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LEAST-SQUARES MIXED FINITE ELEMENT METHODS FOR THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS *

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Abstract

Least-squares mixed finite element methods are proposed and analyzed for the incompressible magnetohydrodynamic equations, where the two vorticities are additionally introduced as independent variables in order that the primal equations are transformed into the first-order systems. We show that there hold the coerciveness and the optimal error bound in appropriate norms for all variables under consideration, which can be approximated by all kinds of continuous element. Consequently, the Babuška-Brezzi condition (i.e. the inf-sup condition) and the indefiniteness are avoided which are essential features of the classical mixed methods.

Mathematics subject classification: 65N30.
Key words: The incompressible magnetohydrodynamic equation, Vorticity, Least-squares mixed finite element method.

1. Introduction

Many problems involve systems of partial differential equations in several variables, variational problems derived in a standard manner often correspond to saddle-point optimization problem. It is now well understood that the finite element spaces approximating different physical quantities can not be chosen independently and have to satisfy the inf-sup condition.

In recent years there has been significant interest in least-squares methods, considered as an alternative to the saddle point formulations and circumventing the inf-sup condition. Many studies have already be devoted to the least-squares method, for theoretical and numerical results, let us just mention those by Z.Cai, T.Manteuffel and S.McCormick [4, 5, 6], A.I.Pehlivanov, G.F.Carey and R.D.Lazarov [15], Dan-Ping Yang [19, 20], B.N.Jiang [12,13] and Huo-yuan Duan [9]. The stationary incompressible magnetohydrodynamics(MHD) we consider here results from a coupling between the stationary incompressible Navier-Stokes equations and the stationary Maxwell equations. It governs the behavior of an incompressible fluid carrying an electrical current in presence of a magnetic field. We study the linear MHD equations (cf. [10]) using the least-squares mixed finite element method.

The paper is organized as follows: In section 2, we introduce some notations, Hilbert spaces and inequalities. Section 3 is concerned with the presentation of the equations and the derivation of the least-squares formulation of the linear MHD equations and its coerciveness. In section 4, we give the finite element approximation and obtain the basic error bounds.

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2. The Preparations

First we recall some notations. Let $\Omega \subset R^3$ is an open bounded domain with boundary $\Gamma = \partial\Omega$, $\mathbf{n}$ is unit normal vector to Γ. We introduce the following Sobolev spaces

$$
\begin{aligned}
L^2(\Omega) &= \{v; \int_\Omega v^2 < \infty\}, \\
L_0^2(\Omega) &= \{q \in L^2(\Omega); \int_\Omega q = 0\}, \\
H^m(\Omega) &= \{\partial^\gamma v \in L^2(\Omega),\ 0 \le |\gamma| \le m\},\ (m \ge 1), \\
H_0^1(\Omega) &= \{v \in L^2(\Omega);\ \nabla v \in (L^2(\Omega))^3,\ v_{|\Gamma} = 0\}, \\
H(\mathbf{curl};\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3;\ \mathbf{curl}\ \mathbf{u} \in (L^2(\Omega))^3\}, \\
H(\mathrm{div};\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3;\ \mathrm{div}\ \mathbf{u} \in L^2(\Omega)\}, \\
H_0(\mathbf{curl};\Omega) &= \{\mathbf{u} \in H(\mathbf{curl};\Omega);\ \mathbf{u} \times \mathbf{n}_{|\Gamma} = \mathbf{0}\}, \\
H_0(\mathrm{div};\Omega) &= \{\mathbf{u} \in H(\mathrm{div};\Omega);\ \mathbf{u} \cdot \mathbf{n}_{|\Gamma} = 0\}.
\end{aligned}
$$

Two Green's formulae of integration by parts (cf. [11]), two equalities and one inequality are as follows:

$$
\begin{aligned}
(\mathbf{u}, \nabla\phi) + (\mathrm{div}\ \mathbf{u}, \phi) &= <\mathbf{u}\cdot\mathbf{n},\ \phi>_{|\Gamma} \quad \forall\ \mathbf{u} \in H(\mathrm{div};\Omega), \quad \forall\ \phi \in H^1(\Omega). && (1)\\
(\mathbf{curl}\ \mathbf{u}, \mathbf{v}) - (\mathbf{u}, \mathbf{curl}\ \mathbf{v}) &= <\mathbf{u}\times\mathbf{n},\ \mathbf{v}>_\Gamma \quad \forall\ \mathbf{u} \in H(\mathbf{curl};\Omega), \ \forall\ \mathbf{v} \in (H^1(\Omega))^3. && (2)\\
(\mathbf{a}\cdot\nabla\mathbf{u}, \mathbf{u}) + \frac{1}{2}(\mathrm{div}\ \mathbf{a}, |\mathbf{u}|^2) &= 0 \quad \forall\ \mathbf{u} \in (H_0^1(\Omega))^3. && (3)\\
(\mathbf{b}\times\mathbf{curl}\ \mathbf{B}, \mathbf{u}) &= (\mathbf{u}\times\mathbf{b}, \mathbf{curl}\ \mathbf{B})\ \forall\ \mathbf{b}, \mathbf{u} \in (L^2(\Omega))^3, \ \forall\ \mathbf{B} \in H(\mathbf{curl}, \Omega). && (4)\\
||\mathbf{a}\times\mathbf{b}||_0^2 &\le C||\mathbf{a}||_{0,\infty}^2 ||\mathbf{b}||_0^2 \quad \forall\ \mathbf{a}, \mathbf{b} \in (L^2(\Omega))^3. && (5)
\end{aligned}
$$

Where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$.

Proposition 2.1 (cf. [2,11,14,18]). *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with a Lipschitz continuous boundary Γ. Then*

$$||\mathbf{u}||_0 \le C\{||\mathbf{curl}\ \mathbf{u}||_0 + ||\mathrm{div}\ \mathbf{u}||_0\} \ \ \forall\ \mathbf{u} \in H_0(\mathrm{div};\Omega) \cap H(\mathbf{curl};\Omega). \tag{6}$$

Proposition 2.2 (cf. [11,8]). *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with $C^{1,1}$ boundary Γ, or is a bounded and convex polyhedron. Then*

$$|\mathbf{u}|_1 \le C\{||\mathbf{curl}\ \mathbf{u}||_0 + ||\mathrm{div}\ \mathbf{u}||_0\} \ \ \forall\ \mathbf{u} \in H_0(\mathrm{div};\Omega) \cap (H^1(\Omega))^3. \tag{7}$$

3. The Least-squares Formulation of the Linear MHD Equations

Now let us consider the linear MHD equations. The linear MHD equations are as follows:

$$\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}-\nu\triangle\mathbf{u}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}=\mathbf{f}\quad in\ \Omega, \tag{8}$$

$$\mathrm{div}\ \mathbf{u}=0\quad in\ \Omega, \tag{9}$$

$$\kappa\mathbf{curl}\ \mathbf{curl}\ \mathbf{B}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=\mathbf{0}\quad in\ \Omega, \tag{10}$$

$$\mathrm{div}\ \mathbf{B}=0\quad in\ \Omega, \tag{11}$$

$$\mathbf{u}=\mathbf{0}\quad on\ \Gamma, \tag{12}$$

$$\mathbf{B}\cdot\mathbf{n}=0\quad on\ \Gamma, \tag{13}$$

$$\mathbf{curl}\ \mathbf{B}\times\mathbf{n}=\mathbf{0}\quad on\ \Gamma, \tag{14}$$

where the unknown variables are the velocity field $\mathbf{u}$, the pressure p in the fluid and the magnetic induction $\mathbf{B}$. $\mathbf{f}$ denotes an external force, $\nu=1/Re$ with Re the Renolds number, $\kappa=1/Rm$ with Rm the magnetic Renolds number, ρ the coupling number. $\mathbf{a}$ and $\mathbf{b}$ are known functions (usually stand for the approximate solutions for $\mathbf{u}$ and $\mathbf{B}$ in the previous iterative step of the Picard iterations and supposed regular).

Introducing

$$\mathbf{w}=\mathbf{curl}\ \mathbf{u},\quad \mathbf{z}=\mathbf{curl}\ \mathbf{B}$$

as independent variables, we may recast problem (8)-(14) as a first-order system as follows:

$$\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}=\mathbf{f}\quad in\ \Omega, \tag{15}$$

$$\mathrm{div}\ \mathbf{u}=0\quad in\ \Omega, \tag{16}$$

$$\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=\mathbf{0}\quad in\ \Omega, \tag{17}$$

$$\mathrm{div}\ \mathbf{B}=0\quad in\ \Omega, \tag{18}$$

$$\mathbf{w}=\mathbf{curl}\ \mathbf{u}\quad in\ \Omega, \tag{19}$$

$$\mathbf{z}=\mathbf{curl}\ \mathbf{B}\quad in\ \Omega, \tag{20}$$

$$\mathbf{u}=\mathbf{0}\quad on\ \Gamma, \tag{21}$$

$$\mathbf{B}\cdot\mathbf{n}=0\quad on\ \Gamma, \tag{22}$$

$$\mathbf{z}\times\mathbf{n}=\mathbf{0}\quad on\ \Gamma, \tag{23}$$

$$\mathbf{w}\cdot\mathbf{n}=0\quad on\ \Gamma. \tag{24}$$

The least-squares functional $J(\mathbf{v},\mathbf{D},q,\mathbf{x},\mathbf{y})$ for the mixed system (15)-(24) is

$$\begin{aligned} J(\mathbf{v},\mathbf{D},q,\mathbf{x},\mathbf{y}) \quad = \quad & (\mathbf{a}\cdot\nabla\mathbf{v}+\frac{1}{2}\mathbf{v}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{x}+\nabla q+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{D}-\mathbf{f},\\ & \mathbf{a}\cdot\nabla\mathbf{v}+\frac{1}{2}\mathbf{v}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{x}+\nabla q+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{D}-\mathbf{f})\\ & +(\kappa\mathbf{curl}\ \mathbf{y}-\mathbf{curl}\ (\mathbf{v}\times\mathbf{b}),\kappa\mathbf{curl}\ \mathbf{y}-\mathbf{curl}\ (\mathbf{v}\times\mathbf{b}))\\ & +(\mathrm{div}\ \mathbf{v},\mathrm{div}\ \mathbf{v})+(\mathbf{y}-\mathbf{curl}\ \mathbf{D},\mathbf{y}-\mathbf{curl}\ \mathbf{D})\\ & +(\mathbf{x}-\mathbf{curl}\ \mathbf{v},\mathbf{x}-\mathbf{curl}\ \mathbf{v})+(\mathrm{div}\ \mathbf{D},\mathrm{div}\ \mathbf{D}). \end{aligned} \tag{25}$$

Define spaces

$$\mathbf{U}=(H_0^1(\Omega))^3,\ \ \mathbf{W}=H_0(\mathrm{div};\Omega)\cap H(\mathbf{curl};\Omega),\ \ \mathbf{Q}=H^1(\Omega)/R,\ \ \mathbf{Z}=H_0(\mathbf{curl};\Omega)\cap H(\mathrm{div};\Omega).$$

Then the least-squares minimization problem is : find $\mathbf{u} \in \mathbf{U},\ \mathbf{B} \in \mathbf{W},\ p \in \mathbf{Q},\ \mathbf{w} \in \mathbf{W},\ \mathbf{z} \in \mathbf{Z}$ such that

$$J(\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}) = \inf J(\mathbf{v}, \mathbf{D}, q, \mathbf{x}, \mathbf{y})$$

for all $\mathbf{v} \in \mathbf{U},\ \mathbf{D} \in \mathbf{W},\ q \in \mathbf{Q},\ \mathbf{x} \in \mathbf{W},\ \mathbf{y} \in \mathbf{Z}$. Taking variations in (25) with respect to $\mathbf{v}$, $\mathbf{D}$, q, $\mathbf{x}$ and $\mathbf{y}$, the weak statement becomes: find $\mathbf{u} \in \mathbf{U},\ \mathbf{B} \in \mathbf{W},\ p \in \mathbf{Q},\ \mathbf{w} \in \mathbf{W},\ \mathbf{z} \in \mathbf{Z}$ such that

$$A((\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}); (\mathbf{v}, \mathbf{D}, q, \mathbf{x}, \mathbf{y})) = (\mathbf{f}, \mathbf{a} \cdot \nabla \mathbf{v} + \frac{1}{2}\mathbf{v}\ \mathrm{div}\mathbf{a} + \nu \mathbf{curl}\ \mathbf{x} + \nabla q + \rho \mathbf{b} \times \mathbf{curl}\ \mathbf{D}) \quad (26)$$

holds for all $\mathbf{v} \in \mathbf{U},\ \mathbf{D} \in \mathbf{W},\ q \in \mathbf{Q},\ \mathbf{x} \in \mathbf{W},\ \mathbf{y} \in \mathbf{Z}$. Where

$$\begin{aligned}
&A((\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}); (\mathbf{v}, \mathbf{D}, q, \mathbf{x}, \mathbf{y})) \\
&= (\mathbf{a} \cdot \nabla \mathbf{u} + \frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a} + \nu \mathbf{curl}\ \mathbf{w} + \nabla p + \rho \mathbf{b} \times \mathbf{curl}\ \mathbf{B}, \\
&\mathbf{a} \cdot \nabla \mathbf{v} + \frac{1}{2}\mathbf{v}\ \mathrm{div}\mathbf{a} + \nu \mathbf{curl}\ \mathbf{x} + \nabla q + \rho \mathbf{b} \times \mathbf{curl}\ \mathbf{D}) \\
&+(\kappa \mathbf{curl}\ \mathbf{z} - \mathbf{curl}\ (\mathbf{u} \times \mathbf{b}), \kappa \mathbf{curl}\ \mathbf{y} - \mathbf{curl}\ (\mathbf{v} \times \mathbf{b})) \\
&+(\mathrm{div}\ \mathbf{u}, \mathrm{div}\ \mathbf{v}) + (\mathbf{z} - \mathbf{curl}\ \mathbf{B}, \mathbf{y} - \mathbf{curl}\ \mathbf{D}) \\
&+(\mathbf{w} - \mathbf{curl}\ \mathbf{u}, \mathbf{x} - \mathbf{curl}\ \mathbf{v}) + (\mathrm{div}\ \mathbf{B}, \mathrm{div}\ \mathbf{D}).
\end{aligned}$$

Now we investigate the coerciveness of the bilinear form $A((\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}); (\mathbf{v}, \mathbf{D}, q, \mathbf{x}, \mathbf{y}))$.

Theorem 3.1. *Under the same conditions as in Proposition 2.2, there exists a constant* $C > 0$ *such that*

$$A((\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}); (\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z})) \geq C\{\|\mathbf{u}\|_1^2 + \|\mathbf{B}\|_1^2 + \|p\|_0^2 + \|\mathbf{w}\|_0^2 + \|\mathbf{z}\|_0^2\}.$$

Proof. It is obvious that

$$\begin{aligned}
&A((\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z}); (\mathbf{u}, \mathbf{B}, p, \mathbf{w}, \mathbf{z})) \\
&= ||\mathbf{a} \cdot \nabla \mathbf{u} + \frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a} + \nu \mathbf{curl}\ \mathbf{w} + \nabla p + \rho \mathbf{b} \times \mathbf{curl}\ \mathbf{B}||_0^2 \\
&+||\kappa \mathbf{curl}\ \mathbf{z} - \mathbf{curl}\ (\mathbf{u} \times \mathbf{b})||_0^2 \\
&+\|\mathrm{div}\ \mathbf{u}\|_0^2 + \|\mathrm{div}\ \mathbf{B}\|_0^2 + \|\mathbf{w} - \mathbf{curl}\ \mathbf{u}\|_0^2 + \|\mathbf{z} - \mathbf{curl}\ \mathbf{B}\|_0^2.
\end{aligned}$$

Let $\alpha > 0$ be a constant to be determined, then

$$\begin{aligned}
\|\mathbf{w} - \mathbf{curl}\ \mathbf{u}\|_0^2 &= \frac{1}{2}\{\|\mathbf{w} - \mathbf{curl}\ \mathbf{u} + \alpha \mathbf{curl}\ \mathbf{u}\|_0^2 + \|\mathbf{w} - \mathbf{curl}\ \mathbf{u} - \alpha \mathbf{w}\|_0^2\} \\
&\quad + \alpha(1 - \frac{\alpha}{2})\{\|\mathbf{curl}\ \mathbf{u}\|_0^2 + \|\mathbf{w}\|_0^2\} - 2\alpha(\mathbf{curl}\ \mathbf{u}, \mathbf{w}) \\
&\geq \alpha(1 - \frac{\alpha}{2})\{\|\mathbf{curl}\ \mathbf{u}\|_0^2 + \|\mathbf{w}\|_0^2\} - 2\alpha(\mathbf{curl}\ \mathbf{u}, \mathbf{w}). \qquad (27)
\end{aligned}$$

With Green's formula (2) and equality (3), we can get

$$\begin{aligned}
&||\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}||_0^2\\
&=||\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}-\frac{\alpha}{\nu}\mathbf{u}||_0^2\\
&-\frac{\alpha^2}{\nu^2}\|\mathbf{u}\|_0^2+\frac{2\alpha}{\nu}(\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\mathrm{div}\mathbf{a},\mathbf{u})+2\alpha(\mathbf{curl}\ \mathbf{w},\mathbf{u})\\
&+\frac{2\alpha}{\nu}(\nabla p,\mathbf{u})+\frac{2\alpha\rho}{\nu}(\mathbf{b}\times\mathbf{curl}\ \mathbf{B},\mathbf{u})\\
&\geq-\frac{\alpha^2}{\nu^2}\|\mathbf{u}\|_0^2+2\alpha(\mathbf{curl}\ \mathbf{u},\mathbf{w})+\frac{2\alpha}{\nu}(\nabla p,\mathbf{u})+\frac{2\alpha\rho}{\nu}(\mathbf{b}\times\mathbf{curl}\ \mathbf{B},\mathbf{u}).
\end{aligned}\tag{28}$$

In a similar way, we have

$$\begin{aligned}
&\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})\|_0^2\\
&=\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})-\frac{\alpha\rho}{\nu}\mathbf{B}\|_0^2\\
&-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2+\frac{2\rho\kappa\alpha}{\nu}(\mathbf{curl}\ \mathbf{z},\mathbf{B})-\frac{2\rho\alpha}{\nu}(\mathbf{curl}\ (\mathbf{u}\times\mathbf{b}),\mathbf{B})\\
&\geq-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2+\frac{2\rho\kappa\alpha}{\nu}(\mathbf{curl}\ \mathbf{z},\mathbf{B})-\frac{2\rho\alpha}{\nu}(\mathbf{curl}\ (\mathbf{u}\times\mathbf{b}),\mathbf{B}).
\end{aligned}$$

Since $\mathbf{u}_{|\Gamma}=\mathbf{0}$, together with Green's formula (2) and equality (4) we can get

$$\begin{aligned}
&\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})\|_0^2\\
&\geq-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2+\frac{2\rho\kappa\alpha}{\nu}(\mathbf{curl}\ \mathbf{z},\mathbf{B})-\frac{2\rho\alpha}{\nu}(\mathbf{b}\times\mathbf{curl}\ \mathbf{B},\mathbf{u}).
\end{aligned}\tag{29}$$

At the same time,

$$\begin{aligned}
&\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}\|_0^2\\
&=\frac{1}{2}\{\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}+\frac{\rho\kappa\alpha}{\nu}\mathbf{curl}\ \mathbf{B}\|_0^2+\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}-\frac{\rho\kappa\alpha}{\nu}\mathbf{z}\|_0^2\}\\
&+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\{\|\mathbf{z}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2\}-\frac{2\rho\kappa\alpha}{\nu}(\mathbf{z},\mathbf{curl}\ \mathbf{B})\\
&\geq\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\{\|\mathbf{z}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2\}-\frac{2\rho\kappa\alpha}{\nu}(\mathbf{curl}\ \mathbf{z},\mathbf{B}).
\end{aligned}\tag{30}$$

In view of (27), (28), (29), (30) and Green's formula (1), we get

$$
\begin{aligned}
&\|\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}\|_0^2\\
&+\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})\|_0^2+\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}\|_0^2+\|\mathbf{w}-\mathbf{curl}\ \mathbf{u}\|_0^2\\
&\geq\alpha(1-\frac{\alpha}{2})\{\|\mathbf{curl}\ \mathbf{u}\|_0^2+\|\mathbf{w}\|_0^2\}-\frac{\alpha^2}{\nu^2}\|\mathbf{u}\|_0^2-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2\\
&+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\{\|\mathbf{z}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2\}+\frac{2\alpha}{\nu}(\mathbf{u},\nabla p)\\
&=\alpha(1-\frac{\alpha}{2})\{\|\mathbf{curl}\ \mathbf{u}\|_0^2+\|\mathbf{w}\|_0^2\}-\frac{\alpha^2}{\nu^2}\|\mathbf{u}\|_0^2-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2\\
&+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\{\|\mathbf{z}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2\}-\frac{2\alpha}{\nu}(\mathrm{div}\mathbf{u},p)\\
&\geq\alpha(1-\frac{\alpha}{2})\{\|\mathbf{curl}\ \mathbf{u}\|_0^2+\|\mathbf{w}\|_0^2\}-\frac{\alpha^2}{\nu^2}\|\mathbf{u}\|_0^2-\frac{\alpha^2\rho^2}{\nu^2}\|\mathbf{B}\|_0^2\\
&+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\{\|\mathbf{z}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2\}-\frac{\alpha^2}{\nu^2\varepsilon_0}\|\mathrm{div}\mathbf{u}\|_0^2-\varepsilon_0\|p\|_0^2. \qquad (31)
\end{aligned}
$$

Since $p\in H^1(\Omega)/R$, there exist a $\mathbf{p}^*\in(H_0^1(\Omega))^3$(cf.[3]), such that

$$
(\mathrm{div}\mathbf{p}^*,p)=\|p\|_0^2,\quad \|\mathbf{p}^*\|_1\leq C\|p\|_0.
$$

Let $\beta>0$ be a constant to be determined, with Green's formulae (1), (2), equalities (3), (4) and inequality (5) we can have

$$
\begin{aligned}
&||\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}||_0^2\\
&=||\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}+\beta\mathbf{p}^*||_0^2\\
&-\beta^2\|\mathbf{p}^*\|_0^2-2\beta(\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a},\mathbf{p}^*)-2\beta\nu(\mathbf{curl}\ \mathbf{w},\mathbf{p}^*)\\
&-2\beta(\nabla p,\mathbf{p}^*)-2\beta\rho(\mathbf{b}\times\mathbf{curl}\ \mathbf{B},\mathbf{p}^*)\\
&\geq-\beta^2\|\mathbf{p}^*\|_1^2-2\beta(\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a},\mathbf{p}^*)-2\beta\nu(\mathbf{curl}\ \mathbf{w},\mathbf{p}^*)\\
&+2\beta(p,\mathrm{div}\mathbf{p}^*)-2\beta\rho(\mathbf{b}\times\mathbf{curl}\ \mathbf{B},\mathbf{p}^*)\\
&\geq\beta(2-C\beta)\|p\|_0^2-\frac{\beta^2}{\varepsilon_1}\|\mathbf{p}^*\|_0^2-\varepsilon_1\|\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}\|_0^2\\
&-\frac{\beta^2\nu^2}{\varepsilon_2}\|\mathbf{curl}\ \mathbf{p}^*\|_0^2-\varepsilon_2\|\mathbf{w}\|_0^2-\frac{\beta^2\rho^2}{\varepsilon_3}\|\mathbf{p}^*\|_0^2-\varepsilon_3\|\mathbf{b}\times\mathbf{curl}\ \mathbf{B}\|_0^2\\
&\geq(2\beta-C\beta^2-\frac{C\beta^2}{\varepsilon_1}-\frac{C_2\beta^2\nu^2}{\varepsilon_2}-\frac{C\rho^2\beta^2}{\varepsilon_3})\|p\|_0^2-C_1\varepsilon_1\|\mathbf{a}\|_{1,\infty}^2\|\mathbf{u}\|_1^2\\
&-\varepsilon_2\|\mathbf{w}\|_0^2-C_3\varepsilon_3\|\mathbf{b}\|_{0,\infty}^2\|\mathbf{curl}\ \mathbf{B}\|_0^2. \qquad (32)
\end{aligned}
$$

Let δ, γ be two positive constants to be determined, with Proposition 2.1 and 2.2, we have

$$
\begin{aligned}
&2\|\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}\|_0^2\\
&+\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})\|_0^2+\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}\|_0^2+\|\mathbf{w}-\mathbf{curl}\ \mathbf{u}\|_0^2\\
&+\delta\|\mathrm{div}\ \mathbf{u}\|_0^2+\gamma\|\mathrm{div}\ \mathbf{B}\|_0^2\\
&\geq\alpha(1-\frac{\alpha}{2})\|\mathbf{curl}\ \mathbf{u}\|_0^2+\{\alpha(1-\frac{\alpha}{2})-\varepsilon_2\}\|\mathbf{w}\|_0^2\\
&+(2\beta-C\beta^2-\frac{C\beta^2}{\varepsilon_1}-\frac{C_2\beta^2\nu^2}{\varepsilon_2}-\frac{C\rho^2\beta^2}{\varepsilon_3}-\varepsilon_0)\|p\|_0^2\\
&-\{C_4\frac{\alpha^2}{\nu^2}+C_1\varepsilon_1\|\mathbf{a}\|_{1,\infty}^2\}\{\|\mathbf{curl}\ \mathbf{u}\|_0^2+\|\mathrm{div}\ \mathbf{u}\|_0^2\}-\frac{\alpha^2}{\varepsilon_0\nu^2}\|\mathrm{div}\ \mathbf{u}\|_0^2\\
&+\{\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})-C_3\varepsilon_3|\mathbf{b}\|_{0,\infty}^2-C_5\frac{\alpha^2\rho^2}{\nu^2}\}\|\mathbf{curl}\ \mathbf{B}\|_0^2\\
&-C_5\frac{\alpha^2\rho^2}{\nu^2}\|\mathrm{div}\ \mathbf{B}\|_0^2+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\|\mathbf{z}\|_0^2+\delta\|\mathrm{div}\ \mathbf{u}\|_0^2+\gamma\|\mathrm{div}\ \mathbf{B}\|_0^2\\
&=\{\alpha[1-\alpha(\frac{1}{2}+\frac{C_4}{\nu^2})]-C_1\varepsilon_1\|\mathbf{a}\|_{1,\infty}^2\}\|\mathbf{curl}\ \mathbf{u}\|_0^2\\
&+\{\delta-C_4\frac{\alpha^2}{\nu^2}-C_1\varepsilon_1\|\mathbf{a}\|_{1,\infty}^2-\frac{\alpha^2}{\varepsilon_0\nu^2}\}\|\mathrm{div}\ \mathbf{u}\|_0^2\\
&+\{\frac{\rho\alpha}{\nu}[\kappa-\frac{\rho\alpha}{\nu}(\frac{\kappa^2}{2}+C_5)]-C_3\varepsilon_3|\mathbf{b}\|_{0,\infty}^2\}\|\mathbf{curl}\ \mathbf{B}\|_0^2\\
&+\{\gamma-C_5\frac{\alpha^2\rho^2}{\nu^2}\}\|\mathrm{div}\ \mathbf{B}\|_0^2+\{\alpha(1-\frac{\alpha}{2})-\varepsilon_2\}\|\mathbf{w}\|_0^2\\
&+\{\beta[2-\beta(C+\frac{C}{\varepsilon_1}+C_2\frac{\nu^2}{\varepsilon_2}+C\frac{\rho^2}{\varepsilon_3})]-\varepsilon_0\}\|p\|_0^2+\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})\|\mathbf{z}\|_0^2.
\end{aligned}
$$

Taking

$$
\begin{aligned}
0<\alpha&<\min\{\frac{2\nu^2}{\nu^2+2C_4}\ ,\frac{\kappa\nu}{\rho(\frac{\kappa^2}{2}+C_5)}\ ,2\ ,\frac{2\nu}{\rho\kappa}\}\\
&=\min\{\frac{2\nu^2}{\nu^2+2C_4}\ ,\frac{2\kappa\nu}{\rho(\kappa^2+2C_5)}\ ,\frac{2\nu}{\rho\kappa}\},
\end{aligned}
$$

$$0<\varepsilon_1<\frac{\alpha[1-\alpha(\frac{1}{2}+\frac{C_4}{\nu^2})]}{C_1\|\mathbf{a}\|_{1,\infty}^2}=\frac{2\alpha\nu^2-\alpha^2\nu^2-2C_4\alpha^2}{2C_1\nu^2\|\mathbf{a}\|_{1,\infty}^2},$$

$$0<\varepsilon_3<\frac{1}{C_3\|\mathbf{b}\|_{0,\infty}^2}\{\frac{\rho\kappa\alpha}{\nu}(1-\frac{\rho\kappa\alpha}{2\nu})-C_5\frac{\alpha^2\rho^2}{\nu^2}\},$$

$$\gamma>C_5\frac{\alpha^2\rho^2}{\nu^2},$$

$$0<\varepsilon_2<\alpha(1-\frac{\alpha}{2}),$$

$$0<\beta<\frac{2}{C+C/\varepsilon_1+C_2\nu^2/\varepsilon_2+C\rho^2/\varepsilon_3},$$

$$0<\varepsilon_0<2\beta-\beta^2(C+\frac{C}{\varepsilon_1}+\frac{C_2\nu^2}{\varepsilon_2}+\frac{C\rho^2}{\varepsilon_3}),$$

$$\delta>\frac{C_4\alpha^2}{\nu^2}-C_1\varepsilon_1\|\mathbf{a}\|_{1,\infty}^2-\frac{\alpha^2}{\varepsilon_0\nu^2}.$$

Therefore

$$\begin{aligned}&2||\mathbf{a}\cdot\nabla\mathbf{u}+\frac{1}{2}\mathbf{u}\mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{w}+\nabla p+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{B}||_0^2\\&+\|\kappa\mathbf{curl}\ \mathbf{z}-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})\|_0^2+\|\mathbf{z}-\mathbf{curl}\ \mathbf{B}\|_0^2+\|\mathbf{w}-\mathbf{curl}\ \mathbf{u}\|_0^2\\&+\delta\|\mathrm{div}\mathbf{u}\|_0^2+\gamma\|\mathrm{div}\mathbf{B}\|_0^2\\\geq\ &C\{\|\mathbf{curl}\ \mathbf{u}\|_0^2+\|\mathrm{div}\mathbf{u}\|_0^2+\|\mathbf{curl}\ \mathbf{B}\|_0^2+\|\mathrm{div}\mathbf{B}\|_0^2+\|p\|_0^2+\|\mathbf{w}\|_0^2+\|\mathbf{z}\|_0^2\}.\end{aligned}$$

With Proposition 2.1 and 2.2, we can obtain

$$A((\mathbf{u},\mathbf{B},p,\mathbf{w},\mathbf{z});(\mathbf{u},\mathbf{B},p,\mathbf{w},\mathbf{z}))\geq C\{\|\mathbf{u}\|_1^2+\|\mathbf{B}\|_1^2+\|p\|_0^2+\|\mathbf{w}\|_0^2+\|\mathbf{z}\|_0^2\}.$$

Theorem 3.2. *Let* $\mathbf{f}\in(L^2(\Omega))^3$*, then the problem (26) has a unique solution* $\mathbf{u}\in\mathbf{U},\ \mathbf{B}\in\mathbf{W},\ p\in\mathbf{Q},\ \mathbf{w}\in\mathbf{W},\ \mathbf{z}\in\mathbf{Z}$.

Proof. The result follows from the Lax-Milgram Lemma.

4. Finite Element Approximation and Error Estimates

Now let us consider the finite element method.

Let T_h be the regular triangulation of Ω into tetrahedrons(cf.[7]). Define

$$V_h=\{v\in H^1(\Omega);v_{|K}\in P_1(K),\forall K\in T_h\}$$

where $P_1(K)$ is the space of linear polynomials. Let $\widetilde{v}\in V_h$ be the standard interpolation to $v\in H^2(\Omega)$, from the standard interpolation theory in [1], we have

$$\|v-\widetilde{v}\|_0+h\|v-\widetilde{v}\|_1\leq Ch^2\|v\|_2. \tag{33}$$

Define

$$U_h=(V_h\cap H_0^1(\Omega))^3,\ W_h=(V_h)^3\cap H_0(\mathrm{div};\Omega),\ Q_h=V_h\cap L_0^2(\Omega),\ Z_h=(V_h)^3\cap H_0(\mathbf{curl};\Omega).$$

The finite element method to (26) is find $(\mathbf{u}_h,\ \mathbf{B}_h,\ p_h,\ \mathbf{w}_h,\ \mathbf{z}_h)\in U_h\times W_h\times Q_h\times W_h\times Z_h$ such that

$$\begin{aligned}&A((\mathbf{u}_h,\mathbf{B}_h,p_h,\mathbf{w}_h,\mathbf{z}_h);(\mathbf{v},\mathbf{D},q,\mathbf{x},\mathbf{y}))\\=\ &(\mathbf{f},\mathbf{a}\cdot\nabla\mathbf{v}+\frac{1}{2}\mathbf{v}\ \mathrm{div}\mathbf{a}+\nu\mathbf{curl}\ \mathbf{x}+\nabla q+\rho\mathbf{b}\times\mathbf{curl}\ \mathbf{D})\end{aligned} \tag{34}$$

$\forall(\mathbf{v},\ \mathbf{D},\ q,\ \mathbf{x},\ \mathbf{y})\in U_h\times W_h\times Q_h\times W_h\times Z_h$.

Theorem 4.1. *Under the same conditions as in theorem 3.1, let* $(\mathbf{u},\ \mathbf{B},\ p,\ \mathbf{w},\ \mathbf{z})$ *and* $(\mathbf{u}_h,\ \mathbf{B}_h,\ p_h,\ \mathbf{w}_h,\ \mathbf{z}_h)$ *be the solutions of the first-order system (15)-(24) and the finite element method (34) respectively. If* $(\mathbf{u},\ \mathbf{B},\ p,\ \mathbf{w},\ \mathbf{z})\in(H^2(\Omega))^3\times(H^2(\Omega))^3\times H^2(\Omega)\times(H^2(\Omega))^3\times(H^2(\Omega))^3$*, then*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_1+\|\mathbf{B}-\mathbf{B}_h\|_1+\|p-p_h\|_0+\|\mathbf{w}-\mathbf{w}_h\|_0+\|\mathbf{z}-\mathbf{z}_h\|_0\\&\leq Ch\{\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|p\|_2+\|\mathbf{w}\|_2+\|\mathbf{z}\|_2\}.\end{aligned}$$

Proof. Clearly, the error has the orthogonality property

$$\begin{aligned}&A(\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);(\mathbf{v},\ \mathbf{D},\ q,\ \mathbf{x},\ \mathbf{y}))=0,\\&\qquad\forall\ (\mathbf{v},\ \mathbf{D},\ q,\ \mathbf{x},\ \mathbf{y})\in U_h\times W_h\times Q_h\times W_h\times Z_h.\end{aligned} \tag{35}$$

Moreover, let $(\widetilde{\mathbf{u}},\ \widetilde{\mathbf{B}},\ \widetilde{p},\ \widetilde{\mathbf{w}},\ \widetilde{\mathbf{z}})\in U_h\times W_h\times Q_h\times W_h\times Z_h$ be the standard interpolations to $(\mathbf{u},\ \mathbf{B},\ p,\ \mathbf{w},\ \mathbf{z})$ respectively.

In light of the orthogonality (35) and the Schwarz inequality, we have

$$\begin{aligned}&A((\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);(\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h))\\&=A((\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);(\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}}))\\&\leq A((\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);(\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h))^{1/2}\end{aligned}$$

$$\times A((\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}});(\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}}))^{1/2}.$$

Now we have

$$\begin{aligned}&A((\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);(\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h))^{1/2}\\ &\quad\le A((\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}});(\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}}))^{1/2}.\end{aligned}$$

Therefor, with Theorem 3.1 and (33) we get

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_1+\|\mathbf{B}-\mathbf{B}_h\|_1+\|p-p_h\|_0+\|\mathbf{w}-\mathbf{w}_h\|_0+\|\mathbf{z}-\mathbf{z}_h\|_0\\ &\le CA((\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h);\\ &\quad(\mathbf{u}-\mathbf{u}_h,\ \mathbf{B}-\mathbf{B}_h,\ p-p_h,\ \mathbf{w}-\mathbf{w}_h,\ \mathbf{z}-\mathbf{z}_h))^{1/2}\\ &\le CA((\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}});(\mathbf{u}-\widetilde{\mathbf{u}},\ \mathbf{B}-\widetilde{\mathbf{B}},\ p-\widetilde{p},\ \mathbf{w}-\widetilde{\mathbf{w}},\ \mathbf{z}-\widetilde{\mathbf{z}}))^{1/2}\\ &\le C\{\|\mathbf{u}-\widetilde{\mathbf{u}}\|_1+\|\mathbf{B}-\widetilde{\mathbf{B}}\|_1+\|p-\widetilde{p}\|_1+\|\mathbf{w}-\widetilde{\mathbf{w}}\|_1+\|\mathbf{z}-\widetilde{\mathbf{z}}\|_1\}\\ &\le Ch\{\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|p\|_2+\|\mathbf{w}\|_2+\|\mathbf{z}\|_2\}.\end{aligned}$$

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