Journal of Computational Mathematics, Vol.23, No.4, 2005, 373–382.

CONVERGENCE ANALYSIS FOR A NONCONFORMING MEMBRANE ELEMENT ON ANISOTROPIC MESHES *1)

Dong-yang Shi Shao-chun Chen
(*Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China*)

Ichiro Hagiwara
(*Tokyo Institute of Technology, 2-12-1 Ohokayama, Megro, Tokyo, 152-8552, Japan*)

Abstract

Regular assumption of finite element meshes is a basic condition of most analysis of finite element approximations both for conventional conforming elements and nonconforming elements. The aim of this paper is to present a novel approach of dealing with the approximation of a four-degree nonconforming finite element for the second order elliptic problems on the anisotropic meshes. The optimal error estimates of energy norm and L^2-norm without the regular assumption or quasi-uniform assumption are obtained based on some new special features of this element discovered herein. Numerical results are given to demonstrate validity of our theoretical analysis.

Mathematics subject classification: 65N30,65N15.
Key words: Anisotropic mesh, Nonconforming finite element, Optimal estimate.

1. Introduction

It is well known that Carey's element [1] is a very famous four-degree triangle nonconforming membrane element. Numerous studies have been advocated to its convergence analysis (see [2],[3] and [4] for details). However, one of the drawbacks of the analysis of convergence of above studies is that the regular assumption of the finite element meshes should be satisfied , i.e. there exists a constant $C > 0$, such that for all element K, $h_K/\rho_K \leq C$, where h_K and ρ_K are the diameter of K and the biggest circle contained in K respectively. Therefore, this restriction limits the applications of many elements of practical problems. In practice, the solution of elliptic boundary problem may have anisotropic behavior in parts of the domain, i.e. it varies significantly only in certain direction. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with small mesh size in the direction of rapid variation of solution and a larger mesh size in the perpendicular direction.

In recent years, some researchers have been interested in the study of theoretical analysis and computations without the above regular assumption, i.e. anisotropic behavior, and paid more attention to the interpolation error estimate of conforming Lagrange type elements [5,6,10] and nonconforming C-R type element [7] with narrow edges or having anisotropic properties. In these cases, the key problem is that the usual Sobolev theories (for example, Hilbert-Bramble Lemma) can not be used directly. An example is given in [7].

In this paper, we focus on the study of convergence analysis of Carey's element with the narrow edges or anisotropic properties. The optimal error estimates are obtained by using Lagrange interpolation results for conforming elements and some new properties discovered

* Received October 16, 2003; final revised June 8, 2004.
1) This research is supported by NSF of China (NO.10371113 and NO. 10471133), Foundation of Oversea Scholar of China(NO.2001(119)), the Project of Creative Engineering of Henan Province of China and NSF of Henan Province of China.

herein. The results and the approach of this paper are also valid for some other elements, such as the Wilson element [4], the arbitrary quadrilateral quasi-Wilson element [8,9] and five-node element [12] and so on . At the same time, we also present some computational results which demonstrate the validity and coincidence of our theoretical analysis very well.

2. Carey's Element and Some Lemmas

Let K be a triangle with vertices $p_i = (x_i, y_i), 1 \le i \le 3$, and λ_i be the area coordinate corresponding to the vertices p_i, $\ell_i = \overrightarrow{p_{i+1}p_{i+2}}$, $(i = 1, 2, 3, mod(3))$ be the three sides. Let S denote the area of the triangle K and set the following remarks

$$\begin{cases} \xi_1 = x_2 - x_3, & \xi_2 = x_3 - x_1, & \xi_3 = x_1 - x_2, \\ \eta_1 = y_2 - y_3, & \eta_2 = y_3 - y_1, & \eta_3 = y_1 - y_2, \\ \xi_i^2 + \eta_i^2 = \ell_i^2, & \ell^2 = \ell_1^2 + \ell_2^2 + \ell_3^2. \end{cases}$$

Then, the shape function on element K may be found by

$$u = \sum_{i=1}^{3} u_i \lambda_i + t(u)\varphi, \tag{1}$$

where

$$\varphi = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \tag{2}$$

and u_i denotes the functional value of u at the vertex p_i (i=1,2,3) of K respectively, and the parameter $t(u)$ is taken as

$$t(u) = \frac{-4S}{\ell^2} \int_K \Delta u dx dy. \tag{3}$$

Obviously, this element is a nonconforming membrane element and it is continuous at each vertex of the element K. Let

$$\overline{u} = u_1\lambda_1 + u_2\lambda_2 + u_3\lambda_3, \quad u^1 = t(u)\varphi. \tag{4}$$

Then, $u = \bar{u} + u^1$, i.e. $\bar{u}$ and u^1 are the conforming part and nonconforming part of u respectively.

Let Ω be the polygonal domain, J_h be a family of decomposition of Ω with $\overline{\Omega} = \bigcup_{K \in J_h} \overline{K}$, and $diam(K) \le h$,$\forall$ $K \in J_h$. For a given element $K \in J_h$, let $\overrightarrow{p_1p_2}$ be the longest edge of K. Then we denote $h_1 = h_{1,K} = meas(\overrightarrow{p_1p_2})$ its length and by $h_2 = h_{2,K} = \frac{2S}{h_{1,K}}$ the thickness of K perpendicularly to $\overrightarrow{p_1p_2}$. We assume that the element satisfies the maximum angle condition and a coordinate system condition [11], but it is not necessary to satisfy the regular assumption or quasi-uniform assumptions on meshes [4]. Let F_K be an affine mapping from $\hat{K}$ to K

$$F_K : \quad \begin{cases} x = \sum_{j=1}^{3} x(p_j)\lambda_j, \\ y = \sum_{j=1}^{3} y(p_j)\lambda_j. \end{cases}$$

Let V_h be the associated Carey's finite element space.

$$V_h = \{\ v : v|_K = \hat{v} \circ F_K^{-1},\ \hat{v} \in P_K,\ v(a) = 0,\ \forall\ node\ a \in \partial\Omega\ \},$$

where $P_K = span\{\lambda_1, \lambda_2, \lambda_3, \varphi\}$ is the shape function space.

Now, let us consider the following Poisson problem

$$\begin{cases} -\triangle u = f, & in \;\; \Omega, \\ u|_\Gamma = 0, & on \;\; \Gamma = \partial\Omega. \end{cases} \tag{5}$$

Let $V = H_0^1(\Omega)$, then the weak form of (5) is

$$\begin{cases} Find\ u \in V, & such\ \ that, \\ a(u,v) = f(v), & \forall\ v \in V, \end{cases} \tag{6}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dxdy, \qquad f(v) = \int_\Omega f v dxdy.$$

The approximation of (6) reads as follows

$$\begin{cases} Find\ u_h \in V_h, & such\ \ that, \\ a_h(u_h,v_h) = f(v_h), & \forall\ v_h \in V_h, \end{cases} \tag{7}$$

where

$$a_h(u_h,v_h) = \sum_{K\in J_h} \int_K \nabla u \cdot \nabla v dxdy.$$

Define $\|\cdot\|_h = (\sum\limits_K |\cdot|_{1,K}^2)^{\frac{1}{2}}$, then it is easy to see that $\|\cdot\|_h$ is the norm over V_h . By Lax-Milgram theorem, (6) and (7) has an unique solution u and u_h respectively. In order to estimate $\|u-u_h\|_h$ and $\|u-u_h\|_0$, we first prove the following very useful lemmas.

Lemma 1. *For* $\forall\ w_h \in V_h$,

$$\int_K \frac{\partial w_h^1}{\partial x} dxdy \;\; = \;\; 0, \tag{8}$$

$$\int_K \frac{\partial w_h^1}{\partial y} dxdy \;\; = \;\; 0. \tag{9}$$

Proof. By (2) and (4),

$$w_h^1 = t(w_h)\varphi = t(w_h)(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1).$$

So

$$\begin{aligned} \textstyle\int_K \frac{\partial w_h^1}{\partial x} dxdy &= t(w_h) \textstyle\int_K \sum\limits_{i=1}^3 \frac{\partial\varphi}{\partial\lambda_i}\frac{\partial\lambda_i}{\partial x} dxdy \\ &= \tfrac{t(w_h)}{2S} \textstyle\int_K [\eta_1(\lambda_2+\lambda_3) + \eta_2(\lambda_1+\lambda_3) + \eta_3(\lambda_1+\lambda_2)]dxdy \\ &= \tfrac{t(w_h)}{2S}\tfrac{2\Delta(\eta_1+\eta_2+\eta_3)}{3} \\ &= 0. \end{aligned} \tag{10}$$

Similarly,

$$\int_K \frac{\partial w_h^1}{\partial y} dxdy = 0, \tag{11}$$

thus lemma 1 is proved.

Lemma 2. *For* $\forall\ w_h \in V_h$, *there holds*

$$\|w_h^1\|_h \;\; \leq \;\; C\|w_h\|_h, \tag{12}$$

$$\|w_h^1\|_0 \;\; \leq \;\; Ch\|w_h\|_h. \tag{13}$$

Here and later $C > 0$ denotes a constant independent of h_K/ρ_K, $\forall\ K \in J_h$ and the function under considered.

Proof. By the equalities (8) and (9) of lemma 1, since $\frac{\partial \bar{w}_h}{\partial x}$ and $\frac{\partial \bar{w}_h}{\partial y}$ are two constants, we can deduce that

$$\begin{array}{rcl} |w_h|^2_{1,K} &=& \int_K[(\frac{\partial w_h}{\partial x})^2 + (\frac{\partial w_h}{\partial y})^2)]dxdy \\ &=& \int_K[(\frac{\partial}{\partial x}(\bar{w}_h + w^1_h))^2 + (\frac{\partial}{\partial y}(\bar{w}_h + w^1_h))^2]dxdy \\ &=& \int_K[(\frac{\partial \bar{w}_h}{\partial x})^2 + (\frac{\partial \bar{w}_h}{\partial y})^2]dxdy + \int_K[(\frac{\partial w^1_h}{\partial x})^2 + (\frac{\partial w^1_h}{\partial y})^2]dxdy \\ &=& |\bar{w}_h|^2_{1,K} + |w^1_h|^2_{1,K}. \end{array} \tag{14}$$

And using the definition of $\|\cdot\|_h$, we have

$$\|w_h\|^2_h = \|\bar{w}_h\|^2_h + \|w^1_h\|^2_h.$$

So, (12) follows.

To prove (13), we only need prove

$$\|w^1_h\|_0 \leq Ch\|w^1_h\|_h.$$

We compute $\|w^1_h\|_0$ and $\|w^1_h\|_h$ as follows

$$\begin{array}{rcl} \|w^1_h\|^2_0 &=& \int_K |w^1_h|^2 dxdy \\ &=& t^2(w_h)\int_K(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)^2 dxdy \\ &=& t^2(w_h)\int_K[(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2) + 2(\lambda_1^2\lambda_2\lambda_3 + \lambda_2^2\lambda_3\lambda_1 + \lambda_3^2\lambda_1\lambda_2)]dxdy \\ &=& t^2(w_h)(\frac{S}{30} + \frac{S}{30}) \\ &=& t^2(w_h)\frac{S}{15}, \end{array} \tag{15}$$

$$\begin{array}{rcl} |w^1_h|^2_{1,K} &=& \int_K[(\frac{\partial w^1_h}{\partial x})^2 + (\frac{\partial w^1_h}{\partial y})^2]dxdy \\ &=& \frac{t^2(w_h)}{4S^2}(\int_K[\eta_1(\lambda_2+\lambda_3) + \eta_2(\lambda_1+\lambda_3) + \eta_3(\lambda_1+\lambda_2)]^2 dxdy \\ &+& \int_K[\xi_1(\lambda_2+\lambda_3) + \xi_2(\lambda_1+\lambda_3) + \xi_3(\lambda_1+\lambda_2)]^2 dxdy) \\ &=& \frac{t^2(w_h)}{4S^2}(\int_K[(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3)^2 + (\xi_1\lambda_1 + \xi_2\lambda_2 + \xi_3\lambda_3)^2]dxdy \\ &=& \frac{t^2(w_h)}{4S^2}(\int_K[\lambda_1^2\ell_1^2 + \lambda_2^2\ell_2^2 + \lambda_3^2\ell_3^2 \\ &+& 2\lambda_1\lambda_2(\xi_1\xi_2 + \eta_1\eta_2) + 2\lambda_2\lambda_3(\xi_2\xi_3 + \eta_2\eta_3) + 2\lambda_3\lambda_1(\xi_3\xi_1 + \eta_3\eta_1)]dxdy) \\ &=& \frac{t^2(w_h)}{48S}\ell^2. \end{array} \tag{16}$$

Thus,

$$\|w^1_h\|^2_0 \leq Ch^2\|w^1_h\|^2_h,$$

and (13) follows.

3. Estimates of $\|u - u_h\|_h$ and $\|u - u_h\|_0$

Let $\gamma = (\gamma_1, \gamma_2)$ be the multi-index, and $|\gamma| = \gamma_1 + \gamma_2$, $D^\gamma v = \frac{\partial^\gamma v}{\partial x^{\gamma_1}\partial y^{\gamma_2}}$.

Theorem 1. *Let u and u_h be the solution of (5) and (7) respectively, $u \in H^1_0(\Omega) \cap H^2(\Omega)$, then, under the anisotropic meshes, we have the following estimates*

$$\|u - u_h\|_h \leq Ch|u|_{2,\Omega}, \tag{17}$$

$$\|u - u_h\|_0 \leq Ch^2|u|_{2,\Omega}. \tag{18}$$

Proof. By Strang second lemma [4], we have

$$\|u-u_h\|_h \leq C\{\inf_{v_h\in V_h}\|u-v_h\|_h+\sup_{w_h\in V_h\setminus\{0\}}\frac{a_h(u,w_h)-f(w_h)}{\|w_h\|_h}\}. \tag{19}$$

Let Π_h:$H^2(\Omega)\rightarrow V_h$ be the linear interpolation operator defined by

$$\Pi_h u = u_1\lambda_1+u_2\lambda_2+u_3\lambda_3.$$

Then by the similar argument of [5,6], the first term on the right hand of (19) can be estimated as

$$\inf_{v_h\in V_h}\|u-v_h\|_h \leq \|u-\Pi_h u\|_h \leq Ch|u|_{2,\Omega}. \tag{20}$$

Now,we begin to estimate the second term on the right hand of (19), i.e. the consistency error. This is the key point of our estimates.
$\forall\ w_h\in V_h,\quad w_h=\bar{w}_h+w_h^1$, then, $\bar{w}_h\in C^0(\Omega)\cap H_0^1(\Omega)$.

$$a_h(u,\bar{w}_h)-f(\bar{w}_h)=a(u,\bar{w}_h)-f(\bar{w}_h)=0,$$

thus,

$$\begin{aligned}|a_h(u,w_h)-f(w_h)| &= |a_h(u,\bar{w}_h+w_h^1)-f(\bar{w}_h+w_h^1)|\\ &= |a_h(u,w_h^1)-f(w_h^1)|\\ &\leq |a_h(u,w_h^1)|+|f(w_h^1)|.\end{aligned} \tag{21}$$

Define the zero order operator P_0 as follows

$$P_0\vec{v}=\frac{1}{|K|}\int_K\vec{v}dxdy,\qquad \forall\ \vec{v}\in H^1(K)^2,\quad K\in J_h.$$

Let $\quad\vec{v}=\nabla u$, from lemma 1, we get

$$\int_K\nabla w_h^1dxdy=0,$$

hence,

$$\begin{aligned}|a_h(u,w_h^1)| &= |\sum_{K\in J_h}\int_K\nabla u\cdot\nabla w_h^1dxdy|\\ &= |\sum_{K\in J_h}\int_K(\nabla u-P_0\nabla u)\cdot\nabla w_h^1dxdy|\\ &= |\sum_{K\in J_h}\int_K(\vec{v}-P_0\vec{v})\cdot\nabla w_h^1dxdy|\\ &\leq \sum_{K\in J_h}\|\vec{v}-P_0\vec{v}\|_{0,K}\|\nabla w_h^1\|_{0,K}\\ &\leq (\sum_{K\in J_h}\|\vec{v}-P_0\vec{v}\|_{0,K}^2)^{\frac{1}{2}}(\sum_{K\in J_h}\|\nabla w_h^1\|_{0,K}^2)^{\frac{1}{2}}.\end{aligned} \tag{22}$$

Since

$$P_0v=\frac{1}{|K|}\int_K vdxdy=\frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}d\hat{x}d\hat{y}=\hat{P}_0\hat{v},$$

we know that P_0 is affine-equivalent.

$$
\begin{aligned}
\|\vec{v}-P_0\vec{v}\|_{0,K}^2 &= \|\hat{\vec{v}}-P_0\hat{\vec{v}}\|_{0,\hat{K}}^2\frac{|K|}{|\hat{K}|}\\
&\leq |\hat{\vec{v}}|_{1,\hat{K}}^2\frac{|K|}{|\hat{K}|}\\
&= \sum_{|\alpha|=1}\|\hat{D}^\alpha\hat{\vec{v}}\|_{0,\hat{K}}^2\frac{|K|}{|\hat{K}|}\\
&= \sum_{|\alpha|=1}(\textstyle\int_K h^{2\alpha}|D^\alpha\vec{v}|^2dxdy)\\
&\leq Ch^2\sum_{|\alpha|=1}\|D^\alpha(\nabla u)\|_{0,K}^2\\
&\leq Ch^2|u|_{2,K}.
\end{aligned}
\tag{23}
$$

Using (14), we have

$$
\begin{aligned}
\|\nabla w_h^1\|_{0,K}^2 &= \textstyle\int_K[(\frac{\partial w_h^1}{\partial x})^2+(\frac{\partial w_h^1}{\partial y})^2]dxdy\\
&= |w_h^1|_{1,K}^2\\
&\leq C|w_h|_{1,K}^2.
\end{aligned}
\tag{24}
$$

Substituting (23) and (24) into (22), we get

$$
\begin{aligned}
|a_h(u,w_h^1)| &\leq Ch(\sum_{K\in J_h}|u|_{2,K}^2)^{\frac{1}{2}}(\sum_{K\in J_h}|w_h|_{1,K}^2)^{\frac{1}{2}}\\
&\leq Ch|u|_{2,\Omega}\|w_h\|_h.
\end{aligned}
\tag{25}
$$

Using (13) yields

$$
\begin{aligned}
|f(w_h^1)| &= |\textstyle\int_\Omega(-\Delta u)w_h^1dxdy|\\
&\leq \|-\triangle u\|_{0,\Omega}\|w_h^1\|_0\\
&\leq Ch|u|_{2,\Omega}\|w_h\|_h.
\end{aligned}
\tag{26}
$$

Substituting (25) and (26) into (21), we obtain

$$|a_h(u,w_h)-f(w_h)|\leq Ch|u|_{2,\Omega}\|w_h\|_h.$$

Therefore the consistency error term can be estimated as

$$\sup_{w_h\in V_h}\frac{|a_h(u,w_h)-f(w_h)|}{\|w_h\|_h} \leq Ch|u|_{2,\Omega}. \tag{27}$$

Thus substituting (20) and (27) into (19) follows (17). Applying Aubin-Nitsche duality argument (refer to [4]) yields (18), the proof of this theorem is completed.

Remark 1. The approach proposed in this paper is novel and is very different from the analysis of C-R element on anisotropic meshes of [7].

Remark 2. It has been shown in [8,9,13] that of any $v_h\in V_h$ for quasi-Wilson arbitrary quadrilateral element or Wilson rectangular element, both lemma 1 and lemma 2 are satisfied and thus the above results are valid for these elements.

Remark 3. It has been proved in [8,9] that for any $v_h\in V_h$ of quasi-Wilson element, there holds,

$$\int_K w_h^1dxdy=0,\quad \int_K q_1(x,y)\frac{\partial w_h^1}{\partial x}dxdy=0,\quad \int_K q_1(x,y)\frac{\partial w_h^1}{\partial y}dxdy=0 \tag{28}$$

where $q_1(x,y)\in P_1$(the set of all linear polynomials on K). At the same time, we have obtained the super-convergence property of consistency error, that is, when the exact solution of (6) $u\in H^3(\Omega)\cap H_0^1(\Omega)$, then the consistency error is of the order $O(h^2)$, which is one order higher than that of Wilson element. But now we find that (28) does not hold for Carey's element and therefore the super-convergence analysis of it on anisotropic meshes is certainly a great challenge.

4. Numerical Examples

In order to investigate the numerical behavior of anisotropic Carey's element, we still consider the second order problem (5) with $f(x,y) = 4 - 2x^2 - 2y^2 \in L^2(\Omega)$, and $\Omega = (-1,1) \times (-1,1)$. It can be verified that the exact solution of problem (5) is $u(x,y) = (1-x^2)(1-y^2)$. Denote the triangulation of Ω as J_h, $h = \max_{K\in J_h} h_K$, $\rho = \max_{K\in J_h} \rho_K$, $h_K = diam(K)$, $\rho_K = \max_M diam(M)$, and M is an arbitrary circle contained in K. Define the norm $\|u-u_h\|_0$ and $\|u-u_h\|_h$ as follows:

$$\|u-u_h\|_0 = (\sum_{K\in J_h} \|u-u_h\|_{0,K}^2)^{\frac{1}{2}}$$

$$\|u-u_h\|_h = (\sum_{K\in J_h} |u-u_h|_{1,K}^2)^{\frac{1}{2}}$$

where u and u_h are the exact solution and the Carey's element solution of problem (5) and (7) respectively.

We consider the above three triangulation types of Ω: mesh 1, mesh 2, and mesh 3 (see Figure 4.1, Figure 4.2 and Figure 4.3).

To obtain mesh 1, we subdivide the boundary of Ω into m and n equal intervals along the $x-axis$ and $y-axis$ respectively, and then triangulate the rectangular with diagonal parallel lines. For mesh 2, the subdivision along the $y-axis$ is as same as mesh 1, but $(m+1)$ points $\cos(\frac{(m+1-i)\pi}{m})(i = 1,2,\cdots,m+1)$ are taken along the $x-axis$. And as for mesh 3, we take $(m+1)$ points $\cos(\frac{(m+1-i)\pi}{m})(i = 1,2,\cdots,m+1)$ along the $y-axis$ and $(n+1)$ points $\sin(\frac{(3n+2-2j)\pi}{2n})(j = 1,2,\cdots,m+1)$ along the $y-axis$ respectively.

For mesh 1, $h/\rho \approx m/n$, we carry out the numerical computing with respect to the mesh with $\frac{m}{n} = 10$ and $\frac{m}{n} = 20$ respectively. The numerical results are listed in Table 4.1 and Table 4.2. Herein, α denotes the convergence order.

Table 4.1

$n \times m$	$\|u-u_h\|_0$	α	$\|u-u_h\|_h$	α
2×20	1.9960394275	/	3.1130196548	/
4×40	0.4897561470	2.020047131	1.5577242956	0.9988747754
8×80	0.1219088342	2.0062609245	0.7797570461	0.9983433215
16×160	0.0304447136	2.0015390292	0.3900143110	0.9994976217
32×320	0.0076091568	2.0003832442	0.1950248846	0.9998688432

Table 4.2

$n \times m$	$\|u-u_h\|_0$	α	$\|u-u_h\|_h$	α
2×40	1.9974853754	/	3.1108349141	/
4×80	0.4903037853	2.0264371357	1.5566062993	0.9988977353
8×160	0.1220588906	2.0060985169	0.7791902179	0.9983566298
16×320	0.0304830769	2.0014969448	0.3897297498	0.999501503
32×640	0.0076188012	2.0003726168	0.1948824551	0.9998698497

Figure 4.1 : mesh 1

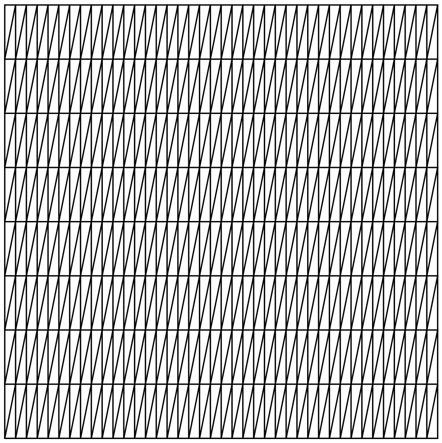


Figure 4.2 : mesh 2

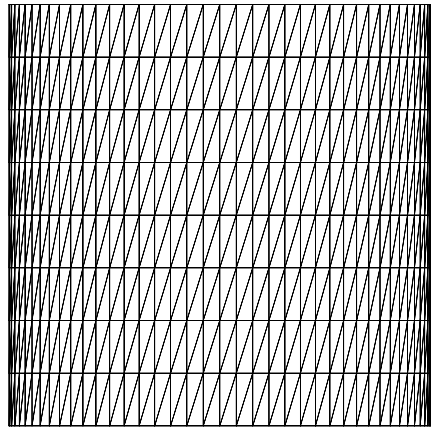
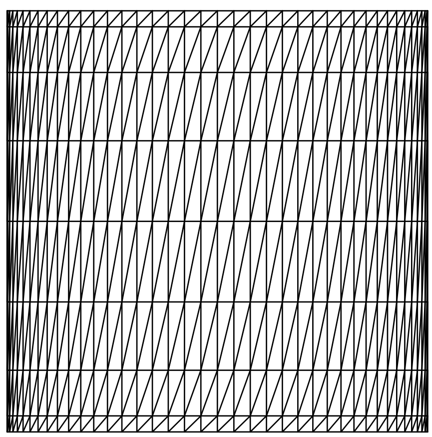


Figure 4.3 : mesh 3



From the above two tables, we can see that the error between u and u_h is indeed independent of h_K/ρ_K. It means that we can get the same order of error estimates whether the subdivision satisfies the regular assumption or not.

We take $\frac{m}{n} = 20$ for mesh 2; and $\frac{m}{n} = 20$, $\frac{m}{n} = 40$ for mesh 3. The results are listed in the following Table 4.3-Table 4.5 respectively.

Table 4.3

$n \times m$	$\max(h_K/\rho_K)$	$\|u-u_h\|_0$	α	$\|u-u_h\|_h$	α
2×40	324.394506	1.9973710247	/	3.1112169145	/
4×80	648.538915	0.4902484469	2.02652	1.5567211777	0.99896945
8×160	1296.952848	0.1220435313	2.00612	0.7792379074	0.9983748
16×320	2593.843135	0.0304791456	2.0015	0.3897523661	0.999506087
32×640	5187.655019	0.0076178127	2.00037374	0.1948936094	0.999870996

Table 4.4

$n \times m$	$\max(h_K/\rho_K)$	$\|u-u_h\|_0$	α	$\|u-u_h\|_h$	α
4×80	917.172529	0.9667297974	/	2.0299411894	/
8×160	1985.289424	0.2550953155	1.922076304	1.0471878693	0.9549176
16×320	4048.269539	0.0645291267	1.983013999	0.5273885898	0.9895820266
32×640	8135.665760	0.0161800230	1.99573684224	0.2641668114	0.997417133

Table 4.5

$n \times m$	$\max(h_K/\rho_K)$	$\|u-u_h\|_0$	α	$\|u-u_h\|_h$	α
2×80	1297.072830	1.9978186102	/	3.1103841296	/
4×160	2593.905636	0.4904269107	2.023155511	1.5563553786	0.9989212396
8×320	5987.686269	0.12209257477	2.00606268	0.7790603672	0.9983644952
16×640	10375.310038	0.0341916872	2.0014875770	0.3896642317	0.9995036142

We note that for mesh 2 and mesh 3, $\max\limits_{K\in J_h}\{h_K/\rho_K\}$ increases rapidly with the increase of the element number. In particular, $\max\limits_{K\in J_h}\{h_K/\rho_K\}$ increases sharply for mesh 3. Hence, both the condition of regularity and quasi-uniform can not be satisfied. Nevertheless, the error order α of $\|u-u_h\|_0$ and $\|u-u_h\|_h$ approach to 2 and 1 respectively, which is as same as the result obtained under the regular condition.

From the above numerical results, we can also conclude that Carey's element is reliable and very stable with respect to the increasing of anisotropy of the meshes. The error between u and u_h is really independent of $h_K/\rho_K, \forall K \in J_h$, which coincides with the theoretical analysis of this paper. Besides, we find another very interesting fact, i.e, the Carey's finite element solution approaches to the exact solution more exactly with the increasing of anisotropy of meshes (see Table 4.5). Thus, it is of importance of using anisotropic Carey's element to deal with the second order elliptic problems.

Acknowledgement. The authors thank the anonymous referees for their valuable suggestions.

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