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A MIXED FINITE ELEMENT METHOD FOR THE CONTACT PROBLEM IN ELASTICITY *

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Abstract

Based on the analysis of [7] and [10], we present the mixed finite element approximation of the variational inequality resulting from the contact problem in elasticity. The convergence rate of the stress and displacement field are both improved from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|logh|^{1/4})$. If stronger but reasonable regularity is available, the convergence rate can be optimal $\mathcal{O}(h)$.

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Key words: Contact problem, Mixed finite element method.

1. Introduction

Variational inequalities arise mainly from the application of mechanics and physics, such as obstacle problem, unilateral problems, contact mechanics. The contact problem in elasticity is one of the mostly used models in the theory of variational inequality(see [6],[8]). Kikuchi and Oden[7] made a detailed analysis of the contact problem in elasticity with the mathematical model and numerical implementation of the models. Wang[10] improved the duality methods in the mixed finite element approximation. In this paper, we make an improvement of the error estimates from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|logh|^{1/4})$. Under stronger but reasonable regularity assumption, the convergence rate can be optimal $\mathcal{O}(h)$.

Throughout the paper all the notation is followed with that in [7]. The notation of Sobolev spaces and the corresponding semi-norms, norms is taken from [1]. In addition, the frequently used constant C is a generic positive constant whose value may be different under different context. Bold Latin letters like $\boldsymbol{u}, \boldsymbol{v}$ represent for vector quantities and the summation convention of repeated indices over $1, 2$ is adopted. The paper is organized as follows: In section 2, we introduce some notation and present the framework of the contact problem. In section 3, mixed problem is derived and its finite element approximation is given. Finally we show our main results and the proofs in section 4.

2. The Framework of the Contact Problem

The contact problem in elasticity arises from deformable solid mechanics. Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain, and its boundary $\partial\Omega$ consists of three non-overlapping parts Γ_D, Γ_C and Γ_F. The displacement field $\boldsymbol{u}$ of Ω is fixed along Γ_D (Dirichlet condition) with $\text{meas}(\Gamma_D) > 0$ while Γ_C is the contact region subjected to a frictionless foundation. Moreover,

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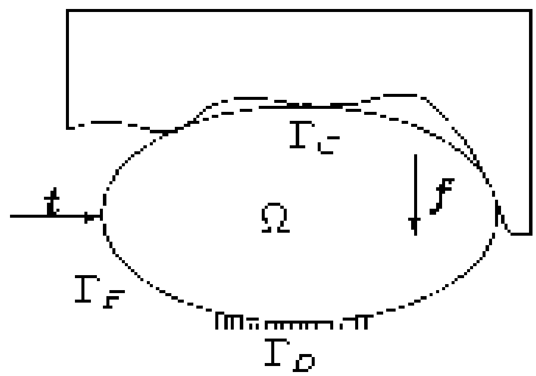

Fig 1.1

Γ_C and Γ_D are not adjacent, and Γ_F is the "glacis" between them with Neumann condition, i.e., the suface traction force $\boldsymbol{t}$ is applied to Γ_F. The body force is denoted by $\boldsymbol{f}$, and $g \in H_{00}^{1/2}(\Gamma_C \cup \Gamma_F)$(see Fig 1.1).

The general continuous setting of the contact problem in elasticity in $\mathbb{R}^2$ can be illustrated as the following mathematical model: to find the displacement field $\boldsymbol{u} \in K$,

$$K = \{\boldsymbol{v} \in \boldsymbol{H^1_{\Gamma_D}(\Omega)} = (H^1_{\Gamma_D}(\Omega))^2 : \boldsymbol{v}\cdot\boldsymbol{n} = v_n \leq g \quad \text{on} \quad \Gamma_C\},$$

such that

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) \geq f(\boldsymbol{v}-\boldsymbol{u}), \qquad \forall \boldsymbol{v} \in K, \tag{2.1}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})dx, \tag{2.2}$$

$$f(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx + \int_{\Gamma_F} \boldsymbol{t}\cdot\boldsymbol{v}ds. \tag{2.3}$$

The notation $H^1_{\Gamma_D}$ stands for the set of functions in $H^1(\Omega)$ which vanish on Γ_D. Besides, $\boldsymbol{\epsilon}$ with $\epsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ denotes the linearized strain tensor field induced by a displacement field $\boldsymbol{v}$ and $\sigma_{ij}(\boldsymbol{v}) = E_{ijkl}\epsilon_{kl}(\boldsymbol{v})$ is the stress tensor with $E = (E_{ijkl})$ denoting the Hooke's tensor of the elastic material. Moreover, the Hooke's tensor E has the following properties :

$$\begin{cases} E_{ijkl} \in L^\infty(\Omega), \quad \|E_{ijkl}\|_{L^\infty} \leq M, \\ E_{ijkl} = E_{klij} = E_{jilk}, \\ E_{ijkl}(\boldsymbol{x})\xi_{ij}\xi_{kl} \geq m\xi_{ij}\xi_{ij}, \quad \text{for all } \boldsymbol{x} \in \Omega,\ \boldsymbol{\xi} = (\xi_{ij}) \in S^2, \end{cases} \tag{2.4}$$

where S^2 denotes the set of the real symmetric matrices of order two. Furthermore, $E = (E_{ijkl})$ is invertible, and its inverse denoted by $C = (C_{ijkl})$ also satisfies similar properties:

$$\begin{cases} C_{ijkl} \in L^\infty(\Omega), \quad \|C_{ijkl}\|_{L^\infty} \leq m_1, \\ C_{ijkl} = C_{klij} = C_{jilk}, \\ C_{ijkl}(\boldsymbol{x})\tau_{ij}\tau_{kl} \geq M_1\tau_{ij}\tau_{ij}, \quad \text{for all } \boldsymbol{x} \in \Omega,\ \boldsymbol{\tau} = (\tau_{ij}) \in S^2, \end{cases} \tag{2.5}$$

with $\tau_{ij} = E_{ijkl}\xi_{kl}$ and $\xi_{ij} = C_{ijkl}\tau_{kl}$. It can be checked that the following differential forms are equivalent to the above variational inequality problem (2.1)-(2.3):

$$\begin{cases} -\boldsymbol{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} = 0 \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{t} \quad \text{on} \quad \Gamma_F, \\ u_n \leq g, \quad \sigma_n \leq 0, \quad \boldsymbol{\sigma}_t = 0, \quad (u_n - g)\sigma_n = 0 \quad \text{on} \quad \Gamma_C, \end{cases} \tag{2.6}$$

where $\sigma_n = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} \cdot \boldsymbol{n}$ and $\boldsymbol{\sigma}_t = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \sigma_n\boldsymbol{n}$.

3. The Mixed Variational Inequality and Its Finite Element Approximation

Notice that the variational inequality (2.1)-(2.3) can also be interpreted as a functional minimization problem, i.e., to find $\boldsymbol{u} \in K$, such that

$$J(\boldsymbol{u}) = \min_{\boldsymbol{v}\in K} J(\boldsymbol{v}) \tag{3.1}$$

where $J(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - f(\boldsymbol{v})$. Using the duality theory in [3], i.e., for a given convex function $s(v)$ defined on a space V, its conjugate function $s'(v')$ is defined by

$$s'(v') = \sup_{v\in V}\langle v, v'\rangle_{V\times V'} - s(v), \quad \forall v' \in V'$$

where V' is the dual space of V, we can easily obtain(see [10])

$$\begin{aligned} J(\boldsymbol{v}) &= \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - f(\boldsymbol{v}) \\ &= \frac{1}{2}\int_\Omega E_{ijkl}\epsilon_{ij}(\boldsymbol{v})\epsilon_{kl}(\boldsymbol{v})dx - f(\boldsymbol{v}) \\ &= \sup_{\boldsymbol{\tau}\in\Lambda}\int_\Omega (\tau_{ij}\epsilon_{ij}(\boldsymbol{v}) - \frac{1}{2}C_{ijkl}\tau_{ij}\tau_{kl})dx - f(\boldsymbol{v}) \end{aligned}$$

with

$$\Lambda = \{\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} \in L^2(\Omega), \quad \tau_{ij} = \tau_{ji}\}.$$

Then, the minimization problem (3.1) induces the following saddle point problem:

$$\inf_{\boldsymbol{v}\in K}\sup_{\boldsymbol{\tau}\in\Lambda}\{\int_\Omega (\tau_{ij}\epsilon_{ij}(\boldsymbol{v}) - \frac{1}{2}C_{ijkl}\tau_{ij}\tau_{kl})dx - f(\boldsymbol{v})\}. \tag{3.2}$$

Following the mixed finite element theory of Brezzi et.al [4], we can show that the corresponding mixed variational inequality is as follows: to find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \Lambda \times K$, such that

$$\begin{cases} a^*(\boldsymbol{\sigma},\boldsymbol{\tau}) - (\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\tau}) = 0 \qquad \forall\boldsymbol{\tau} \in \Lambda, \\ (\boldsymbol{\epsilon}(\boldsymbol{v}-\boldsymbol{u}),\boldsymbol{\sigma}) \geq f(\boldsymbol{v}-\boldsymbol{u}) \qquad \forall\boldsymbol{v} \in K, \end{cases} \tag{3.3}$$

where

$$a^*(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega C_{ijkl}\sigma_{ij}\tau_{kl}dx. \tag{3.4}$$

It is easy to see the above mixed problem has a unique solution due to the ellipticity (2.5) of C_{ijkl} and the following $inf - sup$ condition

$$\sup_{\boldsymbol{\tau}\in\Lambda}\frac{(\boldsymbol{\epsilon}(\boldsymbol{v}),\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{L^2}} \geq \alpha\|\boldsymbol{v}\|_{H^1}, \qquad \forall\boldsymbol{v} \in \boldsymbol{H^1_{\Gamma_D}(\Omega)}. \tag{3.5}$$

which is a direct result of Korn's inequality. Let K_h and Λ_h be the finite element approximation spaces of the mixed variational inequality with respect to the triangulation $\mathcal{J}_h$ and then the discrete mixed finite element approximation is as follows: to find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \Lambda_h \times K_h$, such that

$$\begin{cases} a^*(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\epsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h) = 0 & \forall \boldsymbol{\tau}_h \in \Lambda_h, \\ (\boldsymbol{\epsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h), \boldsymbol{\sigma}_h) \geq f(\boldsymbol{v}_h - \boldsymbol{u}_h) & \forall \boldsymbol{v}_h \in K_h. \end{cases} \tag{3.6}$$

To be exactly, here let $V_h = \{\boldsymbol{v}_h \in \boldsymbol{H}^1_{\boldsymbol{\Gamma_D}}(\boldsymbol{\Omega}) : \boldsymbol{v}_h|_T \in P_1(T), \forall T \in \mathcal{J}_h\}$, and $K_h = \{\boldsymbol{v}_h \in V_h : (\boldsymbol{v}_{hn} - g)(P) \leq 0\}$ with P being the set of the nodes on Γ_C, and $\Lambda_h = \{\boldsymbol{\tau}_h \in \Lambda : \boldsymbol{\tau}_h|_T \in P_0(T), \forall T \in \mathcal{J}_h\}$, then the discrete $inf - sup$ condition can be verified similarly as the continuous one (3.5), i.e., there exists a constant $\beta > 0$, such that

$$\sup_{\boldsymbol{\tau}_h \in \Lambda_h} \frac{(\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{L^2}} \geq \beta \|\boldsymbol{v}_h\|_{H^1} \qquad \forall \boldsymbol{v}_h \in V_h \subset \boldsymbol{H}^1_{\boldsymbol{\Gamma_D}}(\boldsymbol{\Omega}),$$

so the discrete mixed problem (3.6) has a unique solution.

Before we present the improved error estimates for the finite element approximation, we need the following lemmas.

Lemma 3.1[7,10]**.** *Suppose* $(\boldsymbol{\sigma}, \boldsymbol{u})$ *is the solution of the mixed variational problem (3.3) and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ *the solution of the discrete one (3.6) respectively, then for all* $\boldsymbol{\tau}_h \in \Lambda_h, \boldsymbol{v}_h \in K_h$*, and* $\boldsymbol{v} \in K$*, we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \leq (1 + \beta^{-1})\|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1} + m_1\beta^{-1}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} \tag{3.7}$$

$$\begin{aligned} M_1\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L^2} \leq\ & m_1\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{L^2} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1}\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{L^2} \\ & + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} + \int_{\Gamma_C} \sigma_n(v_{hn} - u_{hn} + v_n - u_n)ds. \end{aligned} \tag{3.8}$$

Lemma 3.2. *The following discrete trace inequality holds, for* $1 < p < \infty$*,*

$$\|v\|_{0,p,\partial T} \leq C\{h_T^{-1}\|v\|^p_{0,p,T} + h_T^{p-1}|v|^p_{1,p,T}\}^{1/p}, \qquad \forall v \in W^{1,p}(T), \quad T \in \mathcal{J}_h. \tag{3.9}$$

where C *is a positive constant independent of* v *and* h_T*.*

The proof is same as Stummel's(see [9]).

Next, let us introduce some notation for later use. Let all the line segments $F \subset \partial\Omega$ corresponding to the triangulation $\mathcal{J}_h$ on Γ_C be divided into the following 3 types:

$$\Gamma_C^0 = \{F \subset \Gamma_C : u_n|_F = g\}, \tag{3.10}$$

$$\Gamma_C^- = \{F \subset \Gamma_C : u_n|_F < g\}, \tag{3.11}$$

$$\Gamma_C^{jump} = \{F \subset \Gamma_C : F \cap \Gamma_C^0 \neq \emptyset, F \cap \Gamma_C^- \neq \emptyset\}, \tag{3.12}$$

and

$$\Gamma_C = \Gamma_C^0 \cup \Gamma_C^- \cup \Gamma_C^{jump}. \tag{3.13}$$

Moreover, let

$$P_0^F(v) = \frac{1}{|F|}\int_F v ds, \quad R_0^F(v) = v - P_0^F(v). \tag{3.14}$$

4. Main Results and the Proofs

Now we can establish the error estimates for the finite element approximation to the mixed problem (3.6).

Theorem 4.1. *Suppose* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H^1(\Omega)^2 \times H^2(\Omega)^2$ *is the solution of the mixed variational problem (3.3) and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ *the solution of the discrete one (3.6) respectively,* $\boldsymbol{f} \in L^2(\Omega)^2, \boldsymbol{t} \in H^{-1/2}(\Gamma_F)^2$, *and* $g \in H_{00}^{1/2}(\Gamma_C \cup \Gamma_F) \cap H^{3/2}(\Gamma_C)$ *and that the number of the points on* Γ_C *where the constraints change from binding to non-binding is finite. Then*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} = \mathcal{O}(h|logh|^{1/4}) \tag{4.1}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} = \mathcal{O}(h|logh|^{1/4}) \tag{4.2}$$

where C *is a positive constant depending only on* $\|\boldsymbol{\sigma}\|_{H^1(\Omega)}, |\boldsymbol{u}|_{H^2(\Omega)}$ *and* $|g|_{H^{3/2}(\Gamma_C)}$.

Proof. Let $\boldsymbol{\tau}_I \in \Lambda_h, \boldsymbol{v}_I \in V_h$ represent for the interpolation of $\boldsymbol{\tau}, \boldsymbol{v}$ respectively, then by the standard interpolation error estimates (see [5]), one yields

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_I\|_{L^2} \le Ch|\boldsymbol{\tau}|_{H^1} \tag{4.3}$$

$$\|\boldsymbol{v} - \boldsymbol{v}_I\|_{H^1} \le Ch|\boldsymbol{v}|_{H^2} \tag{4.4}$$

Then choosing $\boldsymbol{\tau}_h = \boldsymbol{\sigma}_I \in \Lambda_h, \boldsymbol{v}_h = \boldsymbol{u}_I \in K_h$ and $\boldsymbol{v} \in K$ such that $v_n = g$ on Γ_C in lemma 3.1, and using Young's inequality, $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \forall \varepsilon > 0$, we then obtain by (2.6)

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}^2 \le Ch^2 |\boldsymbol{\sigma}|_{H^1}^2 + Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} |\boldsymbol{\sigma}|_{H^1} + Ch^2 |\boldsymbol{u}|_{H^2}^2 + \int_{\Gamma_C} \sigma_n (u_{In} - u_{hn}) ds \tag{4.5}$$

Now we consider the last term of the right-hand side of (4.5). By (2.6) and (3.11), it is easy to see $\sigma_n = 0$ on Γ_C^-, so

$$\begin{aligned} \int_{\Gamma_C} \sigma_n (u_{In} - u_{hn}) ds &= \int_{\Gamma_C^0} \sigma_n (u_{In} - u_{hn}) ds + \int_{\Gamma_C^{jump}} \sigma_n (u_{In} - u_{hn}) ds \\ &= \sum_{F \in \Gamma_C^0} \int_F \sigma_n (u_{In} - u_{hn}) ds + \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (u_{In} - u_n) ds \\ &\quad + \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (g - g_I) ds + \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (g_I - u_{hn}) ds \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{4.6}$$

Notice for all $F \in \Gamma_C^0$ with m^F as its midpoint,

$$(u_{In} - u_{hn})(m^F) = (u_{In} - g_I + g_I - u_{hn})(m^F) = (g_I - u_{hn})(m^F) \le 0 \tag{4.7}$$

then by (3.14), (4.7), lemma 3.2 as well as the standard interpolation error estimates, it follows that

$$\begin{aligned} \textstyle\int_F \sigma_n (u_{In} - u_{hn}) ds &= \int_F R_0^F(\sigma_n)(u_{In} - u_{hn}) ds + P_0^F(\sigma_n) \int_F (u_{In} - u_{hn}) ds \\ &= \int_F R_0^F(\sigma_n)(u_{In} - u_{hn}) ds + P_0^F(\sigma_n)(u_{In} - u_{hn})(m^F)|F| \\ &\le \int_F R_0^F(\sigma_n)\{(u_{In} - u_n) + (u_n - u_{hn})\} ds \\ &= \int_F R_0^F(\sigma_n)\{(u_{In} - u_n) + R_0^F(u_n - u_{hn})\} ds \\ &\le \|R_0^F(\sigma_n)\|_{0,F} (\|u_{In} - u_n\|_{0,F} + \|R_0^F(u_n - u_{hn})\|_{0,F}) \\ &\le Ch^{1/2} |\boldsymbol{\sigma}|_{H^1(T)} (Ch^{3/2} |\boldsymbol{u}|_{H^2(T)} + Ch^{1/2} |\boldsymbol{u} - \boldsymbol{u}_h|_{H^1(T)}) \end{aligned}$$

which implies that

$$I_1 = \sum_{F \in \Gamma_C^0} \int_F \sigma_n (u_{In} - u_{hn}) ds \le Ch^2 \|\boldsymbol{\sigma}\|_{H^1} |\boldsymbol{u}|_{H^2} + Ch \|\boldsymbol{\sigma}\|_{H^1} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \tag{4.8}$$

Now we begin to estimate I_2. For all $F \in \Gamma_C^{jump}$, by lemma 3.2 and the standard interpolation error estimates,

$$\begin{aligned} \textstyle\int_F \sigma_n (u_{In} - u_n) ds &\le \|\sigma_n\|_{L^{p'}(F)} \|u_{In} - u_n\|_{L^p(F)} \\ &\le \|\sigma_n\|_{L^{p'}(F)} C h^{1+1/p} |\boldsymbol{u}|_{H^2(T)} \\ &\le C h^{1+1/p} \|\sigma_n\|_{L^{p'}(\Gamma_C)} |\boldsymbol{u}|_{H^2(T)} \end{aligned}$$

where $F \subset \partial T$ and $1/p + 1/p' = 1$. Since $H^{1/2}(\Gamma_C) \hookrightarrow L^{p'}(\Gamma_C)$ for $1 \le p' < +\infty$, and to be exactly (see [2]),

$$\|\sigma_n\|_{L^{p'}(\Gamma_C)} \le C\sqrt{p'} \|\sigma_n\|_{H^{1/2}(\Gamma_C)},$$

thus, by trace theorem we have

$$\|\sigma_n\|_{L^{p'}(\Gamma_C)} \le C\sqrt{p'} \|\boldsymbol{\sigma}\|_{H^1}.$$

Consequently, since the number of the points on Γ_C where constraints change from binding to non-binding is finite,

$$\begin{aligned} I_2 &= \int_{\Gamma_C^{jump}} \sigma_n (u_{In} - u_n) ds = \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (u_{In} - u_n) ds \\ &\le C\sqrt{p'} h^{-1/p'} h^2 \|\boldsymbol{\sigma}\|_{H^1} \sum_{\substack{F \subset \partial T \\ F \in \Gamma_C^{jump}}} |\boldsymbol{u}|_{H^2(T)} \\ &\le C\sqrt{p'} h^{-1/p'} h^2 \|\boldsymbol{\sigma}\|_{H^1} |\boldsymbol{u}|_{H^2}. \end{aligned}$$

Set $p' = |logh|$, it can be easily seen that

$$I_2 = \int_{\Gamma_C^{jump}} \sigma_n (u_{In} - u_n) ds \le Ch^2 |logh|^{1/2} \|\boldsymbol{\sigma}\|_{H^1} |\boldsymbol{u}|_{H^2} \tag{4.9}$$

Moreover, similarly as above, by the standard interpolation error estimates, for all $F \in \Gamma_C^{jump}$,

$$\int_F \sigma_n (g - g_I) ds \le \|\sigma_n\|_{L^{p'}(F)} \|g - g_I\|_{L^p(F)} \le C\sqrt{p'} \|\boldsymbol{\sigma}\|_{H^1} C h^{1+1/p} |g|_{H^{3/2}(F)}$$

Let $p' = |logh|$ again, it easily follows that

$$\begin{aligned} I_3 &= \int_{\Gamma_C^{jump}} \sigma_n (g - g_I) ds = \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (g - g_I) ds \\ &\le C\sqrt{p'} h^{-1/p'} h^2 \|\boldsymbol{\sigma}\|_{H^1} \sum_{F \in \Gamma_C^{jump}} |g|_{H^{3/2}(F)} \\ &\le Ch^2 |logh|^{1/2} \|\boldsymbol{\sigma}\|_{H^1} |g|_{H^{3/2}(\Gamma_C)} \end{aligned} \tag{4.10}$$

Now, we begin to estimate I_4. Note that $u_h \in K_h$, then $(u_{hn} - g_I)(P) = (u_{hn} - g)(P) \le 0$ with P being the set of all the nodes on Γ_C. By the linearity of u_h and g_I, we have $u_{hn} - g_I \le 0$ on

Γ_C, together with (2.6), one gets

$$I_4 = \int_{\Gamma_C^{jump}} \sigma_n (g_I - u_{hn}) ds \leq 0 \tag{4.11}$$

Now combining (4.5), (4.6) and (4.8)-(4.11), one gets

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}^2 \leq\ & Ch^2 \|\boldsymbol{\sigma}\|_{H^1}^2 + Ch \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \|\boldsymbol{\sigma}\|_{H^1} + Ch^2 |\boldsymbol{u}|_{H^2}^2 \\ & + Ch^2 |logh|^{1/2} \|\boldsymbol{\sigma}\|_{H^1} (|u|_{H^2} + |g|_{H^{3/2}(\Gamma_C)})\end{aligned} \tag{4.12}$$

In addition, let $\boldsymbol{v}_h = \boldsymbol{u}_I \in K_h$ in lemma 3.1, we have

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \leq Ch|u|_{H^2} + C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} \tag{4.13}$$

Finally, by (4.12) and (4.13) as well as the Young's inequality, the proof is completed.

Theorem 4.2. *Under the assumptions of theorem 4.1 and the assumption that* $\sigma_n \in L^\infty(\Gamma_C)$*, the following error estimates hold:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} = \mathcal{O}(h), \tag{4.14}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} = \mathcal{O}(h), \tag{4.15}$$

where C *is a positive constant depending only on* $\|\boldsymbol{\sigma}\|_{H^1}, \|\sigma_n\|_{L^\infty(\Gamma_C)}, |\boldsymbol{u}|_{H^2(\Omega)}$ *and* $|g|_{H^{3/2}(\Gamma_C)}$.

Proof. Following the proof of theorem 4.1, in order to improve the convergence rate form $\mathcal{O}(h|logh|^{1/2}$ to $\mathcal{O}(h)$, we only need to re-estimate the terms I_2 and I_3 which lost the optimal convergence rate. By lemma 3.2 for $p = 2$ and the finite number of points on Γ_C where the constraints form binding to non-binding,

$$\begin{aligned} I_2 = & \int_{\Gamma_C^{jump}} \sigma_n (u_{In} - u_n) ds \leq \|\sigma_n\|_{L^\infty(\Gamma_C)} \sum_{F \in \Gamma_C^{jump}} \int_F (u_{In} - u_n) ds \\ & \leq \|\sigma_n\|_{L^\infty(\Gamma_C)} h^{1/2} \sum_{F \in \Gamma_C^{jump}} \|u_{In} - u_n\|_{L^2(F)} \\ & \leq h^{1/2} \|\sigma_n\|_{L^\infty(\Gamma_C)} \sum_{\substack{F \subset \partial T \\ F \in \Gamma_C^{jump}}} Ch^{3/2} |\boldsymbol{u}|_{H^2(T)} \\ & \leq Ch^2 \|\sigma_n\|_{L^\infty(\Gamma_C)} |\boldsymbol{u}|_{H^2} \end{aligned} \tag{4.16}$$

In addition, by standard interpolation error estimates,

$$\begin{aligned} I_3 \ = & \int_{\Gamma_C^{jump}} \sigma_n (g - g_I) ds = \sum_{F \in \Gamma_C^{jump}} \int_F \sigma_n (g - g_I) ds \\ & \leq \|\sigma_n\|_{L^\infty(\Gamma_C)} h^{1/2} \sum_{F \in \Gamma_C^{jump}} \|g - g_I\|_{L^2(F)} \\ & \leq h^{1/2} \|\sigma_n\|_{L^\infty(\Gamma_C)} \sum_{F \in \Gamma_C^{jump}} Ch^{3/2} |g|_{H^{3/2}(F)} \\ & \leq Ch^2 \|\sigma_n\|_{L^\infty(\Gamma_C)} |g|_{H^{3/2}(\Gamma_C)} \end{aligned} \tag{4.17}$$

Combining (4.5), (4.6), (4.8), (4.11), (4.16) and (4.17) together with Young's inequality, the proof is completed.

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