_h - P_{02}v_h)(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x})dy, \\
I_3 &= \int_{l_3} (v_h - P_{03}v_h)(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y})dx, \\
I_4 &= -\int_{l_4} (v_h - P_{04}v_h)(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x})dy.
\end{aligned}
$$

We will note that the conventional consistency error estimate will become invalid under the consideration of anisotropic meshes. Take I_1 for example, in the conventional way,it can be estimated as

$$|I_1| \le h_x h_y^{-1} (\sum_{|\alpha|=1} h_K^{2\alpha} \|D^\alpha v_h\|_{0,K}^2)^{\frac{1}{2}} |\frac{\partial u}{\partial y}|_{1,K}. \qquad (13)$$

When the regularity assumption is satisfied, which yields $\frac{h_x}{h_y} \le C$, then we can get

$$|I_1| \le Ch_K |u|_{2,K} |v_h|_{1,K}. \qquad (14)$$

However, under the anisotropic meshes,$\frac{h_x}{h_y} \longrightarrow \infty$, we can not get the desired convergence result of (14) as usual. Thus it is more difficult for us to estimate anisotropic nonconforming error than conventional one. We will fasten on the consistency error from now on.

Let us see (12) again, we will introduce the following notations

$$Lv_h = \frac{x-(x_k-h_x)}{2h_x} P_{02}v_h - \frac{x-(x_k+h_x)}{2h_x} P_{04}v_h = \frac{1}{2}(1+\xi)\hat{P}_{02}\hat{v}_h - \frac{1}{2}(1-\xi)\hat{P}_{04}\hat{v}_h = \hat{L}\hat{v}_h, \quad (15)$$

$$Nv_h = \frac{y-(y_k-h_y)}{2h_y} P_{03}v_h - \frac{y-(y_k+h_y)}{2h_y} P_{01}v_h = \frac{1}{2}(1+\eta)\hat{P}_{03}\hat{v}_h - \frac{1}{2}(1-\eta)\hat{P}_{01}\hat{v}_h = \hat{N}\hat{v}_h, \quad (16)$$

i.e., L, N are linear interpolations of $P_{02}v_h, P_{04}v_h$, and $P_{01}v_h, P_{03}v_h$, they are also affine equivalent.

By using the definition of these operators, (12) can be written as

$$\begin{aligned} &a_h(u,v_h)-(f,v_h)=\sum_{K\in J_h}[\int_K \tfrac{\partial}{\partial y}[(v_h-Nv_h)(\tfrac{\partial u}{\partial y}-P_0\tfrac{\partial u}{\partial y})]dxdy \\ &+\int_K \tfrac{\partial}{\partial x}[(v_h-Lv_h)(\tfrac{\partial u}{\partial x}-P_0\tfrac{\partial u}{\partial x})]dxdy] \\ &=\sum_{K\in J_h}(A_K+B_K), \end{aligned} \tag{17}$$

where

$$A_K=\int_K \frac{\partial}{\partial y}[(v_h-Nv_h)(\frac{\partial u}{\partial y}-P_0\frac{\partial u}{\partial y})]dxdy,$$

$$B_K=\int_K \frac{\partial}{\partial x}[(v_h-Lv_h)(\frac{\partial u}{\partial x}-P_0\frac{\partial u}{\partial x})]dxdy.$$

Noticed that A_K can be decomposed expressed as

$$A_K=\int_K (v_h-Nv_h)\frac{\partial^2 u}{\partial y^2}dxdy+\int_K (w-P_0w)(\frac{\partial v_h}{\partial y}-\frac{\partial Nv_h}{\partial y})dxdy=A_{K1}+A_{K2}, \tag{18}$$

where

$$A_{K1}=\int_K (v_h-Nv_h)\frac{\partial^2 u}{\partial y^2}dxdy,$$

$$A_{K2}=\int_K (w-P_0w)(\frac{\partial v_h}{\partial y}-\frac{\partial Nv_h}{\partial y})dxdy, w=\frac{\partial u}{\partial y}.$$

Notice that $\hat{N}$ is accurate for zero degree polynomial. By employing interpolation theorem, we have

$$\begin{aligned} A_{K1} &= \int_K (v_h-Nv_h)\frac{\partial^2 u}{\partial y^2}dxdy \\ &\leq (\int_K |v_h-Nv_h|^2dxdy)^{\frac{1}{2}}(\int_K |\frac{\partial^2 u}{\partial y^2}|^2dxdy)^{\frac{1}{2}} \\ &\leq (h_xh_y)^{\frac{1}{2}}\|\hat{v}_h-\hat{N}\hat{v}_h\|_{0,\hat{K}}|u|_{2.K} \\ &\leq C(h_xh_y)^{\frac{1}{2}}|\hat{v}_h|_{1,\hat{K}}|u|_{2,K} \\ &= C(h_xh_y)^{\frac{1}{2}}|u|_{2,K}(\int_{\hat{K}}(|\frac{\partial \hat{v}_h}{\partial \xi}|^2+|\frac{\partial \hat{v}_h}{\partial \eta}|^2)d\xi d\eta)^{\frac{1}{2}} \\ &= C(h_xh_y)^{\frac{1}{2}}|u|_{2,K}(\int_K (h_x^2|\frac{\partial v_h}{\partial x}|^2+h_y^2|\frac{\partial v_h}{\partial y}|^2)(h_xh_y)^{-1}dxdy)^{\frac{1}{2}} \\ &\leq Ch_K|u|_{2,K}|v_h|_{1,K}. \end{aligned} \tag{19}$$

Because

$$\frac{\partial Nv_h}{\partial y}=\frac{1}{2h_y}(P_{03}v_h-P_{03}v_h)=\frac{1}{|K|}\int_K \frac{\partial v_h}{\partial y}dxdy=P_0\frac{\partial v_h}{\partial y}, \tag{20}$$

there holds

$$\|\frac{\partial Nv_h}{\partial y}\|_{0,K}=\frac{1}{|K|}|\int_K \frac{\partial v_h}{\partial y}dxdy|K|^{\frac{1}{2}}\leq(\int_K |\frac{\partial v_h}{\partial y}|^2dxdy)^{\frac{1}{2}}=\|\frac{\partial v_h}{\partial y}\|_{0,K}. \tag{21}$$

Then A_{K2} can be rewritten as

$$\begin{aligned} A_{K2} &= \int_K (w-P_0w)(\frac{\partial v_h}{\partial y}-\frac{\partial Nv_h}{\partial y})dxdy \\ &\le \|w-P_0w\|_{0,K}\|\frac{\partial v_h}{\partial y}-\frac{\partial Nv_h}{\partial y}\|_{0,K} \\ &\le \|w-P_0w\|_{0,K}\|\frac{\partial v_h}{\partial y}\|_{0,K}+\|\frac{\partial Nv_h}{\partial y}\|_{0,K} \\ &\le 2\|w-P_0w\|_{0,K}\|\frac{\partial v_h}{\partial y}\|_{0,K} \\ &\le C(h_xh_y)^{\frac{1}{2}}\|\hat{w}-\hat{P}_0\hat{w}\|_{0,\hat{K}}|v_h|_{1,K} \\ &\le C(h_xh_y)^{\frac{1}{2}}|\hat{w}|_{1,\hat{K}}|v_h|_{1,K} \\ &\le C(h_xh_y)^{\frac{1}{2}}(\int_K(h_x^2(\frac{\partial w}{\partial x})^2+h_y^2(\frac{\partial w}{\partial y})^2)(h_xh_y)^{-1}dxdy)^{\frac{1}{2}}|v_h|_{1,K} \\ &\le Ch|v_h|_{1,K}|w|_{1,K} \\ &\le Ch_K|v_h|_{1,K}|u|_{2,K}. \end{aligned} \tag{22}$$

Substituting (19) and (22) into (18), we obtain

$$|A_K| \le Ch_K|v_h|_{1,K}|u|_{2,K}. \tag{23}$$

Similarly, we can estimate B_K as follows

$$|B_K| \le Ch_K|v_h|_{1,K}|u|_{2,K}. \tag{24}$$

Substituting (23) and (24) into (17) yields

$$|a_h(u,v_h)-(f,v_h)| \le \sum_{K\in J_h} Ch_K|u|_{1,K}|v_h|_{1,K} \le Ch\|v_h\|_h|u|_{2,\Omega}. \tag{25}$$

Then we get the following theorem.

Theorem 3.1. *Under anisotropic meshes, we have the anisotropic error estimate as follows*

$$\|u-u_h\|_h \le Ch|u|_{2,\Omega}, \tag{26}$$

$$\|u-u_h\|_{0,\Omega} \le Ch^2|u|_{2,\Omega}. \tag{27}$$

Proof. Substituting (11) and (25) into (10) we can obtain (26). By the duality argument as standard finite element theory[1] we will get (27). Then the proof is completed.

4. Some Anisotropic Superconvergence Results

In this section, we will focus on studying the superconvergence behavior of the finite element constructed as (1).

Firstly, we can prove the following identical relation

Lemma 4.1. *Under anisotropic meshes, we have*

$$\|u-u_h\|_h^2 = \|u-\Pi_hu\|_h^2+\|\Pi_hu-u_h\|_h^2. \tag{28}$$

Proof. Note that $\triangle(\Pi_hu-u_h)|_K$ and $\frac{\partial(\Pi_hu-u_h)}{\partial n}|_{\partial K}$ are constants, we have

$$a_h(u-\Pi_hu,\Pi_hu-u_h) = \sum_{K\in J_h}\int_K \bigtriangledown(u-\Pi_Ku)\bigtriangledown(\Pi_hu-u_h)dxdy$$

$$
\begin{aligned}
&= \sum_{K\in J_h}\int_K -(u-\Pi_K u)\triangle(\Pi_h u-u_h)dxdy\\
&+ \sum_{K\in J_h}\int_{\partial K} -(u-\Pi_K u)\frac{\partial(\Pi_h u-u_h)}{\partial n}ds\\
&= \sum_{K\in J_h}\triangle(\Pi_h u-u_h)\int_K -(u-\Pi_K u)dxdy\\
&+ \sum_{K\in J_h}\frac{\partial(\Pi_h u-u_h)}{\partial n}\int_{\partial K} -(u-\Pi_K u)ds\\
&= 0.
\end{aligned}
$$

Then it is easy to show that

$$
\begin{aligned}
\|u-u_h\|_h^2 &= a_h(u-u_h,u-u_h)\\
&= a_h(u-\Pi_h u,u-\Pi_h u)+a_h(\Pi_h u-u_h,\Pi_h u-u_h)+2a_h(u-\Pi_h u,\Pi_h u-u_h)\\
&= \|u-\Pi_h u\|_h^2+\|\Pi_h u-u_h\|_h^2.
\end{aligned}
$$

Thus the proof is completed.

Remark 1. The identical relation (28) is obvious for conforming element, but (28) seldom happens for nonconforming element, especially for the element with anisotropic property.

The following theorem shows that the order of anisotropic consistency error is of $O(h^2)$ which is one order higher than the anisotropic interpolation error.

Theorem 4.1. *Under anisotropic meshes, if $u\in H^3(\Omega)$, we have*

$$
|a_h(u,v_h)-(f,v_h)|\le Ch^2|u|_{3,\Omega}\|v_h\|_h,\quad \forall v_h\in V_h. \tag{29}
$$

Proof. We turn back to (12) again and study the following relations,

$$
\begin{aligned}
I_1+I_3 &= \int_{l_1} -(v_h-P_{01}v_h)(\frac{\partial u}{\partial y}-P_{01}\frac{\partial u}{\partial y})dx+\int_{l_3}(v_h-P_{03}v_h)(\frac{\partial u}{\partial y}-P_{03}\frac{\partial u}{\partial y})dx\\
&= -\int_{x_K-h_x}^{x_K+h_x}[\frac{\partial u}{\partial y}(x,y_K-h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\frac{\partial u}{\partial y}(x,y_K-h_y)dx]\cdot\\
&\quad [v_h(x,y_K-h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}v_h(x,y_K-h_y)dx]dx\\
&\quad +\int_{x_K-h_x}^{x_K+h_x}[\frac{\partial u}{\partial y}(x,y_K+h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\frac{\partial u}{\partial y}(x,y_K+h_y)dx]\cdot\\
&\quad [v_h(x,y_K+h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}v_h(x,y_K+h_y)dx]dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& v_h(x, y_K - h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K - h_y)dx \\
&= \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} [v_h(x, y_K - h_y) - v_h(t, y_K - h_y)]dt \\
&= \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial v_h}{\partial z}(z, y_K - h_y)dzdt \\
&= \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial v_h}{\partial z}(z, y_K + h_y)dzdt \\
&= v_h(x, y_K + h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx.
\end{aligned}
$$

By the way, here we have used the specialities: $\frac{\partial v_h}{\partial x} \in \{1, x\}$ and $\frac{\partial v_h}{\partial y} \in \{1, y\}$, then

$$
\begin{aligned}
I_1 + I_3 \quad = \quad & \int_{x_K-h_x}^{x_K+h_x}[-\frac{\partial u}{\partial y}(x, y_K - h_y) + \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\frac{\partial u}{\partial y}(x, y_K - h_y)dx \\
& +\frac{\partial u}{\partial y}(x, y_K + h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\frac{\partial u}{\partial y}(x, y_K + h_y)dx] \cdot \\
& [v_h(x, y_K + h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx]dx \\
= \quad & \int_{x_K-h_x}^{x_K+h_x}[-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial^2 u}{\partial x\partial y}(x, y_K - h_y)dxdt \\
& +\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial^2 u}{\partial x\partial y}(x, y_K + h_y)dxdt] \cdot \\
& [\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial v_h}{\partial z}(z, y_K + h_y)dzdt]dx \\
= \quad & \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}[\int_{x_K-h_x}^{x_K+h_x}\int_t^x\int_{y_K-h_y}^{y_K+h_y}\frac{\partial^3 u}{\partial x\partial y^2}dxdtdy] \cdot \\
& [\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial v_h}{\partial z}(z, y_K + h_y)dzdt]dx \\
= \quad & \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} D_1 D_2 dx.
\end{aligned} \tag{30}
$$

where

$$
D_1 = \int_{x_K-h_x}^{x_K+h_x}\int_t^x\int_{y_K-h_y}^{y_K+h_y}\frac{\partial^3 u}{\partial x\partial y^2}dxdtdy,
$$

$$
D_2 = \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \frac{\partial v_h}{\partial z}(z, y_K + h_y)dzdt.
$$

Since

$$\begin{aligned}|D_1|^2 &\leq \int_{x_K-h_x}^{x_K+h_x}\int_t^x\int_{y_K-h_y}^{y_K+h_y}|\frac{\partial^3 u}{\partial x\partial y^2}|^2 dxdtdy\times 2h_y\int_{x_K-h_x}^{x_K+h_x}|x-t|dt\\ &\leq 2h_x\int_{x_K-h_x}^{x_K+h_x}\int_{y_K-h_y}^{y_K+h_y}|\frac{\partial^3 u}{\partial x\partial y^2}|^2 dxdy\times 2h_y\int_{x_K-h_x}^{x_K+h_x}|x-t|dt\\ &= 4h_xh_y\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}^2\times\int_{x_K-h_x}^{x_K+h_x}|x-t|dt,\end{aligned}$$

$$\begin{aligned}|D_2|^2 &= \frac{1}{4h_x^2}|\int_{x_K-h_x}^{x_K+h_x}\int_t^x\frac{\partial v_h}{\partial z}(z,y_K-h_y)dzdt|^2\\ &\leq \frac{1}{4h_x^2}\int_{x_K-h_x}^{x_K+h_x}\int_t^x|\frac{\partial v_h}{\partial z}|^2 dzdt\int_{x_K-h_x}^{x_K+h_x}|x-t|dt,\end{aligned}$$

then

$$\begin{aligned}\int_{x_K-h_x}^{x_K+h_x}|D_1|^2dx &\leq 4h_xh_y\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}^2\times\int_{x_K-h_x}^{x_K+h_x}\int_{x_K-h_x}^{x_K+h_x}|x-t|dtdx\\ &= 4h_xh_y\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}^2\times\frac{8h_x^3}{3}\\ &= \frac{32h_x^4h_y}{3}\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}^2,\end{aligned}\tag{31}$$

$$\begin{aligned}&\int_{x_K-h_x}^{x_K+h_x}|D_2|^2dx \leq \frac{1}{4h_x^2}\int_{x_K-h_x}^{x_K+h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x|\frac{\partial v_h}{\partial z}|^2dzdt\int_{x_K-h_x}^{x_K+h_x}|x-t|dtdx\\ &\leq \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}|\frac{\partial v_h}{\partial x}|^2dx\int_{x_K-h_x}^{x_K+h_x}\int_{x_K-h_x}^{x_K+h_x}|x-t|dxdt\\ &= \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}|\frac{\partial v_h}{\partial x}|^2dx\times\frac{8h_x^3}{3}\\ &= \frac{2h_x^2}{3h_y}\int_{x_K-h_x}^{x_K+h_x}\int_{y_K-h_y}^{y_K+h_y}|\frac{\partial v_h}{\partial x}|^2dxdy.\end{aligned}\tag{32}$$

By (30), (31),(32) and Cauchy-Schwartz inequality, we have

$$|I_1+I_3|\leq\frac{4h_x^2}{3}\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}\|\frac{\partial v_h}{\partial x}\|_{0,K}.\tag{33}$$

Similarly , we can get

$$|I_2+I_4|\leq\frac{4h_y^2}{3}\|\frac{\partial^3 u}{\partial x^2\partial y}\|_{0,K}\|\frac{\partial v_h}{\partial y}\|_{0,K}.\tag{34}$$

Then (29) follows from (12), (33) and (34). This completes the proof.

Remark 2. We should point out that theorem 4.1 will be hold for the rectangular finite elements whose spaces satisfy the following property: $\frac{\partial v_h}{\partial x}$ and $\frac{\partial v_h}{\partial y}$ have nothing to do with the variable y and x, respectively. One may check that the rotated Q_1 element studied in [14,15] and the elements proposed in [16] have above property. Thus (29) holds for these elements.

Remark 3. The order of consistency error of this element is $O(h^2)$ under anisotropic meshes, which is just one order higher than the anisotropic interpolation error. This convergency property is similar to that of the famous Quasi-Wilson[17,18] element under regularity assumption.

The superclose result will be obtained by theorem 4.1.

Theorem 4.2. *Suppose $u, u_h, \Pi_h u$ are the same as in lemma 4.1, $u \in H^3(\Omega) \cap H_0^1(\Omega)$, then we have the following superclose result under anisotropic meshes*

$$\|\Pi_h u - u_h\|_h \leq Ch^2 |u|_{3,\Omega}. \tag{35}$$

Proof. By lemma 4.1, we have

$$a_h(\Pi_h u - u, \Pi_h u - u_h) = 0,$$

then

$$\begin{aligned}\|\Pi_h u - u_h\|_h^2 &= a_h(\Pi_h u - u_h, \Pi_h u - u_h)\\ &= a_h(\Pi_h u - u, \Pi_h u - u_h) + a_h(u - u_h, \Pi_h u - u_h)\\ &= a_h(u - u_h, \Pi_h u - u_h)\\ &= a_h(u, \Pi_h u - u_h) - f(\Pi_h u - u_h).\end{aligned}$$

By theorem 4.1, we have

$$\|\Pi_h u - u_h\|_h^2 \leq Ch^2 |u|_{3,\Omega} \|\Pi_h u - u_h\|_h. \tag{36}$$

So, (35) follows from (36).

The following superconvergence theorem is the main result of this section.

Theorem 4.3. *Assume O_K to be the central point of element K, $u \in H^3(\Omega) \cap W^{1,\infty}(\Omega)$, then we have*

$$(\sum_{K \in J_h} | \bigtriangledown (u - u_h)(O_K)|^2 h_x h_y)^{\frac{1}{2}} \leq Ch^2 |u|_{3,\Omega}. \tag{37}$$

Proof.

$$\begin{aligned}(\sum_{K \in J_h} | \bigtriangledown (u - u_h)(O_K)|^2 h_x h_y)^{\frac{1}{2}} &\leq (\sum_{K \in J_h} | \bigtriangledown (u - \Pi_h u)(O_K)|^2 h_x h_y)^{\frac{1}{2}}\\ &+(\sum_{K \in J_h} | \bigtriangledown (\Pi_h u - u_h)(O_K)|^2 h_x h_y)^{\frac{1}{2}}\\ &= M + H,\end{aligned} \tag{38}$$

where

$$M = (\sum_{K \in J_h} | \bigtriangledown (u - \Pi_h u)(O_K)|^2 h_x h_y)^{\frac{1}{2}},$$

$$H = (\sum_{K \in J_h} | \bigtriangledown (\Pi_h u - u_h)(O_K)|^2 h_x h_y)^{\frac{1}{2}}.$$

Firstly, we estimate M. Denote $\hat{O}$ by the central point of the reference element $\hat{K}$, then we have

$$| \bigtriangledown (u - \Pi_h u)(O_K)|^2 = |\frac{\partial(u - \Pi_h u)}{\partial x}(O_K)|^2 + |\frac{\partial(u - \Pi_h u)}{\partial y}(O_K)|^2. \tag{39}$$

Let $\hat{Q}(\hat{u}) = |\frac{\partial(\hat{u} - \hat{\Pi}\hat{u})}{\partial \xi}(\hat{O})|$, then $\forall \hat{u} \in P_2(\hat{K}), \hat{Q}(\hat{u}) = 0$. In fact, $\frac{\partial(\hat{u} - \hat{\Pi}\hat{u})}{\partial \xi} \in P_1(\hat{K})$, suppose $\frac{\partial(\hat{u} - \hat{\Pi}\hat{u})}{\partial \xi} = a\xi + b\eta + c$, then

$$\frac{1}{\hat{K}} \int_{\hat{K}} \frac{\partial(\hat{u} - \hat{\Pi}\hat{u})}{\partial \xi} d\xi d\eta = \frac{1}{\hat{K}} \int_{\hat{K}} (a\xi + b\eta + c) d\xi d\eta = c = \frac{\partial(\hat{u} - \hat{\Pi}\hat{u})}{\partial \xi}(\hat{O}), \tag{40}$$

and

$$\frac{1}{\hat{K}}\int_{\hat{K}}\frac{\partial(\hat{u}-\hat{\Pi}\hat{u})}{\partial\xi}d\xi d\eta=\frac{1}{\hat{K}}\int_{\partial\hat{K}}(\hat{u}-\hat{\Pi}\hat{u})n_{\xi}ds=0. \tag{41}$$

Then by Bramble-Hilbert lemma, the first term on the right hand of (39) can be estimated as

$$\begin{aligned}|\frac{\partial(u-\Pi_h u)}{\partial x}(O_K)|^2 &= |\hat{Q}(\hat{u})|^2h_x^{-2}\le \|\frac{\partial(\hat{u}-\hat{\Pi}\hat{u})}{\partial\xi}\|^2_{0,\infty,\hat{K}}h_x^{-2}\\ &\le C|\frac{\partial(\hat{u}-\hat{\Pi}\hat{u})}{\partial\xi}|^2_{2,\hat{K}}h_x^{-2}=C|\frac{\partial\hat{u}}{\partial\xi}|^2_{2,\hat{K}}h_x^{-2}\\ &= C(h_xh_y)^{-1}\sum_{|\alpha|=2}h_K^{2\alpha}\|D^{\alpha}\frac{\partial u}{\partial x}\|^2_{0,K}. \end{aligned}\tag{42}$$

By the same argument, the second term on the right hand of (39) can be estimated as

$$|\frac{\partial(u-\Pi_h u)}{\partial y}(O_K)|^2\le C(h_xh_y)^{-1}\sum_{|\alpha|=2}h_K^{2\alpha}\|D^{\alpha}\frac{\partial u}{\partial y}\|^2_{0,K}. \tag{43}$$

Substituting (42) and (43) into (39), we get

$$|\bigtriangledown(u-\Pi_h u)(O_K)|^2\le C(h_xh_y)^{-1}\sum_{|\alpha|=2}h_K^{2\alpha}(\|D^{\alpha}\frac{\partial u}{\partial x}\|^2_{0,K}+\|D^{\alpha}\frac{\partial u}{\partial y}\|^2_{0,K}), \tag{44}$$

and

$$M\le C\sum_{K\in J_h}[\sum_{|\alpha|=2}h_K^{2\alpha}(\|D^{\alpha}\frac{\partial u}{\partial x}\|^2_{0,K}+\|D^{\alpha}\frac{\partial u}{\partial y}\|^2_{0,K})]^{\frac{1}{2}}\le Ch^2|u|_{3,\Omega}. \tag{45}$$

Now, let us estimate H. $|\bigtriangledown(\Pi_h u-u_h)(O_K)|^2$ can be expressed as

$$\begin{aligned}|\bigtriangledown(\Pi_h u-u_h)(O_K)|^2 &= |\frac{\partial(\Pi_h u-u_h)}{\partial x}(O_K)|^2+|\frac{\partial(\Pi_h u-u_h)}{\partial y}(O_K)|^2\\ &= |\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\xi}(\hat{O})|^2h_x^{-2}+|\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\eta}(\hat{O})|^2h_y^{-2}. \end{aligned}\tag{46}$$

Note that $\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\xi}, \frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\eta}\in P_1(\hat{K})$,and by (40), we have

$$\begin{aligned}|\bigtriangledown(\Pi_h u-u_h)(O_K)|^2 &= |\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\xi}|^2h_x^{-2}+|\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\eta}|^2h_y^{-2}\\ &\le \frac{1}{|\hat{K}|}[\|\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\xi}\|^2_{0,\hat{K}}h_x^{-2}+\|\frac{\partial(\hat{\Pi}\hat{u}-\hat{u}_h)}{\partial\eta}\|^2_{0,\hat{K}}h_y^{-2}]\\ &= (4h_xh_y)^{-1}[\|\frac{\partial(\Pi_h u-u_h)}{\partial x}\|^2_{0,K}+\|\frac{\partial(\Pi_h u-u_h)}{\partial y}\|^2_{0,K}]\\ &= C(h_xh_y)^{-1}|\Pi_h u-u_h|^2_{1,K}. \end{aligned}\tag{47}$$

By (47), theorem 4.2 and Cauchy-Schwarz inequality, we have

$$|N|\le C\|\Pi_h u-u_h\|_h\le Ch^2|u|_{3,\Omega}. \tag{48}$$

Substituting (45) and (48) into (38), we complete the proof of theorem 4.3.

Remark 4. The above superconvergence result only requires $u\in H^3(\Omega)\cap W^{1,\infty}(\Omega)$. However, as to rotated Q_1 element, in order to get our results, reference [19] requires $u\in W^{3,\infty}(\Omega)$ and all the elements to be equal square.

5. Numerical Experiment

In order to investigate the numerical behavior of the element under anisotropic meshes, we still consider the second order problem (7) with $f(x,y) = 4 - 2x^2 - 2y^2 \in L^2(\Omega)$, and $\Omega = (-1,1) \times (-1,1)$. It can be verified that the exact solution of problem (7) is $u(x,y) = (1-x^2)(1-y^2)$. In order to obtain the meshes on Ω, we subdivide the boundary of Ω into n and m equal intervals along the $x-axis$ and $y-axis$, respectively. We carry out the numerical computing with respect to the mesh with $\frac{n}{m} = 10$ and $\frac{n}{m} = 20$, respectively. The numerical results are listed in Table 5.1-5.4. Herein, α denotes the convergence order.

Table 5.1

$m \times n$	$\|u-u_h\|_{0,\Omega}$	α	$\|u-u_h\|_h$	α
2×20	0.1230889753	/	1.3833059703	/
4×40	0.0307053090	2.0031416416	0.7078987621	0.9665054679
8×160	0.0076721561	2.0007841587	0.3559613009	0.9918226004
16×320	0.0019177785	2.0001959801	0.1782315704	0.9979674816
32×640	0.0004794283	2.0000491142	0.0891471325	0.9994925261

Table 5.2

$m \times n$	$\|u-u_h\|_{0,\Omega}$	α	$\|u-u_h\|_h$	α
2×40	0.1220586740	/	1.3786250665	/
4×80	0.0304980042	2.0007882118	0.7053385757	0.9668422341
8×160	0.0076234603	2.0001969337	0.3546467856	0.9919331670
16×320	0.0019058000	2.0000493526	0.1775697988	0.9979966283
32×640	0.0004764459	2.0000123978	0.0888156759	0.9994999766

From the above two tables 5.1 and 5.2, we can see that the optimal energy norm error and L^2 norm error estimates between u and u_h are obtained under large aspect ratio ($\frac{h_K}{\rho_K} = \frac{\sqrt{m^2+n^2}}{m}$). It shows that the optimal error estimates are independent of h_K and of h_K/ρ_K, which means that we can get the same order of error estimates whether the subdivision satisfies the regular assumption or not.

Table 5.3

$m \times n$	$\|\Pi_h u - u_h\|_h$	α	$(\sum_{K\in J_h} \mid \nabla (u-u_h)(O_K)\mid^2 h_x h_y)^{\frac{1}{2}}$	α
2×20	0.2111396436	/	0.0961083012	/
4×40	0.0527266706	2.0015926361	0.0240498704	1.9986319542
8×80	0.0131780286	2.0003983974	0.0060138915	1.9996583462
16×160	0.0032942797	2.0000994205	0.0015035618	1.9999146461
32×320	0.0008235557	2.0000250340	0.0003758960	1.9999787807

Table 5.4

$m \times n$	$\|\Pi_h u - u_h\|_h$	α	$(\sum_{K\in J_h} \mid \nabla (u-u_h)(O_K)\mid^2 h_x h_y)^{\frac{1}{2}}$	α
2×40	0.2108962281	/	0.0961951954	/
4×80	0.0527096083	2.0003952980	0.0240544522	1.9996608496
8×160	0.0131764991	2.0000989437	0.0060139663	1.9999153614
16×320	0.0032940683	2.0000247955	0.0015035137	1.9999787807
32×640	0.0008235136	2.0000061989	0.0003758798	1.9999946356

On the other hand, from table 5.3 and 5.4, we can see that superclose and superconvergence behavior are also coincide with our theoretical analysis.

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