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ON COEFFICIENT POLYNOMIALS OF CUBIC HERMITE-PADÉ APPROXIMATIONS TO THE EXPONENTIAL FUNCTION *1)

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Abstract

The polynomials related with cubic Hermite-Padé approximation to the exponential function are investigated which have degrees at most n, m, s respectively. A connection is given between the coefficients of each of the polynomials and certain hypergeometric functions, which leads to a simple expression for a polynomial in a special case. Contour integral representations of the polynomials are given. By using of the saddle point method the exact asymptotics of the polynomials are derived as n, m, s tend to infinity through certain ray sequence. Some further uniform asymptotic aspects of the polynomials are also discussed.

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Key words: Padé-type approximant, Cubic Hermite-Padé approximation, Hypergeometric function, Saddle point method.

1. Introduction

Hermite-Padé approximation to the exponential function was introduced by Hermite [5] who considered expressions of the form

$$t_k(x)e^{s_kx}+t_{k-1}(x)e^{s_{k-1}x}+\cdots+t_1(x)e^{s_1x}=O(x^h), \tag{1.1}$$

where $t_1(x), t_2(x), \cdots, t_k(x)$ are polynomials, of specified degrees,chosen so that h is as large as possible.

Included, of course, in expressions of type (1.1) are both the ordinary Padé approximations

$$\hat{P}_n(x)e^{-x}+\hat{Q}_m(x)=O(x^{n+m+1}) \tag{1.2}$$

with deg($\hat{P}_n$)$\leq n$, deg($\hat{Q}_m$)$\leq m$, $\hat{P}_n(0)\neq 0$, and the quadratic Hermite-Padé approximations [3,4]

$$\tilde{P}_n(x)e^{-2x}+\tilde{Q}_m(x)e^{-x}+\tilde{R}_s(x)=O(x^{n+m+s+2}) \tag{1.3}$$

with deg($\tilde{P}_n$)$\leq n$,deg($\tilde{Q}_m$)$\leq m$,deg($\tilde{R}_s$)$\leq s$, $\tilde{P}_n(0)\neq 0$.

In this paper, we wish to investigate a number of properties of the polynomials P_n, T_l, Q_m and R_s that arise from the solution of the following cubic Hermite-Padé approximations

$$P_n(x)e^{-3x}+T_l(x)e^{-2x}+Q_m(x)e^{-x}+R_s(x)=O(x^{n+m+s+l+3}), \tag{1.4}$$

with deg(P_n)$\leq n$, deg(T_l)$\leq l$, deg(Q_m)$\leq m$, deg(R_s)$\leq s$, P_n monic. But as is well known, if we set $x=y-\frac{a}{3}$, then any cubic equation $x^3+ax^2+bx+c=0$ can be transformed into the following form

$$y^3+(b-\frac{a^2}{3})y+(\frac{2}{27}a^3-\frac{1}{3}ab+c)=0.$$

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So without loss of generality, in this paper we only consider approximations to e^{-x} generated by finding polynomials P_n, Q_m and R_s so that

$$E_{nms}(x) := P_n(x)e^{-3x} + Q_m(x)e^{-x} + R_s(x) = O(x^{n+m+s+2}). \tag{1.5}$$

The explicit formulae for these unique polynomials are known; in the super-diagonal case $n = m = s$, they were obtained by Wang & Zheng [12] and for arbitrary $n, m, s \in \mathbf{N}$, they can be found in Zheng & Wang [13].

2. The Polynomials P_n, Q_m and R_s

The polynomials P_n, Q_m and R_s with deg(P_n)= n, deg(Q_m)= m, deg(R_s)= s, P_n monic, that satisfy (1.5) are given by (cf. Zheng & Wang [13])

$$P_n(x) = n!\sum_{j=0}^{n} \frac{p_j x^j}{j!}. \tag{2.1}$$

where, for $0 \le j \le n$,

$$p_j = 2^{j-n}\sum_{k=0}^{n-j}\left(\frac{2}{3}\right)^k \binom{n+m-k-j}{m}\binom{s+k}{s}; \tag{2.2}$$

$$Q_m(x) = -\frac{3^{s+1}}{2^n} n!\sum_{j=0}^{m} \frac{q_j x^j}{j!}, \tag{2.3}$$

where, for $0 \le j \le m$,

$$q_j = \sum_{k=0}^{m-j}(-2)^{k+j}\binom{n+m-k-j}{n}\binom{s+k}{s}; \tag{2.4}$$

$$R_s(x) = (-1)^m 2^{m+1} 3^{s-n} n!\sum_{j=0}^{s} \frac{r_j x^j}{j!}, \tag{2.5}$$

where, for $0 \le j \le s$,

$$r_j = (-1)^j \sum_{k=0}^{s-j}\frac{1}{3^k}\binom{s+m-k-j}{m}\binom{n+k}{n}. \tag{2.6}$$

We observe that each of the polynomials P_n, Q_m, and R_s depends on all three positive integers n, m, and s and the subscript merely denotes the degree of the polynomial in each case. Writing $P_n(x) = P(n, m, s; x), Q_m(x) = Q(n, m, s; x)$,and $R_s(x) = R(n, m, s; x)$.

Our first result establishes a connection between the coefficients of P_n, Q_m, R_s and certain ${}_2F_1$ hypergeometric functions.We recall the definition of the Gauss function (cf.[1])

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \tag{2.7}$$

where

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha), & \text{if } \; k \ge 1, \\ 1, & \text{if } \alpha \ne 0, k = 0. \end{cases} \tag{2.8}$$

If $t \in \mathbf{N}$,it follows immediately from (2.8) that

$$(-t)_k = \begin{cases} (-1)^k t!/(t-k)!, & \text{for } \; 0 \le k \le t, \\ 0, & \text{for } k > t. \end{cases} \tag{2.9}$$

Therefore, the hypergeometric series ${}_2F_1(-t,b;c;z), t \in \mathbf{N}$, is a polynomial of degree t in z and, from (2.8) and (2.9), we have for $b, c \in \mathbf{N}$,

$$ {}_2F_1(-t,b+1;c+1;z) := \sum_{k=0}^{t} \binom{t}{k} \frac{(b+k)!c!}{b!(c+k)!}(-z)^k. \tag{2.10}$$

Theorem 1. *Let p_j, q_j, and r_j be given by (2.2),(2.4), and (2.6), respectively. Then*

$$p_j = 2^{j-n}\binom{n+m-j}{m}\ {}_2F_1(j-n,s+1;j-n-m;\frac{2}{3}), j=0,1,\cdots,n; \tag{2.11}$$

$$q_j = (-2)^j\binom{n+m-j}{n}\ {}_2F_1(j-m,s+1;j-n-m;-2), j=0,1,\cdots,m; \tag{2.12}$$

$$r_j = (-1)^j\binom{s+m-j}{m}\ {}_2F_1(j-s,n+1;j-s-m;\frac{1}{3}), j=0,1,\cdots,s. \tag{2.13}$$

Another way of writing this is

$$p_j = 2^{j-n}\binom{n+m+s+1-j}{n-j}\ {}_2F_1(j-n,s+1;m+s+2;\frac{1}{3}), j=0,1,\cdots,n; \tag{2.14}$$

$$q_j = (-2)^j\binom{n+m+s+1-j}{m-j}\ {}_2F_1(j-m,s+1;n+s+2;3), j=0,1,\cdots,m; \tag{2.15}$$

$$r_j = (-1)^j\binom{n+m+s+1-j}{s-j}\ {}_2F_1(j-s,n+1;n+m+2;\frac{2}{3}), j=0,1,\cdots,s. \tag{2.16}$$

Proof. From (2.2) with $n-j=t$ and (2.9) we have, for $0 \le t \le n$,

$$\begin{aligned} p_j &= 2^{j-n}\sum_{k=0}^{t}\frac{(m+t-k)!(s+k)!}{m!s!k!(t-k)!}\left(\frac{2}{3}\right)^k \\ &= 2^{j-n}\frac{(m+t)!}{m!t!}\sum_{k=0}^{t}\frac{(-t)_k(s+1)_k}{(-m-t)_k k!}\left(\frac{2}{3}\right)^k, \end{aligned} \tag{2.17}$$

from which (2.11) immediately follows. The identities (2.12) and (2.13) follow from the same method.

In general the function ${}_2F_1(a,b;c;z)$ is not defined if $c=0,-1,-2,\cdots$, but in (2.11)-(2.13) the a-parameter equal also a non-positive integer value,with $|a| \le |c|$. In that case the F-function is well-defined. We use a well-known transformation of the F-function to obtain (2.14)-(2.16), where the c-parameter is a positive integer and which are more convenient representations. We use [4]

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b;a+b-c+1;1-z), \\ &a=0,-1,-2,\cdots;|\arg(1-z)|<\pi. \end{aligned} \tag{2.18}$$

Applying this formula to (2.11)-(2.13) we obtain that all arguments in the gamma functions in front of the F-function in (2.18) become equal to non-positive integers.Hence some care is needed in applying the transformation. To verify (2.11)→(2.14) we use the property (cf.[1])

$$\frac{\Gamma(z)}{\Gamma(z-k)} = (-1)^k\frac{\Gamma(k+1-z)}{\Gamma(1-z)}, k=0,1,2,\cdots \tag{2.19}$$

and introduce a small parameter ϵ. That is, we write using $a=j-n, b=s+1, c=j-n-m$,

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \lim_{\epsilon\to 0}\frac{\Gamma(c+\epsilon)\Gamma(c+\epsilon-a-b)}{\Gamma(c+\epsilon-a)\Gamma(c+\epsilon-b)}$$

$$= \frac{\Gamma(m+1)\Gamma(n+m+s-j+2)}{\Gamma(n+m-j+1)\Gamma(m+s+2)}. \tag{2.20}$$

This gives the result in (2.14). The results for q_j and r_j follow in a similar way.

As an immediate consequence of Theorem 1,we can express the coefficients p_j, q_j, and r_j in appropriate Jacobi polynomials.

Corollary 1. *For any $n, m, s \in \mathbf{N}$,if $P_k^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree k with parameter α and β, then*

$$p_j = (-6)^{j-n} P_{n-j}^{(j-n-m-1,j-n-s-1)}(5), \tag{2.21}$$

$$q_j = (-1)^m 2^j P_{m-j}^{(j-n-m-1,n+s+1)}(5), \tag{2.22}$$

$$r_j = (-1)^j 3^{j-s} P_{s-j}^{(n+m+1,j-n-s-1)}(5). \tag{2.23}$$

Proof. Firstly we have (cf.[1])

$$P_k^{(\alpha,\beta)}(x) = \binom{k+\alpha}{k} {}_2F_1(-k, \alpha+\beta+k+1; \alpha+1; \frac{1-x}{2}). \tag{2.24}$$

After applying the following relation (cf.[1])

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1}) \tag{2.25}$$

to (2.11),we have

$$p_j = 6^{j-n} \binom{n+m-j}{m} {}_2F_1(j-n, j-m-n-s-1; j-n-m; -2). \tag{2.26}$$

Through a few manipulations with binomial coefficients and gamma functions (again with negative integer arguments), the proof of (2.21) follows from (2.26) and (2.24) with $\alpha = j - n - m - 1, \beta = j - n - s - 1, k = n - j$,and $x = 5$.

The relation (2.22) follows from (2.12) and (2.24) in the same manner. Finally by applying (2.25) we have

$$r_j = (-1)^j 3^{j-s} \binom{n+m+s+1-j}{s-j} {}_2F_1(j-s, m+1; n+m+2; -2). \tag{2.27}$$

Now we can easily prove (2.23) from (2.27) and (2.24). Especially,when $m = s$,the coefficient p_j can be expressed in appropriate Gegenbauer polynomial (cf.[1]).

Corollary 2. *For any $n, m = s \in \mathbf{N}$, if $C_k^{(\gamma)}$ denotes the Gegenbauer polynomial of degree k with parameter γ, then*

$$p_j = (-6)^{j-n} \frac{(s+n-j)!(n+2s+1-j)!}{s!(2n+2s+1-2j)!} C_{n-j}^{(j-n-s-1/2)}(5). \tag{2.28}$$

Proof. We have (cf.[1])

$$C_k^{(\gamma)}(x) = \frac{(2\gamma)_k}{(\gamma+1/2)_k} P_k^{(\gamma-1/2,\gamma-1/2)}(x).$$

It follows from (2.21) that

$$p_j = (-6)^{j-n} \frac{\Gamma(2j-2n-2s-1)\Gamma(-s)}{\Gamma(j-n-2s-1)\Gamma(j-n-s)} C_{n-j}^{(j-n-s-1/2)}(5).$$

A few manipulations with binomial coefficients and gamma functions (again with negative integer arguments) gives the proof of (2.28).

3. Contour Integral Representations and Asymtotics

The polynomials P_n, Q_m, and R_s that satisfy (1.5), and are given by (2.1)-(2.6), admit simple contour integral representations.

Theorem 2. *Let $n, m, s \in \mathbf{N}$ and let C be a circle, centre at the origin,radius $r \in (0,1)$. Let $P_n(x), Q_m(x)$, and $R_s(x)$ be the polynomials given by (2.1),(2.3),and (2.5),respectively.Then*

$$P_n(x) = \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C \frac{e^{-2xv}}{v^{n+1}(v+1)^{m+1}(v+3/2)^{s+1}} dv, \tag{3.1}$$

$$Q_m(x) = \frac{(-1)^{m+1} 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C \frac{e^{2xv}}{v^{m+1}(v+1)^{n+1}(1/2-v)^{s+1}} dv, \tag{3.2}$$

$$R_s(x) = \frac{(-1)^{m+s} 3^{s+1} 2^{m+1} n!}{2\pi i} \oint_C \frac{e^{xv}}{v^{s+1}(v+1)^{m+1}(v+3)^{n+1}} dv. \tag{3.3}$$

Proof. We only prove (3.1). The other two relations can be proved in the same manner. Expanding e^{-2xv} in its Maclaurin series and using Cauchy's integral theorem, we have

$$\begin{aligned}
&\frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C \frac{e^{-2xv}}{v^{n+1}(v+1)^{m+1}(v+3/2)^{s+1}} dv \\
&= \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C \frac{1}{v^{n+1}(v+1)^{m+1}(v+3/2)^{s+1}} \sum_{j=0}^{\infty} \frac{(-2x)^j}{j!} v^j dv \\
&= \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} \left\{ \sum_{j=0}^{n-1} \frac{(-2x)^j}{j!} \oint_C \frac{1}{v^{n-j+1}(v+1)^{m+1}(v+3/2)^{s+1}} dv \right. \\
&\quad + \frac{(-2x)^n}{n!} \oint_C \frac{1}{(v+1)^{m+1}(v+3/2)^{s+1}} \frac{1}{v} dv \\
&\quad \left. + \sum_{j=n+1}^{\infty} \frac{(-2x)^j}{j!} \oint_C \frac{v^{j-n-1}}{(v+1)^{m+1}(v+3/2)^{s+1}} dv \right\} \\
&= x^n + \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1}} \sum_{j=0}^{n-1} \frac{(-2x)^j}{j!} \frac{1}{(n-j)!} \left[\frac{1}{(v+1)^{m+1}(v+3/2)^{s+1}} \right]^{(n-j)} \Bigg|_{v=0}.
\end{aligned}$$

By using of Leibniz' rule, we have

$$\begin{aligned}
&\left[\frac{1}{(v+1)^{m+1}(v+3/2)^{s+1}} \right]^{(n-j)} \Bigg|_{v=0} \\
&= \sum_{k=0}^{n-j} \binom{n-j}{k} \left[\frac{1}{(v+1)^{m+1}} \right]^{(n-j-k)} \Bigg|_{v=0} \left[\frac{1}{(v+3/2)^{s+1}} \right]^{(k)} \Bigg|_{v=0} \\
&= \sum_{k=0}^{n-j} \left(\frac{2}{3}\right)^{s+k+1} \binom{n-j}{k} [-(m+1)][-(m+1)-1] \cdots [-(m+1)-(n-j-k)+1] \cdot \\
&\quad [-(s+1)][-(s+1)-1] \cdots [-(s+1)-k+1] \\
&= \sum_{k=0}^{n-j} \left(\frac{2}{3}\right)^{s+k+1} \binom{n-j}{k} (-1)^{n-j} \frac{(m+n-j-k)!(s+k)!}{m!s!}.
\end{aligned}$$

A comparison of the coefficients of powers of x on the right side of (3.1) with (2.1) completes the proof of (3.1).

In order to analyze the asymptotic behaviour of the polynomials $P_n(x), Q_m(x)$, and $R_s(x)$ given by (3.1), (3.2), and (3.3),respectively,we let

$$N = n+1, M = m+1, S = s+1, \tag{3.4}$$

and assume that all these parameters are large.We write

$$M = \alpha N, S = \beta N, \tag{3.5}$$

where α, β are real, positive constants. We write (3.1) in the form

$$P_n(x) = \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C e^{-N\hat{p}(v)} e^{-2xv} dv, \tag{3.6}$$

where

$$\hat{p}(v) := \ln \left[v(1+v)^\alpha \left(v + \frac{3}{2} \right)^\beta \right]. \tag{3.7}$$

To give the asymptotics, we need the following lemma

Lemma (Saddle Point Method)[7]**.** *Let h, g be analytic function in a simply connected open set Δ and assume that g has no zeros in Δ. Let Γ be a smooth oriented path with a finite length and endpoints a and b,lying in Δ. Moreover, let $I_n = \int_\Gamma h(z)/g(z)^n dz$. Further assume that a point z_0 of Γ, different from an endpoint, is a nondegenerate critical point of g (i.e., $g'(z_0) = 0, g^{''}(z_0) \neq 0$)and let ω be the phase corresponding to the direction of the tangent to the oriented path at z_0. Suppose that* $\min_{z\in\Gamma} |g(z)|$ *is attained at the point z_0 only. Then, if $h(z_0) \neq 0$,*

$$I_n = \sqrt{2\pi g(z_0)/g''(z_0)} \frac{h(z_0)}{\sqrt{n} g(z_0)^n} \left(1 + O\left(\frac{1}{n}\right) \right), \tag{3.8}$$

as $n \to \infty$, where the phase ω_0 of $g^{''}(z_0)/g(z_0)$ is chosen to satisfy $|\omega_0 + 2\omega| \leq \pi/2$.Since $g(z) - g(z_0) \sim (g^{''}(z_0)/2)(z-z_0)^2$ as $z \to z_0$ along Γ, and $|g(z)/g(z_0)| \geq 1$, it is always possible to choose ω_0 uniquely in this way.

Applying the saddle point method to the integral in (3.6), a simple calculation shows that for all real,positive values of α and $\beta, \hat{p}(v)$ has derivative equal to zero at a point,say v_0,lying in $(-1, 0)$,and at another point to the left of -1. The contour C can be chosen to run through v_0. Moreover, $\hat{p}^{''}(v_0) \neq 0$ and, in fact, $\hat{p}^{''}(v_0)$ is real and negative for all $\alpha, \beta > 0$. Therefore, as $N \to \infty$, we deduce from (3.6) that

$$P_n(x) \sim \frac{(-1)^n 3^{s+1} n!}{2^{n+s+1} 2\pi i} 2e^{-N\hat{p}(v_0)} \sqrt{\frac{\pi}{N}} \frac{e^{-2xv_0}}{\sqrt{2\hat{p}''(v_0)}}, \tag{3.9}$$

where if $\hat{p}^{''}(v_0) = -k_0^2$ say, $k_0 > 0$, we choose the branch of $\sqrt{2\hat{p}''(v_0)} = i\sqrt{2}k_0$, in accordance with [7]. In Theorem 3 more details are given for a special case.

Similarly, for $Q_m(x)$, we have from (3.2)

$$Q_m(x) = \frac{(-1)^{m+1} 3^{s+1} n!}{2^{n+s+1} 2\pi i} \oint_C e^{-N\hat{q}(v)} e^{2xv} dv, \tag{3.10}$$

where

$$\hat{q}(v) := \ln \left[v^\alpha (1+v) \left(\frac{1}{2} - v \right)^\beta \right] \tag{3.11}$$

Now for $0 < \beta < 4$ we can choose C to run through two saddle points:$\hat{q}(v)$ has derivative equal to zero at two distinct points, $v_1 \in (-1, 0), v_2 \in (0, 1)$ for all $\alpha > 0, 0 < \beta < 4$.

Applying the saddle point method to the integral in (3.10), as $N \to \infty$, we have

$$Q_m(x) \sim \frac{(-1)^{m+1}3^{s+1}n!}{2^{n+s+1}2\pi i}\left\{2e^{-N\hat{q}(v_1)}\sqrt{\frac{\pi}{N}}\frac{e^{2xv_1}}{\sqrt{2\hat{q}''(v_1)}} + 2e^{-N\hat{q}(v_2)}\sqrt{\frac{\pi}{N}}\frac{e^{2xv_2}}{\sqrt{2\hat{q}''(v_2)}}\right\}. \tag{3.12}$$

When $\beta \geq 4$, we can choose C to run through one saddle point: $\hat{q}(v)$ has derivative equal to zero at one point $v_1 \in (-1,0)$ (at another point $v_2 \leq -1$) for all $\alpha > 0, \beta \geq 4$.

In this case, as $N \to \infty$, we have

$$Q_m(x) \sim \frac{(-1)^{m+1}3^{s+1}n!}{2^{n+s+1}2\pi i}2e^{-N\hat{q}(v_1)}\sqrt{\frac{\pi}{N}}\frac{e^{2xv_1}}{\sqrt{2\hat{q}''(v_1)}}. \tag{3.13}$$

For $R_s(x)$, we have from (3.3)

$$R_s(x) = \frac{(-1)^{m+s}3^{s+1}2^{m+1}n!}{2\pi i}\oint_C e^{-N\hat{r}(v)}e^{xv}dv, \tag{3.14}$$

where

$$\hat{r}(v) := \ln[v^{\beta}(v+1)^{\alpha}(v+3)]. \tag{3.15}$$

Then we can choose C to run through one saddle point: $\hat{r}(v)$ has derivative equal to zero at one point $v_3 \in (-1,0)$ (at another point to the left of -1) for all $\alpha, \beta > 0$.

Applying the saddle point method,as $N \to \infty$,we have

$$R_s(x) \sim \frac{(-1)^{m+s}3^{s+1}2^{m+1}n!}{2\pi i}2e^{-N\hat{r}(v_3)}\sqrt{\frac{\pi}{N}}\frac{e^{xv_3}}{\sqrt{2\hat{r}''(v_3)}}. \tag{3.16}$$

The asymptotic formulae for P_n, Q_m,and R_s are rather cumbersome arithmetically for arbitrary $\alpha, \beta > 0$. We shall, therefore, restrict ourselves to the (rather natural) case when $\beta = 2$ in (3.5), although the method works for all $\alpha, \beta > 0$.

Theorem 3. *Let $P_n(x), Q_m(x)$, and $R_s(x)$ be given by (3.1),(3.2), and (3.3), respectively, and assume that (3.5) holds with $\beta = 2$. Set*

$$\sigma := \sqrt{9\alpha^2 + 30\alpha + 9}\ , \tag{3.17}$$

$$\varsigma := \sqrt{3\alpha^2 + 19\alpha + \frac{299}{9} + \frac{9}{\alpha} - \sigma\left(\frac{\alpha}{3} + 2 + \frac{3}{\alpha}\right)}\ , \tag{3.18}$$

$$\tau := \sqrt{3\alpha^2 + 19\alpha + \frac{299}{9} + \frac{9}{\alpha} + \sigma\left(\frac{\alpha}{3} + 2 + \frac{3}{\alpha}\right)}\ , \tag{3.19}$$

$$A_{n,\alpha} := \frac{n!\{(\alpha+3)[3(\alpha+3)-\sigma]\}^{n+1}}{2^{m+3n+3}\sqrt{(n+1)\pi}\cdot\varsigma}\left[\frac{\sigma-(\alpha+3)}{\alpha}\right]^{\alpha(n+1)}, \tag{3.20}$$

$$B_{n,\alpha} := \frac{n!\{(\alpha+3)[3(\alpha+3)+\sigma]\}^{n+1}}{2^{m+3n+3}\sqrt{(n+1)\pi}\cdot\tau}\left[\frac{\sigma+(\alpha+3)}{\alpha}\right]^{\alpha(n+1)}. \tag{3.21}$$

Then, as $n \to \infty$, we have

$$P_n(x) \sim A_{n,\alpha}e^{\frac{3(\alpha+3)-\sigma}{2(\alpha+3)}x}, \tag{3.22}$$

$$Q_m(x) \sim -A_{n,\alpha}e^{-\frac{\alpha+3+\sigma}{2(\alpha+3)}x} + (-1)^{m+1}B_{n,\alpha}e^{-\frac{\alpha+3-\sigma}{2(\alpha+3)}x}, \tag{3.23}$$

$$R_s(x) \sim (-1)^m B_{n,\alpha}e^{-\frac{3(\alpha+3)-\sigma}{2(\alpha+3)}x}. \tag{3.24}$$

The asymptotics are uniform with respect to x on compact subset of $\mathbf{C}$.

Proof. Putting $\beta = 2$ in (3.7) and differentiating with respect to v, we see that $\hat{p}'(v) = 0$ when $v = -\frac{3}{4} \pm \frac{\sigma}{4(\alpha+3)}$.

Setting

$$v_0 := -\frac{3}{4} + \frac{\sigma}{4(\alpha+3)}, \tag{3.25}$$

it follows from (3.7) with $\beta = 2$ that

$$e^{-N\hat{p}(v_0)} = \frac{(-1)^N}{9^N 2^M}\{(\alpha+3)[3(\alpha+3)-\sigma]\}^N \left[\frac{\sigma-(\alpha+3)}{\alpha}\right]^M, \tag{3.26}$$

and

$$2\hat{p}^{''}(v_0) = -\varsigma^2. \tag{3.27}$$

Therefore from (3.9), as $N \to \infty$, recalling that $N = n+1$, we deduce (3.22) from (3.26) and (3.27).

Turning to the polynomial $Q_m(x)$, with $\beta = 2$ in (3.11). Then $\hat{q}'(v) = 0$ when $v = -\frac{1}{4} \pm \frac{\sigma}{4(\alpha+3)}$, so that the integral in (3.10) has two (simple) saddle points inside C at the real axis given by

$$v_1 := -\frac{1}{4} - \frac{\sigma}{4(\alpha+3)}, \qquad v_2 := -\frac{1}{4} + \frac{\sigma}{4(\alpha+3)}. \tag{3.28}$$

Moreover, from (3.11) with $\beta = 2$, we have

$$2\hat{q}^{''}(v_1) = -\varsigma^2, \qquad 2\hat{q}^{''}(v_2) = -\tau^2, \tag{3.29}$$

while

$$e^{-N\hat{q}(v_1)} = (-1)^{M+N} e^{-N\hat{p}(v_0)}, \tag{3.30}$$

$$e^{-N\hat{q}(v_2)} = \frac{1}{9^N 2^M}\{(\alpha+3)[3(\alpha+3)+\sigma]\}^N \left[\frac{\sigma+(\alpha+3)}{\alpha}\right]^M. \tag{3.31}$$

We must choose the branches of $\sqrt{2\hat{q}''(v_1)}$ and $\sqrt{2\hat{q}''(v_2)}$ in accordance with [7], namely,

$$\sqrt{2\hat{q}''(v_1)} = i\varsigma, \qquad \sqrt{2\hat{q}''(v_2)} = -i\tau. \tag{3.32}$$

Then,from (3.12) together with (3.30)-(3.32), we deduce (3.23).

Now we investigate the polynomial $R_s(x)$ with $\beta = 2$.Then $\hat{r}'(v) = 0$ when $v = -\frac{3}{2} \pm \frac{\sigma}{2(\alpha+3)}$. Setting

$$v_3 = -\frac{3}{2} + \frac{\sigma}{2(\alpha+3)}, \tag{3.33}$$

it follows from (3.15) with $\beta = 2$ that

$$e^{-N\hat{r}(v_3)} = \frac{\{(\alpha+3)[3(\alpha+3)+\sigma]\}^N}{9^N 2^{2M+3N}} \left[\frac{\sigma+(\alpha+3)}{\alpha}\right]^M, \tag{3.34}$$

and

$$8\hat{r}^{''}(v_3) = -\tau^2. \tag{3.35}$$

In accordance with [7],we choose

$$\sqrt{2\hat{r}''(v_3)} = \frac{i}{2}\tau. \tag{3.36}$$

Then,from (3.16) together with (3.34) and (3.36), we deduce (3.24).

The results hold uniformly with respect to x on compact subsets of C, because we assume that x is independent of the large parameter n(cf.[7]).

The contour integrals for the polynomials P_n, Q_m, R_s, and E_{nms} can be written in the form

$$P_n(x) = -\frac{3^{s+1} n! e^{2x}}{2^{n+s+1} 2\pi i} \oint_{C_1} \frac{e^{-2xw}}{(1-w)^{n+1} w^{m+1} (w+1/2)^{s+1}} dw, \tag{3.37}$$

$$Q_m(x) = -\frac{3^{s+1}n!}{2^{n+s+1}2\pi i}\oint_{C_0}\frac{e^{-2xw}}{(1-w)^{n+1}w^{m+1}(w+1/2)^{s+1}}dw, \tag{3.38}$$

$$R_s(x) = -\frac{3^{s+1}n!e^{-x}}{2^{n+s+1}2\pi i}\oint_{C_{-\frac{1}{2}}}\frac{e^{-2xw}}{(1-w)^{n+1}w^{m+1}(w+1/2)^{s+1}}dw, \tag{3.39}$$

$$E_{nms}(x) = -\frac{3^{s+1}n!e^{-x}}{2^{n+s+1}2\pi i}\oint_{C}\frac{e^{-2xw}}{(1-w)^{n+1}w^{m+1}(w+1/2)^{s+1}}dw, \tag{3.40}$$

where C_ν is a circle, center at $w=\nu$, radius $r \in (0,1/2)$, and C is a circle, center at the origin,radius $r > 1$. The result for the remainder E_{nms} defined in (1.5) follows from adding up the results in (3.37)-(3.39). So, in fact, we have the same integral representation for the quantities P_n, Q_m, R_s, and E_{nms}, but with different contours of integration. Of course, all contours can be deformed without crossing the poles.

To obtain the asymptotic behavior of the remainder,we cannot simply use the results in (3.22)-(3.24). Adding up these results gives

$$E_{nms}(x) = P_n(x)e^{-3x} + Q_m(x)e^{-x} + R_s(x) \sim 0,$$

which does not give useful information,but is in agreement with the approximating property of the Hermite-Padé method. A better estimate for E_{nms} follows from (3.40), by taking into the account the exponential function when computing the saddle point.

Theorem 4. *Let E_{nms} be defined by (3.40); assume that n, m, s tend to ∞ and $x = o(n+m+s)$. Then*

$$E_{nms} = \frac{(-1)^{m+s}3^{s+1}2^{m+1}n!}{(n+m+s+2)!}e^{-x}x^{n+m+s+2}[1+o(1)]. \tag{3.41}$$

Proof. We write (3.40) in the form

$$E_{nms}(x) = \frac{(-1)^n 3^{s+1}n!e^{-x}}{2^{n+s+1}2\pi i}\oint_C \frac{e^{-2xw}}{w^{n+m+s+3}}t(w)dw, \tag{3.42}$$

where

$$t(w) = \frac{1}{(1-\frac{1}{w})^{n+1}(1+\frac{1}{2w})^{s+1}}.$$

The function $e^{-2xw}/w^{n+m+s+3}$ has a saddle point at $w_0 = -\frac{n+m+s+3}{2x}$,which tends to infinity, and $t(w) = 1 + \frac{2n-s-1}{2w} + \cdots = 1 + o(1)$ in a neighborhood of the saddle point, and in fact on a circle with radius $|w_0|$. This proves the theorem.

4. Some Aspects of Uniform Asymptotic Method

The asymptotic estimates given in (3.22)-(3.24) and (3.41) cannot be used to obtain detailed information on the zeros, because the zeros occur outside compact sets as the orders n, m, s tend to infinity.

As explained at the end of the previous section,the four quantities P_n, Q_m, R_s, and E_{nms} all have the same integral representation

$$\oint e^{-\phi(w)}\frac{dw}{w(1-w)(\frac{1}{2}+w)}, \tag{4.1}$$

with different contours and with

$$\phi(w) = 2zw + n\ln(1-w) + m\ln w + s\ln(\frac{1}{2}+w), \tag{4.2}$$

where we now write z instead of x, to underline that the argument is complex. The saddle points of the integrand are the zeros of the derivative of ϕ. There are three saddle points defined by the cubic equation

$$\psi(w) := 2zw^3 + (m+n+s-z)w^2 - (\frac{m}{2} - \frac{n}{2} + s + z)w - \frac{m}{2} = 0. \tag{4.3}$$

Obvoiusly, $\psi(-\frac{1}{2}) = \frac{3}{4}s, \psi(0) = -\frac{m}{2}, \psi(1) = \frac{3}{2}n$, so the saddle points are real when z is real. When $z > 0$, the saddle points occur in $(-\infty, -\frac{1}{2}), (-\frac{1}{2}, 0)$,and $(0, 1)$.When $z < 0$,the saddle points occur in $(-\frac{1}{2}, 0), (0, 1)$,and $(1, +\infty)$.

For certain complex values of z, two or three saddle points may coincide. It is known from uniform asymptotic (cf.[7] or [10])that Airy functions can describe the asymptotic behaviour of the integrals when two saddle points coincide. It is also known that in the z-plane strings of zeros arise near z-values that make the saddle points coalesce.

In the case $s = 2n$, when the resultant expression of $\psi(w)$ and $\psi'(w)$ equals zero, namely $Res(\psi, \psi', w) = 0$, two saddle points coincide. It is easy to prove that z solves the equation

$$\begin{aligned} &36z^4 + (132n - 36m)z^3 + (189n^2 + 306mn - 27m^2)z^2 + (162n^3 + 270mn^2 \\ &+126m^2n + 18m^3)z + 81n^4 + 324mn^3 + 270m^2n^2 + 84m^3n + 9m^4 = 0. \end{aligned} \tag{4.4}$$

When $s = 2n = 4m$ and

$$z^4 + 23mz^3 + 189m^2z^2 + 637m^3z + 686m^4 = 0 \tag{4.5}$$

three saddle points coincide at

$$w = -\frac{1}{4} \pm \frac{\sqrt{33}}{12}. \tag{4.6}$$

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