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THE OPTIMAL ORDER ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS TO HYPERBOLIC PROBLEMS*1)

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Abstract

In this paper, the linear finite element approximation to the positive and symmetric, linear hyperbolic systems is analyzed and an $O(h^2)$ order error estimate is established under the conditions of strongly regular triangulation and the H^3-regularity for the exact solutions. The convergence analysis is based on some superclose estimates derived in this paper. Our method and result here are also applicable to general hyperbolic problems. Finally, we discuss the linearized shallow water system of equations.

Mathematics subject classification: 65M, 65N.
Key words: Hyperbolic problems, Finite element approximations, Optimal error estimates.

1. Introduction

Since 1970's, finite element method for solving partial differential equations has been successfully applied to elliptic and parabolic problems, however, it is still not very popular for hyperbolic problems. In view of that compared with the difference method, finite element method is more flexible and adaptive, and easier to mathematically analyze, recently finite element methods for hyperbolic problems have attracted more and more attention; see, e.g., [1-5] for the Galerkin method; [6-13] for the discontinuous Galerkin method; [14-17] for the Petrov-Galerkin method; and [18-21] for the streamline diffusion method.

It is well known that for the k-th order finite element approximations to elliptic or parabolic problems, the optimal order error estimate in L_2 norm is of $O(h^{k+1})$ order with the exact solution u in $H^{k+1}(\Omega)$. However, for linear hyperbolic problems, it is still an unsolved completely problem that whether or not the finite element solutions admit this optimal order estimate. Generally speaking, the convergence order of Galerkin method for hyperbolic problems is of $O(h^k)$ order, that is one order lower than the approximation order of finite element space; cf. [1] and [2]. And in [1], Dupont gave a counterexample by using third order Hermit element to indicate that this convergence order is sharp. Since then, in order to obtain the high accuracy and cope with the lower regularity of hyperbolic problems, the discontinuous Galerkin method is proposed and used extensively in this area; cf. [6],[7],[8],[9],[12] and [13]. By this method, the convergence order can be improved to $O(h^{k+\frac{1}{2}})$, and recently some superconvergence results are also given in [22] for elliptic problem by using discontinuous Galerkin method.

In the context of Galerkin method, under some assumptions on the finite element partition and regularity of the exact solution, it is possible to obtain the optimal order error estimates when linear finite elements are used; see, e.g.,[3] for bilinear rectangular element; and [5] for linear triangular element imposed on uniform mesh partition. Obviously, the condition of uniform mesh partition is not very interesting in the practical case.

In this paper, we will discuss the linear finite element approximation to positive and symmetric hyperbolic systems. Under the conditions of strongly regular triangulation (cf. [23]) and

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H^3-regularity for the exact solutions, the optimal order error estimates are established. The theoretical tools for the error analysis are some superclose estimates that are also derived in this paper. Our method and result here are also applicable to general hyperbolic problems. To author's knowledge, very few optimal convergence order can be reached for hyperbolic problems, even in one dimensional case. Hence, our research work in this paper is theoretically significant.

Let $\Omega \subset R^2$ be a polygonal domain, $J_h = \{e\}$ be the finite element partition of domain Ω parameterized by mesh size h so that $\overline{\Omega} = \cup_{e \in J_h} \{\,\overline{e}\,\}$. Introduce the linear finite element space S_h defined by

$$S_h = \{\, v \in C(\overline{\Omega}) \bigcap H^1(\Omega) : v|_e \text{ is linear, } \forall e \in J_h \,\}.$$

We will use the standard notation for the Sobolev spaces $W_p^m(\Omega)$ with corresponding norms and seminorms, and when $p = 2$, $W_2^m(\Omega) = H^m(\Omega)$, $\|\cdot\|_{m,2} = \|\cdot\|_m$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner product and norm in $L_2(\Omega)$ space. Let X be a Banach space, constant $T > 0$, we will also use the space,

$$L_p(0,T;X) = \{\, v(t) : (0,T) \to X : \ \|v\|_{L_p(X)} = \Big(\int_0^T \|v(t)\|_X^p \, dt\,\Big)^{\frac{1}{p}} < \infty \,\}.$$

In this paper, letter C represents a generic constant independent of mesh size h.

The plan of this paper is as follows. In section 2, some superclose estimates for interpolation are established. In section 3, the linear finite element approximations are analyzed for steady and nonsteady positive and symmetric hyperbolic systems, respectively, and the optimal order error estimates are derived. Finally, we will discuss the linearized shallow water system of equations.

2. Superclose Estimates

Definition 2.1. *Let $e = \triangle p_1p_2p_3$, $e' = \triangle p_1'p_2'p_3'$, and e and e' be two adjacent triangle elements sharing a common edge in J_h. The quadrilateral $\overline{e} \cup \overline{e'}$ is called as an approximate parallelogram if (see figure 1)*

$$|\, \overrightarrow{p_1p_2} + \overrightarrow{p_1'p_2'} \,| = O(h^2)\,, \qquad |\, \overrightarrow{p_2p_3} + \overrightarrow{p_2'p_3'} \,| = O(h^2). \tag{2.1}$$

Definition 2.2. *A triangulation J_h is called as strongly regular, if any two adjacent triangular elements in J_h form an approximate parallelogram (see figure 1).*

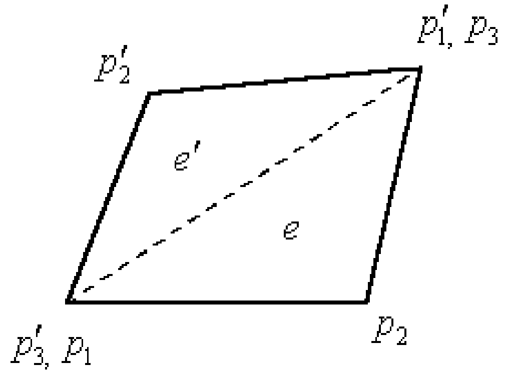

Figure 1. approximating parallelogram

Remark 2.1. Strongly regular triangulation must be quasi-uniform. And any domain composed of several convex quadrilaterals can be subdivided into strongly regular triangulation [23,24].

Lemma 2.1. *Let triangulation J_h be strongly regular, $e = \triangle p_1p_2p_3$ and $e' = \triangle p_1'p_2'p_3'$ be two adjacent triangle elements (see figure 1), vectors $\overrightarrow{L} = \overrightarrow{p_1p_2}$ (or $\overrightarrow{p_2p_3}$) and $\overrightarrow{L'} = \overrightarrow{p_1'p_2'}$ (or $\overrightarrow{p_2p_3}$), the lengths $l = |\overrightarrow{L}|$, $l' = |\overrightarrow{L'}|$, and the unit direction vectors $n = \frac{1}{l}\overrightarrow{L}$, $n' = \frac{1}{l'}\overrightarrow{L'}$. Then*

$$|l - l'| = O(h^2), \qquad |n + n'| = O(h). \tag{2.2}$$

Proof. From (2.1) we have

$$\begin{aligned}
|l - l'| &= \frac{1}{l+l'}|\overrightarrow{L}\cdot\overrightarrow{L} - \overrightarrow{L'}\cdot\overrightarrow{L'}| = \frac{1}{l+l'}|(\overrightarrow{L} + \overrightarrow{L'})\cdot(\overrightarrow{L} - \overrightarrow{L'})| \\
&\le |\overrightarrow{L} + \overrightarrow{L'}| = O(h^2), \\
|n + n'| &= |\frac{1}{l}\overrightarrow{L} + \frac{1}{l'}\overrightarrow{L'}| = \frac{1}{l\,l'}|(l'-l)\overrightarrow{L} + l(\overrightarrow{L} + \overrightarrow{L'})| \\
&\le \frac{1}{l'}(|l'-l| + |\overrightarrow{L} + \overrightarrow{L'}|) = O(h).
\end{aligned}$$

Let triangular element $e = \triangle p_1p_2p_3$ with three edge vectors $\overrightarrow{L_1} = \overrightarrow{p_2p_3}$, $\overrightarrow{L_2} = \overrightarrow{p_3p_1}$, $\overrightarrow{L_3} = \overrightarrow{p_1p_2}$, $l_i = |\overrightarrow{L_i}|$ and $n_i = \frac{1}{l_i}|\overrightarrow{L_i}|$ denote the lengths and unit direction vectors of $\overrightarrow{L_i}$ $(i = 1, 2, 3)$, respectively, and $D_i = n_i \cdot \nabla$ be the direction derivatives along $\overrightarrow{L_i}$ (see figure 2).

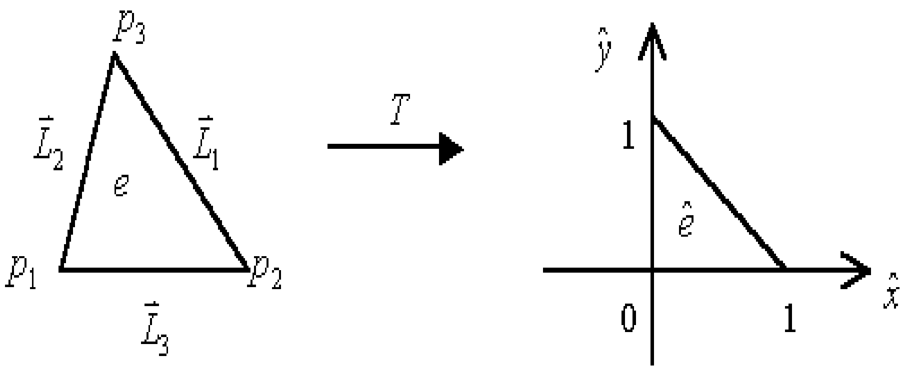

Figure 2. triangular element and unit triangular element

Lemma 2.2. *Let $e = \triangle p_1p_2p_3$ in J_h. Then*

$$\int_e (w - w_I)\phi = -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w\,\phi + O(h^3)(\|w\|_{2,e}\|\phi\|_{1,e} + \|w\|_{3,e}\|\phi\|_{0,e}). \tag{2.3}$$

Where w_I is the piecewise linear interpolation approximation of function w in S_h.

Proof. Let $\hat{e}$ be the unit triangle with vertices $\hat{p}_1 = (0,0)$, $\hat{p}_2 = (1,0)$ and $\hat{p}_3 = (0,1)$. Set $\hat{l}_1 = \sqrt{2}$, $\hat{l}_2 = \hat{l}_3 = 1$, $\hat{D}_1 = (\partial_y - \partial_x)/\sqrt{2}$, $\hat{D}_2 = -\partial_y$ and $\hat{D}_3 = \partial_x$. By straightforward calculation, we can see that for any quadratic polynomial q,

$$\int_{\hat{e}} q = \frac{\hat{e}}{3}\sum_{i=1}^{3} q(\hat{p}_i) - \frac{1}{24}\int_{\hat{e}}\sum_{i=1}^{3}(\hat{l}_i)^2(\hat{D}_i)^2 q, \tag{2.4}$$

which is invariant under affine-linear transformations. Define the linear bounded functional F on $W_1^3(\hat{e})$ by

$$F(\hat{w}) = \int_{\hat{e}} (\hat{w} - \hat{w}_I) + \frac{1}{24}\int_{\hat{e}} \sum_{i=1}^{3} (\hat{l}_i)^2 (\hat{D}_i)^2\, \hat{w}\,. \tag{2.5}$$

From (2.4) and note that $(w - w_I)(p_i) = 0$ and $D_i^2 w_I = 0$, we obtain

$$F(q) = 0, \quad \forall\, q \in P_2(\hat{e}).$$

Then, by using the Bramble-Hilbert lemma,

$$|\, F(\hat{w})| \le C|\, \hat{w}\,|_{3,1,\hat{e}}. \tag{2.6}$$

Combining (2.5) and (2.6) and utilizing the affine-linear transformation, we have

$$\int_e (w - w_I) = -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w + O(h^3)|\, w\,|_{3,1,e}. \tag{2.7}$$

Let $\overline{\phi} = \dfrac{1}{|e|}\displaystyle\int_e \phi$. Writing

$$\int_e (w - w_I)\phi = \int_e (w - w_I)\overline{\phi} + \int_e (w - w_I)(\phi - \overline{\phi}).$$

It follows from (2.7) and the interpolation approximation properties that

$$\begin{aligned}\int_e (w - w_I)\phi &= -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w(\overline{\phi} - \phi + \phi) + O(h^3)|\, w\,|_{3,1,e}\overline{\phi} + O(h^3)\|w\|_{2,e}\|\phi\|_{1,e}\\ &= -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w\phi + O(h^3)(\|\, w\,\|_{2,e}\|\phi\|_{1,e} + \|w\|_{3,e}\|\phi\|_{0,e}).\end{aligned}$$

The proof is completed.

Lemma 2.3. *Let* $\vec{\beta} \in [W_\infty^1(\Omega)]^2$, $w \in H^3(\Omega)$, $v \in S_h$. *Then*

$$|\int_\Omega (w - w_I)\,\vec{\beta}\cdot\nabla v\,| \le Ch^2\|w\|_3(\,\|v\| + (\int_{\partial\Omega} |v|^2|\,\vec{\beta}\cdot n|\,)^{\frac{1}{2}}\,). \tag{2.8}$$

Where n denotes the outward unit normal vector along $\partial\Omega$.

Proof. By using Lemma 2.2, the *Green's* formula and inverse inequality, and noting that v is piecewise linear, we have

$$\begin{aligned}\int_\Omega (w - w_I)\,\vec{\beta}\cdot\nabla v &= \sum_e \int_e (w - w_I)\,\vec{\beta}\cdot\nabla v\\ &= \sum_e -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w\,\vec{\beta}\cdot\nabla v + O(h^2)\|\, w\,\|_3\|v\|\\ &= -\frac{1}{24}\sum_e \int_{\partial e} \sum_{i=1}^{3} l_i^2 D_i^2 w\, v\,\vec{\beta}\cdot n + O(h^2)\|\, w\,\|_3\|v\|\\ &= -\frac{1}{24}\sum_{l\in\partial e, l\notin\partial\Omega} \int_l \sum_{i=1}^{3} [\, l_i^2 D_i^2 w - (l_i')^2(D_i')^2 w]\, v\,\vec{\beta}\cdot n\\ &\quad -\frac{1}{24}\int_{\partial\Omega} \sum_{i=1}^{3} l_i^2 D_i^2 w\, v\,\vec{\beta}\cdot n + O(h^2)\|\, w\,\|_3\|v\|.\end{aligned} \tag{2.9}$$

Where $l = \partial e \cap \partial e'$ (see figure 1). Now it follows from Lemma 2.1 that

$$\begin{aligned} |\, l_i^2 D_i^2 w - (l_i')^2 (D_i')^2 w \,| &= |\, (\, l_i^2 - (l_i')^2 \,) D_i^2 w + (l_i')^2 (\, D_i^2 - (D_i')^2 \,) w \,| \\ &= |\, (l_i + l_i')(l_i - l_i') D_i^2 w + (l_i')^2 (D_i + D_i')(D_i - D_i') w \,| \\ &\le Ch^3 |D^2 w| + (l_i')^2 |(n_i + n_i') \cdot \nabla (D_i - D_i') w \,| \le Ch^3 |\, D^2 w \,|. \end{aligned}$$

Substituting this into (2.9) to yield

$$\begin{aligned} |\int_\Omega (w - w_I) \vec{\beta} \cdot \nabla v| &\le Ch^3 \sum_e \int_{\partial e} |\, D^2 w \,| \, |v| \\ &\quad + Ch^2 \|w\|_{2,\partial\Omega} (\int_{\partial\Omega} |v|^2 |\, \vec{\beta} \cdot n| \,)^{\frac{1}{2}} + Ch^2 \|w\|_3 \|v\| \\ &\le Ch^2 \|w\|_3 (\, \|v\| + (\int_{\partial\Omega} |v|^2 |\, \vec{\beta} \cdot n| \,)^{\frac{1}{2}} \,). \end{aligned}$$

Where we have used the following trace inequalities and inverse inequality:

$$(\int_{\partial e} w^2)^{\frac{1}{2}} \le Ch^{-\frac{1}{2}} (\, h \|\nabla w\|_{0,e} + \|w\|_{0,e}), \quad w \in H^1(e),$$

$$(\int_{\partial\Omega} |w|^2)^{\frac{1}{2}} \le C(\Omega) \|w\|_1, \quad \|v\|_1 \le Ch^{-1} \|v\|, \quad v \in S_h.$$

The proof is completed.

3. First Order Hyperbolic Problems

3.1. Steady problems

Consider the following first order hyperbolic problem:

$$\mathbf{A}(x) \cdot \nabla \mathbf{u} + B(x) \mathbf{u} = \mathbf{f}(x), \;\; x \in \Omega, \tag{3.1}$$

$$N(x) \mathbf{u} = \frac{1}{2} (M - D) \mathbf{u} = \mathbf{0}, \;\; x \in \partial\Omega. \tag{3.2}$$

Where $\mathbf{A} = (A_1, A_2)$, $A_k = (a_{ij}^{(k)}(x))$, $B = (b_{ij}(x))$ and $M = (m_{ij}(x))$ are some given $m \times m$ order matrices, $a_{ij} \in W_\infty^1(\Omega)$, $b_{ij}, m_{ij} \in L_\infty(\Omega)$, $D = \mathbf{A} \cdot n$, $n = (n_x, n_y)$ is the outward unit normal on $\partial\Omega$, $\mathbf{u} = (u_1, \cdots, u_m)^T$ and $\mathbf{f} = (f_1, \cdots, f_m)^T$ are m-dimensional vector functions. Problem (3.1)-(3.2) is called as a positive and symmetric hyperbolic system if

$$A_1 = A_1^T, \; A_2 = A_2^T, \; x \in \Omega, \tag{3.3}$$

$$B + B^T - div\mathbf{A} \ge \sigma_0 I, \;\; constant\; \sigma_0 > 0, \; x \in \Omega, \tag{3.4}$$

$$M + M^T \ge 0, \; x \in \partial\Omega, \tag{3.5}$$

$$Ker(M - D) + Ker(M + D) = R^m, \; x \in \partial\Omega. \tag{3.6}$$

Introduce the bilinear form:

$$A(\mathbf{u}, \mathbf{v}) = (\mathbf{A} \cdot \nabla \mathbf{u}, \mathbf{v}) + (B\mathbf{u}, \mathbf{v}) + < N\mathbf{u}, \mathbf{v} >_{\partial\Omega} . \tag{3.7}$$

By the $Green's$ formula and (3.4) we have

$$A(\mathbf{u}, \mathbf{u}) \ge \frac{1}{2} \sigma_0 \|\mathbf{u}\|^2 + \frac{1}{2} < M\mathbf{u}, \mathbf{u} >_{\partial\Omega}, \; \forall \, \mathbf{u} \in [H^1(\Omega)]^m. \tag{3.8}$$

Now we define the linear finite element approximation to problem (3.1)-(3.2) by finding $\mathbf{u}_h \in [S_h]^m$ such that

$$A(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \;\; \forall \, \mathbf{v}_h \in [S_h]^m. \tag{3.9}$$

It is easy to see from (3.8) that $\mathbf{u}_h$ uniquely exists and satisfies the stability estimate

$$\frac{1}{2}\sigma_0\|\mathbf{u}_h\|^2+ < M\mathbf{u}_h, \mathbf{u}_h >_{\partial\Omega} \le \frac{2}{\sigma_0}\| \mathbf{f} \|^2. \tag{3.10}$$

From (3.1)-(3.2) and (3.9), we have the error equation

$$A(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h) = 0,\ \forall\, \mathbf{v}_h \in [S_h]^m. \tag{3.11}$$

In order to do the error analysis here we assume a more strong condition than condition (3.5), which can be satisfied by many hyperbolic problems. There exists a constant $\sigma_1 > 0$ such that

$$(H) \qquad < (M+M^T)\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega} \ge\ \sigma_1 < \mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega},\ \forall\, \mathbf{v}_h \in [S_h]^m.$$

Theorem 3.1. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of problems (3.1)-(3.2) and (3.9) respectively, $\mathbf{u} \in [H^3(\Omega)]^m$, triangulation J_h be strongly regular and hypothesis (H) hold. Then $\mathbf{u}_h$ satisfies the following optimal order error estimates*

$$\|\mathbf{u}-\mathbf{u}_h\| + \|\mathbf{u}-\mathbf{u}_h\|_{0,\partial\Omega}\ \le Ch^2\|\mathbf{u}\|_3.$$

Proof. It follows from (3.11) and the *Green's* formula that

$$\begin{aligned} & A(\mathbf{u}_I-\mathbf{u}_h, \mathbf{v}_h) = A(\mathbf{u}_I-\mathbf{u}, \mathbf{v}_h) = -(\mathbf{u}_I-\mathbf{u},\, \mathbf{A}\cdot\nabla\mathbf{v}_h) \\ + \quad & ((B-div\mathbf{A})(\mathbf{u}_I-\mathbf{u}), \mathbf{v}_h) + \frac{1}{2} < (M+D)(\mathbf{u}_I-\mathbf{u}), \mathbf{v}_h >_{\partial\Omega},\ \forall\, \mathbf{v}_h \in [S_h]^m. \end{aligned}$$

Taking $\mathbf{v}_h = \mathbf{u}_I - \mathbf{u}_h$, by using (3.8), hypothesis (H), Lemma 2.3 and interpolation approximation property we obtain

$$\begin{aligned} & \frac{1}{2}\sigma_0\|\mathbf{u}_I-\mathbf{u}_h\|^2 + \frac{1}{4}\sigma_1 < \mathbf{u}_I-\mathbf{u}_h, \mathbf{u}_I-\mathbf{u}_h >_{\partial\Omega} \\ \le \quad & Ch^2\|\mathbf{u}\|_3(\,\|\mathbf{u}_I-\mathbf{u}_h\| + (\int_{\partial\Omega}|\mathbf{u}_I-\mathbf{u}_h|^2|\mathbf{A}\cdot n|)^{\frac{1}{2}}\,) \\ + \quad & Ch^2\|\mathbf{u}\|_2\|\mathbf{u}_I-\mathbf{u}_h\| + C\|\mathbf{u}-\mathbf{u}_I\|_{0,\partial\Omega}\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\partial\Omega}. \end{aligned}$$

Combining this with the inequality

$$\|\mathbf{u}-\mathbf{u}_I\|_{0,\partial\Omega}\ \le Ch^2\|u\|_{2,\partial\Omega}\ \le Ch^2\|u\|_3,$$

we complete the proof.

Below we will briefly discuss the single equation case. Consider a hyperbolic problem of the form

$$\begin{aligned} \vec{\beta}\cdot\nabla u + \alpha\, u \quad &= \quad f,\ in\ \Omega, && (3.12) \\ u \quad &= \quad g,\ on\ \partial\Omega_-. && (3.13) \end{aligned}$$

Where $\vec{\beta}\in [W^1_\infty(\Omega)]^2$ and $\alpha \in L_\infty(\Omega)$, f and g are some given smooth functions, $\partial\Omega_- = \{\, x \in \partial\Omega : \vec{\beta}\cdot n < 0 \,\}$ and $\partial\Omega_+ = \partial\Omega\backslash\partial\Omega_-$. We assume that

$$\alpha - \frac{1}{2}div\ \vec{\beta} \ge \sigma_0,\ \sigma_0 > 0,\ in\ \Omega. \tag{3.14}$$

Set a finite element space by

$$S_h^0 = \{\, v_h \in S_h : v_h = 0,\ on\ \partial\Omega_- \,\}.$$

The finite element approximation for problem (3.12)-(3.13) reads: Find $u_h \in S_h$ such that

$$\begin{aligned} (\vec{\beta}\cdot\nabla u_h, v_h) + (\alpha\, u_h, v_h) \quad &= \quad (f, v_h),\ \forall\, v_h \in S_h^0, && (3.15) \\ u_h \quad &= \quad g_I, \quad on\ \partial\Omega_-. && (3.16) \end{aligned}$$

From (3.14) we see that

$$(\vec{\beta}\cdot\nabla w, w) + (\alpha\, w, w) \geq \sigma_0\|w\|^2 + \frac{1}{2}\int_{\partial\Omega} w^2\, \vec{\beta}\cdot n,\ \forall\, w \in H^1(\Omega). \tag{3.17}$$

This implies that the solution u_h uniquely exists and satisfies

$$\sigma_0\|u_h\|^2 + \int_{\partial\Omega_+} |\,u_h|^2|\,\vec{\beta}\cdot n\,| \leq \frac{1}{\sigma_0}\|f\|^2 + \int_{\partial\Omega_-} |\,g_I|^2|\,\vec{\beta}\cdot n\,|. \tag{3.18}$$

From (3.17), Lemma 2.3 and the identity

$$(\vec{\beta}\cdot\nabla(u_h - u_I), v_h) + (\,\alpha(u_h - u_I), v_h) = -(u - u_I, \vec{\beta}\cdot\nabla v_h) \tag{3.19}$$

$$+ \quad (\,(\alpha - div\,\vec{\beta})(u - u_I), v_h) + \int_{\partial\Omega_+} (u - u_I)v_h|\,\vec{\beta}\cdot n\,|,\ v_h \in S_h^0, \tag{3.20}$$

we immediately obtain the following theorem.

Theorem 3.2. *Let u and u_h be the solutions of problems (3.12)-(3.13) and (3.15)-(3.16) respectively, $u \in H^3(\Omega)$, triangulation J_h be strongly regular. Then*

$$\|u - u_h\| + (\int_{\partial\Omega_+} |\,u - u_h|^2|\,\vec{\beta}\cdot n\,|\,)^{\frac{1}{2}} \leq Ch^2\|u\|_3. \tag{3.21}$$

Remark 3.1. For a single equation case, the hypothesis (H) can be removed.

Remark 3.2. The optimal order error estimates for bilinear and linear finite element approximations to hyperbolic problems have been obtained in [3] and [5], respectively, under some restrictive conditions such as uniform mesh partitions.

3.2. Nonsteady problems

Consider the time-dependent first order hyperbolic problem:

$$\mathbf{u}_t + \mathbf{A}\cdot\nabla\mathbf{u} + B\mathbf{u} \;=\; \mathbf{f}(t),\ \ (t, x) \in [0, T) \times \Omega, \tag{3.22}$$

$$N\mathbf{u} = \frac{1}{2}(M - D)\mathbf{u} \;=\; \mathbf{0},\ \ \ (t, x) \in [0, T) \times \partial\Omega, \tag{3.23}$$

$$\mathbf{u}(0, x) \;=\; \mathbf{u}_0(x),\ x \in \Omega. \tag{3.24}$$

Where the notation representations in (3.22)-(3.23) are the same as those in (3.1)-(3.2).

Define the finite element approximation for problem (3.22)-(3.24) by finding $\mathbf{u}_h : [0, T) \to [S_h]^m$ such that

$$(\mathbf{u}_{h,t}, \mathbf{v}_h) + A(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h),\ \forall\, \mathbf{v}_h \in [S_h]^m, \tag{3.25}$$

$$\mathbf{u}_h(0) \in [S_h]^m. \tag{3.26}$$

Where the bilinear form $A(u, v)$ is given by (3.7). Taking $\mathbf{v}_h = \mathbf{u}_h$ in (3.25), from (3.8) we obtain

$$\frac{d}{dt}\|\mathbf{u}_h(t)\| + \frac{\sigma_0}{2}\|\mathbf{u}_h(t)\| \leq \|\mathbf{f}(t)\|. \tag{3.27}$$

This implies the stability estimate

$$\|\mathbf{u}_h(t)\| \leq e^{-\frac{\sigma_0}{2}t}(\|\mathbf{u}_h(0)\| + \int_0^t e^{\frac{\sigma_0}{2}\tau}\|\mathbf{f}(\tau)\|\,d\tau\,),\ t > 0. \tag{3.28}$$

Theorem 3.3. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of problems (3.22)-(3.24) and (3.25)-(3.26) respectively, $\mathbf{u}(0) \in [H^3(\Omega)]^m$, $\mathbf{u}_t(t) \in L_1(0, T; [H^3(\Omega)]^m\,)$, triangulation J_h be strongly regular and hypothesis (H) hold. Then, there exists a constant C independent of $t \in [0, T)$ such that*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq e^{-\frac{\sigma_0}{2}t}\|\mathbf{u}(0) - \mathbf{u}_h(0)\| + Ch^2(\|\mathbf{u}(0)\|_3 + \int_0^t \|\mathbf{u}_t(\tau)\|_3\,d\tau\,),\ t > 0.$$

Proof. First introduce the projection approximation of solution $\mathbf{u}$ in $[S_h]^m$ by setting $\mathbf{w}_h(t) : [0,T) \to [S_h]^m$ such that

$$A(\mathbf{u}(t) - \mathbf{w}_h(t), \mathbf{v}_h) = 0, \ \forall \mathbf{v}_h \in [S_h]^m.$$

From Theorem 3.1 we know that

$$\|D_t^j(\mathbf{u} - \mathbf{w}_h)(t)\| \leq Ch^2 \|D_t^j \mathbf{u}(t)\|_3, \ t \in [0,T), \ j = 0,1. \tag{3.29}$$

Now we write the error function

$$\mathbf{u}(t) - \mathbf{u}_h(t) = \mathbf{u}(t) - \mathbf{w}_h(t) + \mathbf{w}_h(t) - \mathbf{u}_h(t) = \eta + \theta.$$

Then, from the equations satisfied by $\mathbf{u}(t)$, $\mathbf{u}_h(t)$ and $\mathbf{w}_h(t)$, we see that $\theta \in [S_h]^m$ satisfies

$$(\theta_t, \mathbf{v}_h) + A(\theta, \mathbf{v}_h) = -(\eta_t, \mathbf{v}_h), \ \forall \mathbf{v}_h \in [S_h]^m. \tag{3.30}$$

Taking $\mathbf{v}_h = \theta$, similar to the argument of (3.28), and using triangular inequality and (3.29), the proof is completed.

Remark 3.3. For nonsteady problems, the condition (3.4) is not necessary for error analysis. In fact, we may use the transformation: $\mathbf{u} = e^{\sigma t}\mathbf{w}$ with σ satisfying $\sigma > \|B + B^T - div\, \mathbf{A}\|_\infty$, so that (3.4) holds.

4. Linearized Shallow Water Systems

Many physical problems in meteorology and oceanography can be expressed in the mathematical model of shallow water systems. In this paper we consider the linearized and symmetrized shallow water equations in the form [9]

$$\mathbf{w}_t + A_1 \partial_x \mathbf{w} + A_2 \partial_y \mathbf{w} + B\mathbf{w} = \mathbf{0}, \ t > 0, \ (x,y) \in \Omega. \tag{4.1}$$

Where $\Omega = [0,a] \times [0,b]$ is a rectangular domain in R^2, $\mathbf{w} = (u, v, \varphi)$ is the vector value function, u/φ_0 and v/φ_0 represent the airflow velocities in x and y directions respectively, $\varphi_0^2 = gH$, g is the acceleration of gravity, H is the average value of the thickness of air layer, φ is a function relating to the thickness of air layer. Furthermore, constant matrices

$$A_1 = \begin{pmatrix} u_0 & 0 & \varphi_0 \\ 0 & u_0 & 0 \\ \varphi_0 & 0 & u_0 \end{pmatrix}, \ A_2 = \begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & \varphi_0 \\ 0 & \varphi_0 & v_0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where u_0 and $v_0 > 0$ are the average values of the airflow velocities, and $u_0^2 + v_0^2 < \varphi_0^2$.

Now we introduce the initial-boundary value conditions. In order to ensure the well-posedness of equations (4.1), the boundary value conditions should be given reasonably according to the characteristic directions of matrices A_1 and A_2. Here we consider equations (4.1) with the following boundary value conditions:

$$\mathbf{w}(t,0,y) = \mathbf{w}(t,a,y), \ \ 0 \leq y \leq b, \ t > 0, \tag{4.2}$$

$$v = 0, \ \ y = b, \ 0 \leq x \leq a, \ t > 0, \tag{4.3}$$

$$u = 0, \ v = -\alpha\varphi, \ \ v_0/\varphi_0 < \alpha < \varphi_0/v_0, \ \ y = 0, \ 0 \leq x \leq a, \ t > 0. \tag{4.4}$$

and the initial value condition $\mathbf{w}(0,x,y) = \mathbf{w}_0(x,y)$. Let $D = A_1 n_x + A_2 n_y$, $n = (n_x, n_y)$ is the outward unit vector on $\partial\Omega$. For function $\mathbf{w}$ satisfying (4.2)-(4.4), we have

$$< D\mathbf{w}, \mathbf{w} >_{\partial\Omega} = \int_0^a (v_0 u^2 + v_0 v^2 + 2\varphi_0 \varphi v + v_0 \varphi^2)|_{y=0}^{y=b} dx \geq 0.$$

Hence, for the solution of problem (4.1)-(4.4), we can derive the stability estimate

$$\|\mathbf{w}(t)\| \leq \|\mathbf{w}(0)\|, \ t > 0. \tag{4.5}$$

This shows that problem (4.1)-(4.4) is well-posed. Introduce the periodic function space

$$[\,H_p^1\,]^3 = \{\,\mathbf{w} \in [H^1(\Omega)]^3 : \mathbf{w}(0,y) = \mathbf{w}(a,y),\, 0 \le y \le b\,\}.$$

Then problem (4.1)-(4.4) can be rewritten as follows: Find $\mathbf{w}(t) : [0,\infty) \to [\,H_p^1\,]^3$ such that

$$\begin{aligned} \mathbf{w}_t + A_1\partial_x\mathbf{w} + A_2\partial_y\mathbf{w} + B\mathbf{w} &= \mathbf{0},\ t>0,\,(x,y)\in\Omega, &(4.6)\\ N\mathbf{w} = \frac{1}{2}(M-D)\mathbf{w} &= \mathbf{0},\ t>0,\,(x,y)\in\partial\Omega. &(4.7) \end{aligned}$$

in which N is the boundary value matrix corresponding to the boundary value conditions (4.2)-(4.4). In fact, we can set $N \equiv \mathbf{O}$, $(x,y) \in \Gamma_2 \cup \Gamma_4$, and

$$N = \frac{1}{2}\begin{pmatrix} 2v_0 & 0 & 0 \\ 0 & \varphi_0/\alpha & \varphi_0 \\ 0 & \varphi_0 & \alpha\varphi_0 \end{pmatrix},\ (x,y)\in\Gamma_1;\ N = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\varphi_0^2/v_0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ (x,y)\in\Gamma_3.$$

Where $\partial\Omega = \bigcup_{i=1}^4 \Gamma_i$, and

$$\begin{aligned} \Gamma_1 &= \{\,(x,y)\in\partial\Omega : y=0\,\},\ \Gamma_2 = \{\,(x,y)\in\partial\Omega : x=a\,\},\\ \Gamma_3 &= \{\,(x,y)\in\partial\Omega : y=b\,\},\ \Gamma_4 = \{\,(x,y)\in\partial\Omega : x=0\,\}. \end{aligned}$$

By straightforward calculation, it is easy to see that $M = 2N + D$ satisfies

$$\begin{aligned} &< M\mathbf{w},\mathbf{v} >_{\Gamma_2\cup\Gamma_4} = 0,\ \forall\,\mathbf{w},\mathbf{v}\in[\,H_p^1]^3, &(4.8)\\ &< M\mathbf{w},\mathbf{w} >_{\Gamma_1\cup\Gamma_3} \ge \sigma_1 < \mathbf{w},\mathbf{w} >_{\Gamma_1\cup\Gamma_3}. &(4.9) \end{aligned}$$

where

$$\sigma_1 = \min\{\,v_0/2,\ \alpha\varphi_0 - v_0,\ \varphi_0/\alpha - v_0\,\} > 0.$$

This implies that the hypothesis (H) in Section 3.1 holds for problem (4.6)-(4.7), because by (4.8) we may neglect the boundary value on $\Gamma_2 \cup \Gamma_4$.

Set the finite element space $[S_{h,p}]^3 = [S_h]^3 \cap [H_p^1]^3$. Now the finite element approximation to problem (4.6)-(4.7) reads: Find $\mathbf{w}_h(t) : [0,\infty) \to [S_{h,p}]^3$ such that

$$\begin{aligned} &(\mathbf{w}_{h,t},\mathbf{v}_h) + A(\mathbf{w}_h,\mathbf{v}_h) = 0,\ \mathbf{v}_h \in [S_{h,p}]^3,\ t>0, &(4.10)\\ &\mathbf{w}(0) \in [S_{h,p}]^3. &(4.11) \end{aligned}$$

According to Theorem 3.3, we immediately obtain the following theorem.

Theorem 4.1. *Let* $\mathbf{w}$ *and* $\mathbf{w}_h$ *be the solutions of problems (4.6)-(4.7) and (4.10)-(4.11) respectively,* $\mathbf{w}(0) \in [H^3(\Omega)]^m$, $\mathbf{w}_t(t) \in L_1(0,T;[H^3(\Omega)]^m)$*, triangulation* J_h *be strongly regular. Then*

$$\|\mathbf{w}(t) - \mathbf{w}_h(t)\| \le \|\mathbf{w}(0) - \mathbf{w}_h(0)\| + Ch^2(\|\mathbf{w}(0)\|_3 + \int_0^t \|\mathbf{w}_t(\tau)\|_3\,d\tau\,),\ t>0.$$

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