Journal of Computational Mathematics, Vol.23, No.5, 2005, 503–512.

PIECEWISE SEMIALGEBRAIC SETS *1)

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Abstract

Semialgebraic sets are objects which are truly a special feature of real algebraic geometry. This paper presents the piecewise semialgebraic set, which is the subset of R^n satisfying a boolean combination of multivariate spline equations and inequalities with real coefficients. Moreover, the stability under projection and the dimension of C^μ piecewise semialgebraic sets are also discussed.

Mathematics subject classification: 14C17, 14G40, 41A15, 41A46, 65D07, 65D10.
Key words: Algebraic geometry, Semialgebraic geometry, Tarski-Seidenberg Principle, Multivariate splines, Piecewise semialgebraic sets.

1. Introduction

Semialgebraic sets and semialgebraic functions(i.e. functions having semialgebraic graph) are objects which are truly a special feature of real algebraic geometry. This class of sets has remarkable stability properties, of which the most important is stability under projection. Practically all the useful constructions with semialgebraic sets have also a very pleasant topological structure: they have good stratifications. Furthermore, semialgebraic functions grow in a very well controlled way. The recent researches of the semialgebraic sets refer to [1,2,3,4,8]

A piecewise algebraic variety(curve) is the zero set of some multivariate(bivariate) splines. As the generalization of the classical algebraic variety(curve), the piecewise algebraic variety(curve) is not only very important for several practical areas such as CAD(Computer-Aided Design), CAM(Computer-Aided Manufacture), CAE(Computer-Aided Engineering), and image processing, but also a useful tool for studying other subjects[13,14,17]. Wang et al. have done a lot of work concerning the piecewise algebraic varieties and curves[7,9,10,12−22].

A piecewise semialgebraic subset of R^n is the subset of $(x_1, \cdots, x_n)$ in R^n satisfying a boolean combination of spline equations and inequalities with real coefficients. It is the generalization of the semialgebraic set. The piecewise algebraic variety(curve) is the degeneration of the piecewise semialgebraic set.

This paper deals with piecewise semialgebraic sets over real closed field R. First of all, we present the piecewise algebraic varieties and piecewise semialgebraic sets. In section 3, we discuss the stability of the piecewise semialgebraic sets under projection and several applications of this property are also investigated. The dimension of piecewise semialgebraic sets are discussed in section 4.

2. Piecewise Algebraic Varieties and Piecewise Semialgebraic Sets

Let R be a real closed field. Using finite number of hypersurfaces in R^n, we partition R^n into finite number of simply connected regions, which are called the partition cells. Denote by Δ the partition of the region R^n which is the union of all partition cells $\delta_1, \cdots, \delta_T$ and their

* Received April 16, 2004; final revised May 18, 2005.
1) Supported by the National Natural Science Foundation of China (Grant Nos. 10271022 and 60373093).

edges $S_1, \cdots, S_E$. $S_1, \cdots, S_E$ are the algebraic hypersurfaces or algebraic families of dimension $\leq n$ which are called the partition net surfaces.

Denote by $P(\Delta)$ the piecewise polynomial ring with respect to partition Δ on R^n as follows

$$P(\Delta) := \{f | f|_{\delta_i} = f_i \in R[x_1, \cdots, x_n], i = 1, 2, \cdots, T\}.$$

For integer $\mu \geq 0$, let

$$S^\mu(\Delta) := \{f | f \in C^\mu(\Delta) \cap P(\Delta)\}.$$

$S^\mu(\Delta)$ is called the C^μ *spline ring*. In fact, it is a Nöther ring[13,14].

Definition 2.1. [10,14] *A subset $X \subset R^n$ is called a C^μ* piecewise algebraic variety *if there exist $f_1, f_2, \cdots, f_r \in S^\mu(\Delta)$, then*

$$X = Z(f_1, f_2, \cdots, f_r) = \{x \in R^n | f_i(x) = 0, i = 1, 2, \cdots, r\}. \tag{1}$$

Theorem 2.1. [10,14] *Let $X \subset R^n$ be a C^μ piecewise algebraic variety. Then the set*

$$I(X) := \{f \in S^\mu(\Delta) | f(x) = 0, \forall x \in X\} \tag{2}$$

yield an ideal of $S^\mu(\Delta)$.

Theorem 2.2. [10,14] *The union of two C^μ piecewise algebraic varieties is a C^μ piecewise algebraic variety. The intersection of many C^μ piecewise algebraic varieties is a C^μ piecewise algebraic variety. The empty set and R^n are C^μ piecewise algebraic varieties.*

Definition 2.2. [10,14] *For any partition Δ of R^n. The topology $\Im^\mu_\Delta = \{R^n \setminus X | X \subset R^n$ is a C^μ piecewise algebraic variety $\}$ is called C^μ* Zariski topology *by taking the closed subsets as the C^μ piecewise algebraic varieties.*

Definition 2.3. [10,14] *Let $X \subset R^n$ be a nonempty C^μ piecewise algebraic variety. If X can be expressed as the union of two nonempty proper closed sets X_1 and X_2, then X is called reducible, otherwise X is called irreducible.*

Theorem 2.3. [10,14]

(1) $\Im^\mu_\Delta$ is a Nöther topology, that is, any irreducible descent closed set sequence $X_1 \supset X_2 \supset \cdots$ is a finite sequence.

(2) Every C^μ piecewise algebraic variety can be expressed as the union of finite number of irreducible C^μ piecewise algebraic varieties.

(3) Let $X \subset R^n$ be a C^μ piecewise algebraic variety. Then X is irreducible if and only if $I(X)$ is a prime ideal of $S^\mu(\Delta)$.

(4) A maximal ideal of $S^\mu(\Delta)$ corresponds to a minimal irreducible closed subset of C^μ piecewise algebraic varieties.

Definition 2.4. *A subset of R^n satisfying*

$$\cup_i \cap_j \{x \in R^n | f_{ij}(x) *_{ij} \quad 0\}$$

is called a C^μ piecewise semialgebraic set, *where $f_{ij} \in S^\mu(\Delta)$ and the $*_{ij}$ are either $=$ or $\neq$ or $>$ or $\geq$.*

In other words, for every partition Δ on R^n, the C^μ piecewise semialgebraic subsets of R^n form the smallest class PSA_n^μ of subsets of R^n such that:

(1) If $p \in S^\mu(\Delta)$, then $\{x \in R^n | p(x) = 0\} \subseteq PSA_n^\mu$ and $\{x \in R^n | p(x) > 0\} \subseteq PSA_n^\mu$.

(2) If $A \subseteq PSA_n^\mu$ and $B \subseteq PSA_n^\mu$, then $A \cup B$, $A \cap B$ and $R^n \setminus A$ are in PSA_n^μ.

The fact that a subset of R^n is piecewise semialgebraic does not depend on the choice of affine coordinates.

Proposition 2.1. *Every C^μ piecewise semialgebraic subset of R^n is the union of finitely many C^μ piecewise semialgebraic subsets of the form*

$$\{x \in R^n | P(x) = 0, Q_1(x) > 0, \cdots, Q_l(x) > 0\},$$

where $l \in \mathbb{N}$ and $P, Q_1, \cdots, Q_l \in S^\mu(\Delta)$.

Proof. Check that the class of finite unions of such subsets satisfies the above properties (1) and (2).

We give now some examples of piecewise semialgebraic sets.

- The piecewise semialgebraic subsets of R are the unions of finitely many points and open intervals.
- A C^μ piecewise algebraic variety of R^n is C^μ piecewise semialgebraic.
- If A is a piecewise semialgebraic subset of R^n and $L \subset R^n$ is a piecewise linear algebraic curve, then $L \cap A$ is the union of finitely many points and open intervals.
- The piecewise semialgebraic sets can take various and pleasant shapes, like Fig. 1 which is a C^1 piecewise semialgebraic set defined by $\{(x, y) \in \mathbb{R}^2 | f(x, y) <= 0 \text{ and } g(x, y) >= 0\}$, where $f(x, y), g(x, y) \in S^1(\Delta)$ are defined as follows and using two lines $l_1 : x + 1 = 0$ and $l_2 : x - 1 = 0$ partition $\mathbb{R}^2$ into three cells d_1, d_2, d_3. Obviously, the pleasant shape cannot be generated by any two quadratic polynomials.

$$f(x,y) = \begin{cases} (x+2)^2 + y^2 - 2.25, (x,y) \in d_1 \\ y^2 - x^2 + 0.25, (x,y) \in d_2 \\ (x-2)^2 + y^2 - 2.25, (x,y) \in d_3 \end{cases}$$

$$g(x,y) = \begin{cases} (x+1)^2 + y^2 - 1, (x,y) \in d_1 \\ y^2 - x^2 + 1, (x,y) \in d_2 \\ (x-1)^2 + y^2 - 1, (x,y) \in d_3 \end{cases}$$

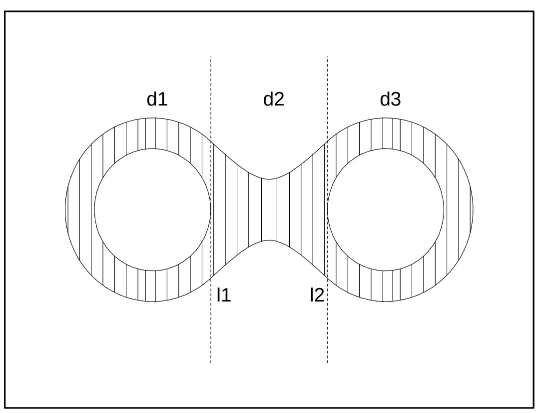

Fig. 1. A piecewise semialgebraic set generated by two C^1 quadratic splines.

3. Projection of Piecewise Semialgebraic Sets

We have seen that the class of all C^μ piecewise semialgebraic subsets is closed under finite unions and intersections, taking complement. It is also closed under projection if the partition Δ on R^n satisfies some conditions.

Definition 3.1. *If $f \in S^\mu(\Delta)$ in n variables $(x_1, \cdots, x_n)$, then the degree of f is defined as*

$$\deg(f) = \max\{\deg(f|_{\delta_i}), i = 1, \cdots, T\},$$

where $\deg(f|_{\delta_i})$ is the total degree of $f|_{\delta_i}$, and the degree of f with respect to x_1 is defined as

$$\deg_{x_1}(f) = \max\{\deg_{x_1}(f|_{\delta_i}), i = 1, \cdots, T\},$$

where $\deg_{x_1}(f|_{\delta_i})$ is the degree of $f|_{\delta_i}$ with respect to x_1.

Definition 3.2. *Taking $-\infty < x_1^{(1)} < \cdots < x_{m_1}^{(1)} < +\infty, \cdots, -\infty < x_1^{(n)} < \cdots < x_{m_n}^{(n)} < +\infty$, and making use hypersurface family $x_1 - x_{i_1}^{(1)} = 0, i_1 = 1, \cdots, m_1; \cdots; x_n - x_{i_n}^{(n)} = 0, i_n = 1, \cdots, m_n$ to yield a partition on R^n, called hyper-rectangular partition, denoted by Δ_{HR}.*

Definition 3.3. *Let $f_1(x_1, Y), f_2(x_1, Y), \cdots, f_s(x_1, Y) \in S^\mu(\Delta)$ in $n+1$ variables, where $Y = (x_2, \cdots, x_{n+1})$, $d = \max\{\deg(f_i), i = 1, \cdots, s\}$, and $d_j = \max\{\deg_{x_j}(f_i), i = 1, \cdots, s\}, j = 1, \cdots, n+1$. If there exists $j \in \{1, \cdots, n+1\}$ such that $d_j > d - (\mu + 1)$, then we say that the sequence $\{f_1, \cdots, f_s\}$ satisfies Degree-condition with respect to variable x_j.*

Definition 3.4. *Let $f_1(x_1, Y), f_2(x_1, Y), \cdots, f_s(x_1, Y) \in S^\mu(\Delta)$ in $n+1$ variables, where $Y = (x_2, \cdots, x_{n+1})$. Suppose $\deg_{x_1}(f_s) = \max\{\deg_{x_1}(f_i), i = 1, \cdots, s\}$. If the piecewise polynomials $g_1, g_2, \cdots, g_s$ satisfy*

$$f_s = f_i h_i + g_i, i = 1, \cdots, s-1, \tag{3}$$

$$f_s = f_s^{'} h_s + g_s, \tag{4}$$

where $f_s^{'}$ is derivative of f_s in x_1, $\deg_{x_1}(g_i) < \deg_{x_1}(f_s)$, and there exists a partition Δ^{x_1} on R^n such that $g_i \in S^\mu(\Delta^{x_1}), i = 1, \cdots, s$ regarding x_1 as a parameter. We say that the sequence $\{f_1, \cdots, f_s\}$ satisfies SR-condition with respect to variable x_1 and g_i is the remainder of the Euclidean-type division of f_s by f_i.

Lemma 3.1. *Let Δ_{HR} be a hyper-rectangular partition on R^{n+1}. If $f_1(x_1, Y)$, $f_2(x_1, Y)$, $\cdots$, $f_s(x_1, Y) \in S^\mu(\Delta_{HR})$ in $n+1$ variables, where $Y = (x_2, \cdots, x_{n+1})$. If the sequence $\{f_1, \cdots, f_s\}$ satisfies Degree-condition respect to variable x_1, then it must satisfy SR-condition with respect to variable x_1.*

Proof. Without loss of generality, we assume that Δ_{HR} is a partition on R^3, the sequence $\{f_1, \cdots, f_s\}$ in three variables (x, y, z), and $\deg_x(f_s) = \max\{\deg_x f_m, m = 1, \cdots, s\}$. Obviously, we can get a rectangular partition Δ_{HR}^x on R^2 regarding the variable x as a parameter. Suppose that four arbitrary adjacent cells $\delta_i, \delta_j, \delta_k$, and δ_l of Δ_{HR}^x share the vertex V(without loss of generality, we assume that V be the origin O) as the common vertex in the counter-clockwise order respectively(as Fig. 2 shown), and the representation of the spline f_m on the cells be

$$f_{m_i} = f_m|_{\delta_i}, f_{m_j} = f_m|_{\delta_j}, f_{m_k} = f_m|_{\delta_k}, f_{m_l} = f_m|_{\delta_l}, m = 1, \cdots, s.$$

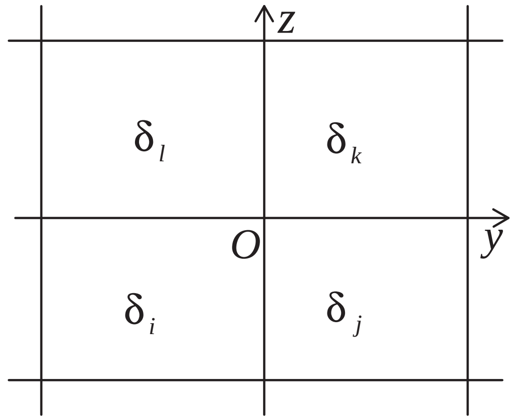

Fig. 2. The cells $\delta_i, \delta_j, \delta_k, \delta_l$ with the common vertex O.

Let the smoothing cofactors of $f_m (m \in \{1, \cdots, s\})$ corresponding to the above four net segments be $q_m^{ij}, q_m^{jk}, q_m^{kl}, q_m^{lj}$ respectively, i.e.,

$$f_{m_j} = f_{m_i} + q_m^{ij} z^{\mu+1}, f_{m_k} = f_{m_j} + q_m^{jk} y^{\mu+1}, \tag{5}$$
$$f_{m_l} = f_{m_k} + q_m^{kl} z^{\mu+1}, f_{m_i} = f_{m_l} + q_m^{li} y^{\mu+1}. \tag{6}$$

According to the conformality condition at point O, we have

$$(q_m^{ij} + q_m^{kl}) z^{\mu+1} + (q_m^{jk} + q_m^{li}) y^{\mu+1} \equiv 0, m = 1, \cdots, s.$$

Since $z^{\mu+1}$ and $y^{\mu+1}$ are prime to each other, therefore we find quantity relations between smoothing cofactors

$$q_m^{ij} + q_m^{kl} = y^{\mu+1} d_m, \tag{7}$$
$$q_m^{jk} + q_m^{li} = -z^{\mu+1} d_m, \tag{8}$$

where d_m is a polynomial and $m = 1, \cdots, s$. Let g_{m_i} be the remainder of the Euclidean division of f_{s_i} by f_{m_i} respect to x, i.e.,

$$f_{s_i} = f_{m_i} h_{m_i} + g_{m_i}, m \in \{1, \cdots, s-1\},$$

where $\deg_x(g_{m_i}) < \deg_x(f_{s_i}) \leq \deg_x(f_s)$. Using the equations (5), (6), (7) and (8), we have

$$\begin{aligned} f_{s_j} &= f_{s_i} + q_s^{ij} z^{\mu+1} \\ &= f_{m_i} h_{m_i} + g_{m_i} + q_s^{ij} z^{\mu+1} \\ &= (f_{m_j} - q_m^{ij} z^{\mu+1}) h_{m_i} + g_{m_i} + q_s^{ij} z^{\mu+1} \\ &= f_{m_j} h_{m_i} + g_{m_i} + (q_s^{ij} - q_m^{ij}) z^{\mu+1} \\ &= f_{m_j} h_{m_j} + g_{m_i} + g^{'} z^{\mu+1}, \end{aligned} \tag{9}$$

$$\begin{aligned} f_{s_k} &= f_{s_j} + q_s^{jk} y^{\mu+1} \\ &= f_{m_j} h_{m_i} + g_{m_i} + g^{'} z^{\mu+1} + q_s^{jk} y^{\mu+1} \\ &= (f_{m_k} - q_m^{jk} y^{\mu+1}) h_{m_i} + g_{m_i} + g^{'} z^{\mu+1} + q_s^{jk} y^{\mu+1} \\ &= f_{m_k} h_{m_i} + g_{m_i} + g^{'} z^{\mu+1} + (q_s^{jk} - q_m^{jk}) y^{\mu+1} \\ &= f_{m_k} h_{m_k} + g_{m_i} + g^{'} z^{\mu+1} + g^{''} y^{\mu+1}, \end{aligned} \tag{10}$$

$$\begin{aligned} f_{s_l} &= f_{s_k} + q_s^{kl} z^{\mu+1} \\ &= f_{m_k} h_{m_i} + g_{m_i} + g^{'} z^{\mu+1} + g^{''} y^{\mu+1} + q_s^{kl} z^{\mu+1} \\ &= (f_{m_l} - q_m^{kl} z^{\mu+1}) h_{m_i} + g_{m_i} + g^{'} z^{\mu+1} + g^{''} y^{\mu+1} + q_s^{kl} z^{\mu+1} \\ &= f_{m_l} h_{m_i} + g_{m_i} + g^{''} y^{\mu+1} + (q_s^{kl} - q_m^{kl} + g^{'}) z^{\mu+1} \\ &= f_{m_l} h_{m_l} + g_{m_i} + (g^{''} + d^{'} z^{\mu+1}) y^{\mu+1}, \end{aligned} \tag{11}$$

where $h_{m_j} = h_{m_k} = h_{m_l} = h_{m_i}, g^{'} = q_s^{ij} - q_m^{ij}, g^{''} = q_s^{jk} - q_m^{jk}$, and $d^{'} z^{\mu+1} = q_s^{kl} - q_m^{kl} + g^{'}$(using equation (7)). Since the sequence $\{f_1, \cdots, f_s\}$ satisfies Degree-condition respect to variable x, we have $\deg_x(g^{'}), \deg_x(g^{''})$ and $\deg_x(d^{'})$ are lower than $\deg_x(f_s)$. Then degrees of $g_{m_i}, g_{m_i} + g^{'} z^{\mu+1}, g_{m_i} + g^{'} z^{\mu+1} + g^{''} y^{\mu+1}$, and $g_{m_i} + (g^{''} + d^{'} z^{\mu+1}) y^{\mu+1}$ are lower than the degree of f_s respect with x respectively. Let

$$g_{m_j} = g_{m_i} + g^{'} z^{\mu+1}, g_{m_k} = g_{m_i} + g^{'} z^{\mu+1} + g^{''} y^{\mu+1}, g_{m_l} = g_{m_i} + (g^{''} + d^{'} z^{\mu+1}) y^{\mu+1}.$$

Thus we have $g_{m_e}, e = i, j, k, l$, satisfy conformality condition[12,14] on the $St(O)$(the star region of O). Furthermore, we have

$$f_s = f_m h_m + g_m,$$

where $g_m \in S^\mu(\Delta_{HR}^x)$ and $deg_x(f_s) > deg_x(g_m), m = 1, \cdots, s-1$. Similarity, we can get

$$f_s = f_s^{'} h_s + g_s,$$

where $g_s \in S^\mu(\Delta_{HR}^x)$ regarding x as a parameter and $deg_x(f_s) > deg_x(g_s)$. By the Definition 3.4, the spline sequence $\{f_1, \cdots, f_s\}$ satisfies SR-condition with respect to variable x. By the same idea, for hyper-rectangular partition on R^{n+1}, we get $\{f_1, \cdots, f_s\}$ satisfies SR-condition with respect to variable x_1.

Theorem 3.1. (Tarski-Seidenberg Principle) *Let Δ_{HR} be a hyper-rectangular partition on R^{n+1}, and $f_1(x_1, Y), f_2(x_1, Y), \cdots, f_s(x_1, Y) \in S^\mu(\Delta_{HR})$ be a sequence in $n+1$ variables, where $Y = (x_2, \cdots, x_{n+1})$. If the sequence $\{f_1, \cdots, f_s\}$ satisfies Degree-condition respect to variable x_1, then there exists a sequence of splines $W \subset S^\mu(\Delta'_{HR})$, where $\Delta_{HR}^{'}$ is a hyper-rectangular partition of R^n, and a boolean combination $B(Y)$ of spline equations and inequations generated by W, such that, for every $y \in R^n$, the system*

$$\begin{cases} f_1(x_1, y) *_1 & 0 \\ \cdots & \\ f_s(x_1, y) *_s & 0 \end{cases}$$

has a solution x in R if and only if $B(y)$ holds true in R^n.

Proof. Since Δ_{HR} is a hyper-rectangular partition on R^{n+1} and the sequence $\{f_1, f_2, \cdots, f_s\} \subset S^\mu(\Delta_{HR})$, then the sequence $\{f_1, f_2, \cdots, f_s\}$ satisfies SR-condition with respect to variable x_1 by Lemma 3.1. Thus for parameter $x_1 \in R$, there exists a partition $\Delta_{HR}^{x_1}$ on R^n such that

$f_s^{'}, g_i \in S^\mu(\Delta_{HR}^{x_1})$ where $f_s^{'}$ is derivative of f_s with respect to x_1 and g_i is the remainder of the Euclidean-type division of f_s by f_i. Using the same idea of the proof of Tarski-Seidenberg Principle theorem of semialgebraic sets in [2,6], the result is held.

Theorem 3.2. (Tarski-Seidenberg -second form-) *Suppose Δ_{HR} be a hyper-rectangular partition on R^{n+1}. Let $A \in PSA^\mu_{n+1}$ generated by the sequence $\{f_1, \cdots, f_s\} \subset S^\mu(\Delta_{HR})$, and $\pi : R^{n+1} \longrightarrow R^n$, the projection on the first n coordinates. If the sequence $\{f_1, \cdots, f_s\}$ satisfies Degree-condition respect to variable x_{n+1}, then there exists a hyper-rectangular partition of R^n, such that $\pi(A) \in PSA^\mu_n$.*

Proof. Since A is the union of finitely many subsets of the form

$$\{x = (x_1, \cdots, x_{n+1}) \in R^{n+1} | P(x) = 0, Q_1(x) > 0, \cdots, Q_l(x) > 0\},$$

where $l \in \mathbb{N}$ and $P(x), Q_1(x), \cdots, Q_l(x) \in S^\mu(\Delta_{HR})$, we may assume that A itself is of this form. It follows from the Tarski-Seidenberg Principle(Theorem 3.1) that there exists a sequence of splines $W \subset S^\mu(\Delta'_{HR})$, where $\Delta^{'}_{HR}$ is a hyper-rectangular partition of R^n, and a boolean combination $B(x_1, \cdots, x_n)$ of spline equations and inequations generated by W, such that

$$\pi(A) = \{(x_1, \cdots, x_n) \in R^n | \exists x_{n+1} \in R, (x_1, \cdots, x_n, x_{n+1}) \in A\}$$

is the set of $(x_1, \cdots, x_n)$ which satisfy $B(x_1, \cdots, x_n)$. This means that $\pi(A) \in PSA^\mu_n$.

Corollary 3.1. *Suppose Δ_{HR} be a hyper-rectangular partition on R^{n+k}. If $A \in PSA^\mu_{n+k}$ and the spline sequence which generate the piecewise semialgebraic set A satisfies Degree-condition respect to variables $x_{n+1}, \cdots, x_{n+k}$, then its image by the projection on the space of the first n coordinates is a C^μ piecewise semialgebraic subset of R^n.*

Proof. The statement is easily obtained by induction on k by using Theorem 3.2.

Corollary 3.2. *Suppose R^n be considered with Euclidean topology and Δ_{HR} be a hyper-rectangular partition on R^n. If $A \in PSA^\mu_n$, its closure and interior are again C^μ piecewise semialgebraic.*

Proof. The closure of A is

$$clos(A) = \{x \in R^n | \forall \varepsilon \in R, \varepsilon > 0 \Rightarrow \exists y \in A, \|x - y\|^2 < \varepsilon^2\}$$

and can be written as

$$clos(A) = R^n \backslash (\pi_1(\{(x, \varepsilon) \in R^n \times R | \varepsilon > 0\} \backslash \pi_2(B))),$$

where

$$B = \{(x, \varepsilon, y) \in R^n \times R \times R^n | y \in A, \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2\},$$

$\pi_1(x, \varepsilon) = x$ and $\pi_2(x, \varepsilon, y) = (x, \varepsilon)$. Since $\sum_{i=1}^n (x_i - y_i)^2 < \varepsilon^2$ is a polynomial and Δ_{HR} is a hyper-rectangular partition on R^n, then there must exist a hyper-rectangular partition on R^{2n+1} such that $B \in PSA^\mu_{2n+1}$. Obviously, the spline sequences which generate B and $\{(x, \varepsilon) \in R^n \times R | \varepsilon > 0\} \backslash \pi_2(B)$ satisfy Degree-condition respect to last n variables and ε respectively. By Theorem 3.2, $clos(A)$ is C^μ piecewise semialgebraic. By taking complements, one sees that the interior is also C^μ piecewise semialgebraic. □

4. Dimension of Piecewise Semialgebraic Sets

We shall compare the dimension of piecewise semialgebraic sets, defined via the dimension of semialgebraic sets, with the dimension of piecewise algebraic varieties. We recall the results concerning the dimension of piecewise algebraic varieties that we need[21].

The dimension of an algebraic variety V is, by definition, the Krull dimension of its coordinate ring $A(V)$, i.e. the maximal length of chains of prime ideals in $A(V)$: $dim(A(V))$ is the maximum of d such that there exist prime ideals $p_0, p_1, \cdots, p_d$ of $A(V)$, with $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_d$.

Definition 4.1. [14] *Let $X \subset R^n$ be a C^μ piecewise algebraic variety. Then its dimension is*

$$\dim(X) := \max_{1 \le i \le T} \dim(clos_{Zar}(X \cap \delta_i)), \tag{12}$$

where $clos_{Zar}(X \cap \delta_i)$ is the Zariski closure of $X \cap \delta_i$ and $\dim(clos_{Zar}(X \cap \delta_i)) = -1$ if $X \cap \delta_i = \phi$.

Since the dimension of C^μ piecewise algebraic varieties described above is not convenient for application, we present definition of the krull dimension of C^μ piecewise algebraic varieties.

Definition 4.2. [21] *Let $X \subset R^n$ be a C^μ piecewise algebraic variety. If X is a topological space, the* krull dimension *of X is the supremum of all integers m such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_m$ of distinct irreducible closed subsets of X, denoted by $kr(X)$.*

Theorem 4.1. [21] *If $X \subset R^n$ be a C^μ piecewise algebraic variety, then* $\dim(X) = kr(X)$.

Now we present the the dimension of piecewise semialgebraic sets, defined via the dimension of semialgebraic sets.

Definition 4.3. *Let $S \in PSA_n^\mu$. Then its dimension is*

$$\dim(S) := \max_{1 \le i \le T} \dim(clos(S \cap \delta_i)), \tag{13}$$

where $clos(S \cap \delta_i)$ is the Euclidean closure of $S \cap \delta_i$, $\dim(clos(S \cap \delta_i))$ is dimension of semialgebraic set $clos(S \cap \delta_i)$, and $\dim(clos(S \cap \delta_i)) = -1$ if $S \cap \delta_i = \phi$.

For $S \in PSA_n^\mu$, we denote by $I(S) \subset S^\mu(\Delta)$ the subset of splines as

$$I(S) = \{P \in S^\mu(\Delta) | P(x) = 0, \forall x \in S\}.$$

It is clearly that $I(S)$ is an ideal of $S^\mu(\Delta)$. The quotient ring $A(S) = S^\mu(\Delta)/I(S)$ is called the coordinate ring of S. If $S \in PSA_n^\mu$, then $Z(I(S))$ is the smallest C^μ piecewise algebraic variety of R^n containing S. It is called the *Zariski closure* of S, and it will be denoted by $\overline{S}^Z$.

Theorem 4.2. *Let $S \in PSA_n^\mu$. Its dimension as a C^μ piecewise semialgebraic set is equal to the dimension, as a C^μ piecewise algebraic variety, of its Zariski closure $\overline{S}^Z$. In particular, if $V \in PSA_n^\mu$ is a C^μ piecewise algebraic variety, its dimension as a piecewise semialgebraic set is equal to its dimension as a piecewise algebraic variety(i.e. the Krull dimension of V).*

Proof. It is clearly that

$$Z(I(clos(S \cap \delta_i))) = \overline{clos(S \cap \delta_i)}^Z = clos_{Zar}(\overline{S}^Z \cap \delta_i).$$

Then

$$\dim(\overline{clos(S \cap \delta_i)}^Z) = \dim(clos_{Zar}(\overline{S}^Z \cap \delta_i)).$$

By using the result of semialgebraic sets[2,3], we get

$$\dim(\overline{clos(S \cap \delta_i)}^Z) = \dim(clos(S \cap \delta_i)).$$

Thus we obtain

$$\dim(clos(S \cap \delta_i)) = \dim(clos_{Zar}(\overline{S}^Z \cap \delta_i)).$$

By using Definition 4.1, 4.3, we have

$$\dim(S) = \dim(\overline{S}^Z).$$

Furthermore, if $V \in PSA_n^\mu$ is a C^μ piecewise algebraic variety, its dimension as a piecewise semialgebraic set is equal to its dimension as a piecewise algebraic variety(i.e. the Krull dimension of V).

Corollary 4.1. *Let $S \in PSA_n^\mu$. Then the dimension of S is equal to dimension of its coordinate ring $A(S)$.*

Proof. By Theorem 4.2, the dimension of S is equal to the dimension of its Zariski closure $\overline{S}^Z$. From [21], the dimension of a $\overline{S}^Z$ is equal to the dimension of its coordinate ring $A(\overline{S}^Z)$. Since $I(S) = I(\overline{S}^Z)$, then the result is proved.

Acknowledgment. The authors appreciate the useful comments from the anonymous referees. Their advice helped improve the presentation of this paper.

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