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ON HERMITIAN POSITIVE DEFINITE SOLUTION OF NONLINEAR MATRIX EQUATION $X + A^*X^{-2}A = Q$ [*1)]

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Abstract

Based on the fixed-point theory, we study the existence and the uniqueness of the maximal Hermitian positive definite solution of the nonlinear matrix equation $X + A^*X^{-2}A = Q$, where Q is a square Hermitian positive definite matrix and A^* is the conjugate transpose of the matrix A. We also demonstrate some essential properties and analyze the sensitivity of this solution. In addition, we derive computable error bounds about the approximations to the maximal Hermitian positive definite solution of the nonlinear matrix equation $X + A^*X^{-2}A = Q$. At last, we further generalize these results to the nonlinear matrix equation $X + A^*X^{-n}A = Q$, where $n \geq 2$ is a given positive integer.

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1. Introduction

Consider the *nonlinear matrix equation* **(NME)**

$$X + A^*X^{-2}A = Q, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix, and $Q \in \mathbb{C}^{n\times n}$ is a *Hermitian positive definite* **(HPD)** matrix. Here, we use $\mathbb{C}^{n\times n}$ to denote the set of all $n\times n$ complex matrices, and A^* the conjugate transpose of the matrix A.

Nonlinear matrix equations of type (1.1) often arise in dynamic programming, stochastic filtering, control theory, statistics, and so on. See [9]. It can be categorized into a general system of nonlinear equations in the $\mathbb{C}^{n^2}$ space (see [25, 4, 6, 7]), which includes the linear and nonlinear matrix equations recently discussed in [1, 18, 19, 20, 21, 14, 8] as special cases.

The nonlinear matrix equations $X \pm A^*X^{-1}A = Q$ have been extensively studied by several authors[9, 13, 10, 11, 24, 26], and some properties of their solutions have been obtained. In addition, Xu[23] has given the perturbation analysis of the maximal solution of the nonlinear matrix equation $X + A^*X^{-1}A = Q$. When $Q = I$, the identity matrix, many authors have studied the properties of the (Hermitian) positive definite solutions of the NME(1.1). See [16, 17, 27, 22, 15] for details. However, when Q is a general HPD matrix, the NME(1.1) becomes more complicated and little is known yet about properties of its solutions.

In this paper, we discuss existence and uniqueness of the HPD solutions of the NME(1.1). Moreover, we reveal essential properties of these HPD solutions and their maximal one. In particular, for the maximal HPD solution of the NME(1.1), we investigate its sensitivity property in detail and, based upon this, we derive a computable bound for its numerical approximations. Besides, we further generalize all of these results to the nonlinear matrix equation $X + A^*X^{-n}A = Q$, where $n \geq 2$ is a given positive integer.

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The following notations are used throughout this paper. For $A \in \mathbb{C}^{n\times n}$, we use $\lambda(A)$ to denote its eigenvalue set and $\|A\|$ its spectral norm, i.e.,

$$\lambda(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of the matrix } A\}$$

and

$$\|A\| = \sqrt{\max_i \lambda_i(A^*A)},$$

where $\lambda_i(\cdot)$ is the i-th eigenvalue of the corresponding matrix. In particular, we use $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ to represent the minimal and the maximal eigenvalues of a Hermitian matrix, respectively. For $A, B \in \mathbb{C}^{n\times n}$, we write $A \succ B$ ($A \succeq B$) if both A and B are Hermitian and $A - B$ is positive definite (semidefinite). In particular, $A \succ 0$ ($A \succeq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If an HPD matrix $X \in \mathbb{C}^{n\times n}$ satisfies $A \preceq X \preceq B$, then we may also write it as $X \in [A, B]$. See [2, 3, 5] for more details about these matrix orderings.

2. The HPD Solutions

In this section, we will investigate existence, uniqueness and some essential properties of the HPD solutions of the NME(1.1).

To simplify discussion, we let

$$Y = Q^{-\frac{1}{2}} X Q^{-\frac{1}{2}}, \quad B = Q^{-\frac{1}{2}} A Q^{-\frac{1}{2}} \quad \text{and} \quad P = Q^{-1}, \tag{2.1}$$

where $Q^{\frac{1}{2}}$ denotes the HPD square root of the HPD matrix Q and $Q^{-\frac{1}{2}} := (Q^{\frac{1}{2}})^{-1}$. Then the NME(1.1) can be equivalently rewritten as the nonlinear matrix equation

$$Y + B^* Y^{-1} P Y^{-1} B = I. \tag{2.2}$$

We remark that only when $P = I$, i.e., $Q = I$, the NMEs (1.1) and (2.2) describe the same type of nonlinear matrix equations. Otherwise, they are substantially different from each other. However, as these two equations possess intrinsic relationships and their solutions are internally connected through the matrix transformations (2.1), we can investigate the properties of the solutions of the NME(1.1) by the aid of the NME(2.2).

2.1 The HPD solutions of the NME(2.2)

To have an intuitive understanding of the HPD solutions of the NME(2.2), we first investigate its simplest case when $n = 1$. Now, the NME(2.2) obviously reduces to the form

$$b^2 p = y^2 - y^3. \tag{2.3}$$

Define

$$\varphi(y) = y^2 - y^3.$$

As

$$\max_{y\in[0,1]} \varphi(y) = \varphi(\frac{2}{3}) = \frac{4}{27},$$

we know that a necessary condition about the existence of a positive root of the nonlinear equation (2.3) is

$$b^2 p \le \frac{4}{27}.$$

Hence, in general, we may assume that the matrices B and P involved in the NME(2.2) satisfy

$$\|B\|^2 \|P\| \le \frac{4}{27}.$$

Let α_1 and α_2, with $\alpha_1 \leq \alpha_2$, be the only two nonnegative solutions of the nonlinear equation

$$x^2(1-x) = \lambda_{\min}(P) \cdot \lambda_{\min}(B^*B),$$

and let β_1 and β_2, with $\beta_1 \leq \beta_2$, be those of the nonlinear equation

$$x^2(1-x) = \lambda_{\max}(P) \cdot \lambda_{\max}(B^*B).$$

Then we easily see that

$$0 < \alpha_1 \leq \beta_1 \leq \frac{2}{3}$$

and

$$\frac{2}{3} \leq \beta_2 \leq \alpha_2 < 1.$$

The following theorem presents bounds about the smallest and the largest eigenvalues of an HPD solution of the NME(2.2).

Theorem 2.1. *Let the matrices B and P satisfy $\|B\|^2\|P\| \leq \frac{4}{27}$. If Y is an HPD solution of the NME(2.2), then it holds that*

$$\alpha_1 \leq \lambda_{\min}(Y) \leq \beta_1 \quad or \quad \alpha_2 \leq \lambda_{\min}(Y) \leq \beta_2, \tag{2.4}$$

and

$$\alpha_1 \leq \lambda_{\max}(Y) \leq \beta_1 \quad or \quad \alpha_2 \leq \lambda_{\max}(Y) \leq \beta_2. \tag{2.5}$$

Proof. Because Y is an HPD solution of the NME(2.2), we easily have the estimates

$$0 \prec \lambda_{\min}(Y)I \preceq Y \preceq \lambda_{\max}(Y)I \preceq I, \tag{2.6}$$

and hence,

$$\frac{1}{\lambda_{\max}(Y)}I \preceq Y^{-1} \preceq \frac{1}{\lambda_{\min}(Y)}I.$$

It then follows that

$$\lambda_{\max}(B^*Y^{-1}PY^{-1}B) \leq \lambda_{\max}(B^*B) \cdot \lambda_{\max}(Y^{-1}PY^{-1}) \leq \frac{\lambda_{\max}(P) \cdot \lambda_{\max}(B^*B)}{[\lambda_{\min}(Y)]^2}$$

and

$$\lambda_{\min}(B^*Y^{-1}PY^{-1}B) \geq \lambda_{\min}(B^*B) \cdot \lambda_{\min}(Y^{-1}PY^{-1}) \geq \frac{\lambda_{\min}(P) \cdot \lambda_{\min}(B^*B)}{[\lambda_{\max}(Y)]^2}.$$

Noticing that

$$\begin{aligned} \lambda_{\min}(Y) &= \lambda_{\min}(I - B^*Y^{-1}PY^{-1}B) \\ &= 1 - \lambda_{\max}(B^*Y^{-1}PY^{-1}B) \\ &\geq 1 - \frac{\lambda_{\max}(P) \cdot \lambda_{\max}(B^*B)}{[\lambda_{\min}(Y)]^2} \end{aligned}$$

and

$$\begin{aligned} \lambda_{\max}(Y) &= \lambda_{\max}(I - B^*Y^{-1}PY^{-1}B) \\ &= 1 - \lambda_{\min}(B^*Y^{-1}PY^{-1}B) \\ &\geq 1 - \frac{\lambda_{\min}(P) \cdot \lambda_{\min}(B^*B)}{[\lambda_{\max}(Y)]^2}, \end{aligned}$$

we can obtain the inequalities

$$\begin{cases} (\lambda_{\min}(Y))^2(1-\lambda_{\min}(Y)) & \leq & \lambda_{\max}(B^*B)\cdot\lambda_{\max}(P), \\ (\lambda_{\max}(Y))^2(1-\lambda_{\max}(Y)) & \geq & \lambda_{\min}(B^*B)\cdot\lambda_{\min}(P). \end{cases} \tag{2.7}$$

On the other hand, since the matrix B is nonsingular and

$$YP^{-1}Y = B(I-Y)^{-1}B^*,$$

from (2.6) we get

$$\frac{1}{1-\lambda_{\min}(Y)}BB^* \preceq B(I-Y)^{-1}B^* \preceq \frac{1}{1-\lambda_{\max}(Y)}BB^*,$$

which immediately gives the inequalities

$$\begin{cases} \frac{1}{1-\lambda_{\min}(Y)}BB^* \preceq YP^{-1}Y \preceq \frac{Y^2}{\lambda_{\min}(P)}, \\ \frac{Y^2}{\lambda_{\max}(P)} \preceq YP^{-1}Y \preceq \frac{1}{1-\lambda_{\max}(Y)}BB^*. \end{cases} \tag{2.8}$$

From (2.8) we obtain

$$\begin{cases} (\lambda_{\min}(Y))^2(1-\lambda_{\min}(Y)) & \geq & \lambda_{\min}(B^*B)\cdot\lambda_{\min}(P), \\ (\lambda_{\max}(Y))^2(1-\lambda_{\max}(Y)) & \leq & \lambda_{\max}(B^*B)\cdot\lambda_{\max}(P). \end{cases} \tag{2.9}$$

Now, by combining (2.7) and (2.9) we have the estimates

$$\lambda_{\min}(B^*B)\cdot\lambda_{\min}(P) \leq [\lambda_{\min}(Y)]^2(1-\lambda_{\min}(Y)) \leq \lambda_{\max}(B^*B)\cdot\lambda_{\max}(P) \tag{2.10}$$

and

$$\lambda_{\min}(B^*B)\cdot\lambda_{\min}(P) \leq [\lambda_{\max}(Y)]^2(1-\lambda_{\max}(Y)) \leq \lambda_{\max}(B^*B)\cdot\lambda_{\max}(P). \tag{2.11}$$

Obviously, (2.10) and (2.11) are equivalent to (2.4) and (2.5), respectively.

Theorem 2.1 clearly shows that when the matrices B and P satisfy $\|B\|^2\|P\| \leq \frac{4}{27}$, a necessary condition for an HPD matrix Y to be a solution of the NME(2.2) is that Y satisfies the following constraint:

$$Y \in [\alpha_1 I, \beta_1 I] \cup [\alpha_2 I, \beta_2 I] \cup \{Y \mid \alpha_1 \leq \lambda_{\min}(Y) \leq \beta_1, \quad \beta_2 \leq \lambda_{\max}(Y) \leq \alpha_2\}. \tag{2.12}$$

The following example confirms the correctness of Theorem 2.1 and the sharpness of the involved bounds.

Example 2.1. Consider the NME(2.2) with

$$B = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.6 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

After direct computations, we have

$$\alpha_1 = 0.1740, \quad \beta_1 = 0.4386, \quad \alpha_2 = 0.9736 \quad \text{and} \quad \beta_2 = 0.8508.$$

In fact, the NME(2.2) exactly has the following four different *symmetric positive definite* **(SPD)** solutions:

$$Y_1 = \begin{pmatrix} 0.3361 & 0 \\ 0 & 0.2140 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0.9093 & 0 \\ 0 & 0.9610 \end{pmatrix}$$

and

$$Y_3 = \begin{pmatrix} 0.9093 & 0 \\ 0 & 0.2140 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0.9610 & 0 \\ 0 & 0.3361 \end{pmatrix}.$$

Evidently, these four solutions satisfy

$$Y_1 \in [\alpha_1 I, \beta_1 I], \quad Y_2 \in [\beta_2 I, \alpha_2 I]$$

and

$$Y_3, Y_4 \in \{Y \mid \alpha_1 \le \lambda_{\min}(Y) \le \beta_1, \quad \beta_2 \le \lambda_{\max}(Y) \le \alpha_2\}.$$

Theorem 2.2. *Let the matrices B and P satisfy $\|B\|^2\|P\| < \frac{4}{27}$. Then the following facts hold:*

(i) the NME(2.2) has a unique HPD solution Y_L which satisfies $\beta_2 I \preceq Y_L \preceq \alpha_2 I$;

(ii) the iteration sequence

$$Y_{k+1} = I - B^*Y_k^{-1}PY_k^{-1}B, \quad k = 0, 1, 2, \ldots,$$

converges to Y_L for any $Y_0 \in [\frac{2}{3}I, I]$;

(iii) any HPD solution Y of the NME(2.2) such that $Y \succeq Y_L$ must satisfy $Y = Y_L$.

Proof. We first prove (i). Define $\Omega \subset \mathbb{C}^{n\times n}$ and $f : \Omega \to \mathbb{C}^{n\times n}$ by

$$\Omega = \{Y \mid \frac{2}{3}I \preceq Y \preceq I\} \quad \text{and} \quad f(Y) = I - B^*Y^{-1}PY^{-1}B, \qquad \forall Y \in \Omega.$$

Clearly, Ω is a nonempty convex closed set and $f(Y)$ is a continuous matrix-valued function.

As for $\forall Y \in \Omega$, $\frac{2}{3}I \preceq Y \preceq I$ holds, we easily have

$$I \preceq Y^{-1} \preceq \frac{3}{2}I$$

and

$$\lambda_{\max}(B^*Y^{-1}PY^{-1}B) \le \frac{9}{4} \cdot \lambda_{\max}(P) \cdot \lambda_{\max}(B^*B) < \frac{9}{4} \cdot \frac{4}{27} = \frac{1}{3}.$$

The latter immediately implies that

$$B^*Y^{-1}PY^{-1}B \preceq \frac{1}{3}I.$$

Therefore

$$f(Y) = I - B^*Y^{-1}PY^{-1}B \succ I - \frac{1}{3}I = \frac{2}{3}I.$$

Noticing that $f(Y) \prec I$, we see that $f(\Omega) \subset \Omega$. By Brouwers fixed-point theorem [12], we know that f has a fixed point $Y_L \in \Omega$. This shows that Y_L is an HPD solution of the NME(2.2). It follows from (2.12) that $Y_L \in [\beta_2 I, \alpha_2 I]$.

We now turn to verify the uniqueness of the HPD solution of the NME(2.2). To this end, we assume that there exist two HPD solutions Y_1^* and Y_2^* of the NME(2.2) such that $Y_1^*, Y_2^* \in \Omega$. Without loss of generality, we assume that $Y_1^* \preceq Y_2^*$.

Because

$$\begin{aligned}
\|Y_2^* - Y_1^*\| &= \|(I - B^*Y_2^{*-1}PY_2^{*-1}B) - (I - B^*Y_1^{*-1}PY_1^{*-1}B)\| \\
&= \|B^*(Y_1^{*-1}PY_1^{*-1} - Y_2^{*-1}PY_2^{*-1})B\| \\
&= \|B^*Y_1^{*-1}PY_1^{*-1}(Y_2^* - Y_1^*)Y_2^{*-1}B + B^*Y_1^{*-1}(Y_2^* - Y_1^*)Y_2^{*-1}PY_2^{*-1}\| \\
&\le 2\|B\|^2\|P\| \cdot \frac{27}{8}\|Y_2^* - Y_1^*\| \\
&= \frac{27}{4}\|B\|^2\|P\|\|Y_2^* - Y_1^*\|,
\end{aligned}$$

we have

$$\left(1-\frac{27}{4}\|B\|^2\|P\|\right)\|Y_2^*-Y_1^*\|\leq 0.$$

Recalling that

$$\frac{27}{4}\|B\|^2\|P\|<1,$$

we immediately know that

$$\|Y_2^*-Y_1^*\|=0,$$

or in other words, $Y_2^*=Y_1^*$.

Hence, Y_L is a unique HPD solution of the NME(2.2) in Ω. By (2.12) again, we have

$$\beta_2 I\preceq Y_L\preceq \alpha_2 I.$$

We now turn to demonstrate (ii). According to (i) we know that

$$\frac{2}{3}I\preceq Y_k\preceq I,\qquad k=1,2,\ldots$$

hold when $\frac{2}{3}I\preceq Y_0\preceq I$. Hence, for any positive integer s, it holds that

$$\begin{aligned}\|Y_{k+s}-Y_k\| &= \|B^*[Y_{k-1}^{-1}PY_{k-1}^{-1}-Y_{k+s-1}^{-1}PY_{k+s-1}^{-1}]B\|\\ &= \|B^*[(Y_{k-1}^{-1}PY_{k-1}^{-1}-Y_{k-1}^{-1}PY_{k+s-1}^{-1})+(Y_{k-1}^{-1}PY_{k+s-1}^{-1}-Y_{k+s-1}^{-1}PY_{k+s-1}^{-1})]B\|\\ &\leq \|B^*Y_{k-1}^{-1}PY_{k-1}^{-1}(Y_{k+s-1}-Y_{k-1})Y_{k+s-1}^{-1}B\|\\ &\quad +\|B^*Y_{k-1}^{-1}(Y_{k+s-1}-Y_{k-1})Y_{k+s-1}^{-1}PY_{k+s-1}^{-1}B\|\\ &\leq \frac{27}{4}\|B\|^2\|P\|\|Y_{k+s-1}-Y_{k-1}\|.\end{aligned}$$

Consequently

$$\|Y_{k+s}-Y_k\|\leq\left(\frac{27}{4}\|B\|^2\|P\|\right)^k\|Y_s-Y_0\|.$$

Because

$$q:=\frac{27}{4}\|B\|^2\|P\|<1$$

and

$$\begin{aligned}\|Y_s-Y_0\| &\leq \|Y_s-Y_{s-1}\|+\|Y_{s-1}-Y_{s-2}\|+\ldots+\|Y_1-Y_0\|\\ &\leq \left(q^{s-1}+q^{s-2}+\ldots+1\right)\|Y_1-Y_0\|\\ &< \frac{1}{1-q}\|Y_1-Y_0\|,\end{aligned}$$

we obtain

$$\|Y_{k+s}-Y_k\|\leq\frac{q^k}{1-q}\|Y_1-Y_0\|.$$

It then follows that the matrix sequence $\{Y_k\}_{k=0}^{\infty}$ forms a Cauchy sequence in the Banach space $\mathbb{C}^{n\times n}\cap\Omega$. Therefore, $\{Y_k\}_{k=0}^{\infty}$ has a limit point $Y^*\in[\frac{2}{3}I,I]$. By making use of (i), we immediately know that $Y^*=Y_L$.

Finally, we verify the validity of (iii). In fact, if the NME(1.1) exists an HPD solution Y satisfying $Y \succeq Y_L$, then we have $Y \succeq \beta_2 I$, as $Y_L \succeq \beta_2 I$. Because (2.12) implies $\beta_2 I \preceq Y \preceq \alpha_2 I$, noticing the uniqueness of Y_L, we hence get $Y = Y_L$.

The unique HPD solution Y_L of the NME(2.2) is called the maximal solution. From Theorems 2.1-2.2, we know that the following fact holds true: If Y is an HPD solution of the NME(2.2) and $Y \neq Y_L$, then $\lambda_{\min}(Y) < \beta_1 < \frac{2}{3}$, i.e., $\|Y^{-1}\|_2 > \frac{1}{\beta_1}$. This implies that Y^{-1} may be ill-conditioned when β_1 is small. Therefore, the maximal HPD solution Y_L of the NME(2.2) uniquely satisfies $1 < \|Y_L^{-1}\| < \frac{3}{2}$.

2.2 The maximal HPD solution of the NME(1.1)

According to the relationship between the NME(1.1) and the NME(2.2), by making use of Theorem 2.2 we can easily demonstrate the following result.

Theorem 2.3. *Let the matrices A and Q satisfy $\|A\|^2\|Q^{-1}\|^3 < \frac{4}{27}$. Then the maximal HPD solution X_L of the NME(1.1) exists and satisfies*

$$\|X_L^{-1}\| < \frac{3}{2}\|Q^{-1}\|. \tag{2.13}$$

Moreover, any other HPD solution X of the NME(1.1) must obey

$$\|X^{-1}\| > \frac{3}{2}\|Q\|^{-1}. \tag{2.14}$$

Proof. By considering (2.1) we can obtain

$$\|B\|^2\|P\| = \|Q^{-\frac{1}{2}}AQ^{-\frac{1}{2}}\|^2\|Q^{-1}\| \leq \|A\|^2\|Q^{-1}\|^3.$$

When

$$\|A\|^2\|Q^{-1}\|^3 < \frac{4}{27},$$

the NME(2.2) exists the maximal HPD solution Y_L, which satisfies $\|Y_L^{-1}\| < \frac{3}{2}$. Moreover, any other HPD solution of the NME(2.2) must obey $\|Y^{-1}\| > \frac{3}{2}$.

Consequently, the NME(1.1) exists the maximal HPD solution X_L, which satisfies

$$\|X_L^{-1}\| = \|Q^{-\frac{1}{2}}Y_L^{-1}Q^{-\frac{1}{2}}\| \leq \|Q^{-1}\|\|Y_L^{-1}\| < \frac{3}{2}\|Q^{-1}\|.$$

In addition, by (2.1) again we know that any other HPD solution of the NME(1.1) must obey

$$\|X^{-1}\| \geq \|Q\|^{-1}\|Q^{\frac{1}{2}}X^{-1}Q^{\frac{1}{2}}\| = \|Q\|^{-1}\|Y^{-1}\| > \frac{3}{2}\|Q\|^{-1}.$$

We can also estimate the bounds of the maximal HPD solution of the NME(1.1).

Theorem 2.4. *Let the matrices A and Q satisfy $\|A\|^2\|Q^{-1}\|^3 < \frac{4}{27}$. Then the maximal HPD solution of the NME(1.1) satisfies*

$$\frac{2}{3}\|Q\| \leq \|X_L\| \leq \|Q\|. \tag{2.15}$$

Proof. Noticing $\|Q^{-1}\|^{-1} \le \|Q\|$, by Theorem 2.3 we have

$$\begin{aligned}\|X_L\| &= \|Q - A^* X_L^{-2} A\| \\ &\ge \|Q\| - \|A\|^2 \|X_L^{-2}\| \\ &> \|Q\| - \|A\|^2 \cdot \frac{9}{4}\|Q^{-1}\|^2 \\ &\ge \|Q\| - \frac{1}{3}\|Q^{-1}\|^{-1} \\ &\ge \|Q\| - \frac{1}{3}\|Q\| \\ &= \frac{2}{3}\|Q\|.\end{aligned}$$

On the other hand, from

$$X_L = Q - A^* X_L^{-2} A$$

we easily have $X_L \preceq Q$. Thus

$$\|X_L\| \le \|Q\|.$$

3. Sensitivity Analysis of the NME(1.1)

The following theorem describes the sensitivity of the maximal HPD solution of the NME(1.1).

Theorem 3.1. *Let $A, \widetilde{A} \in \mathbb{C}^{n\times n}$, and $Q, \widetilde{Q} \in \mathbb{C}^{n\times n}$ be HPD matrices. If*

$$\theta_2 := \frac{3\sqrt{3}}{2}\|A\|\|Q^{-1}\|^{\frac{3}{2}} < 1, \tag{3.1}$$

$$\|\widetilde{A} - A\| \le \frac{2}{9\sqrt{3}}(1-\theta_2)\|Q^{-1}\|^{-\frac{3}{2}} \tag{3.2}$$

and

$$\|\widetilde{Q} - Q\| \quad \le \quad \left(1 - \left[\frac{1}{3}(1+2\theta_2)\right]^{\frac{2}{3}}\right)\|Q^{-1}\|^{-1}, \tag{3.3}$$

then the maximal solutions X_L and $\widetilde{X}_L$ of the nonlinear matrix equations

$$X + A^* X^{-2} A = Q \quad \text{and} \quad \widetilde{X} + \widetilde{A}^* \widetilde{X}^{-2} \widetilde{A} = \widetilde{Q} \tag{3.4}$$

exist and satisfy

$$\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} \le \frac{1}{1-\theta_2^{\frac{2}{3}}}\left(\frac{3}{2}\frac{\|\widetilde{Q}-Q\|}{\|Q\|} + \frac{\|\widetilde{A}-A\|}{\|A\|}\right). \tag{3.5}$$

Proof. Let

$$\Delta A = \widetilde{A} - A \quad \text{and} \quad \Delta Q = \widetilde{Q} - Q.$$

Obviously, it holds that

$$\widetilde{Q}^{-1} = Q^{-1} - Q^{-1}\,\Delta Q\,\widetilde{Q}^{-1}.$$

Taking norms on both sides of this equality and using (3.3) we obtain

$$\begin{aligned}\|\widetilde{Q}^{-1}\| &\le \|Q^{-1}\| + \|Q^{-1}\|\|\Delta Q\|\|\widetilde{Q}^{-1}\| \\ &\le \|Q^{-1}\| + \left[1 - \left(\frac{1}{3}(1+2\theta_2)^{\frac{2}{3}}\right)^{\frac{2}{3}}\right]\|\widetilde{Q}^{-1}\|.\end{aligned}$$

Therefore,

$$\|\widetilde{Q}^{-1}\| \le \frac{\|Q^{-1}\|}{\left[\frac{1}{3}(1+2\theta_2)\right]^{\frac{2}{3}}}. \tag{3.6}$$

It follows from (3.2) and (3.6) that

$$\begin{aligned}
\|\widetilde{A}\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}} &\le (\|A\| + \|\Delta A\|)\cdot \frac{\|Q^{-1}\|^{\frac{3}{2}}}{\frac{1}{3}(1+2\theta_2)} \\
&< \left(\|A\| + \frac{2(1-\theta_2)}{9\sqrt{3}\|Q^{-1}\|^{\frac{3}{2}}}\right)\cdot \frac{\|Q^{-1}\|^{\frac{3}{2}}}{\frac{1}{3}(1+2\theta_2)} \\
&= \frac{6\theta_2 + 2(1-2\theta_2)}{9\sqrt{3}\|Q^{-1}\|^{\frac{3}{2}}}\cdot \frac{\|Q^{-1}\|^{\frac{3}{2}}}{\frac{1}{3}(1+2\theta_2)} \\
&= \frac{2}{3\sqrt{3}}.
\end{aligned}\tag{3.7}$$

According to Theorems 2.3 and 2.4, we know that under the condition (3.1) both maximal solutions X_L and $\widetilde{X}_L$ of the nonlinear matrix equations (3.4) exist, and satisfy

$$\|X_L^{-1}\| < \frac{3}{2}\|Q^{-1}\|, \quad \|\widetilde{X}_L^{-1}\| < \frac{3}{2}\|\widetilde{Q}^{-1}\|$$

and

$$\frac{2}{3}\|Q\| \le \|X_L\| \le \|Q\|, \quad \frac{2}{3}\|\widetilde{Q}\| \le \|\widetilde{X}_L\| \le \|\widetilde{Q}\|.$$

In addition, based on (3.7) and (3.1) we have

$$\|A\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}} \le (\|A\| + \|\Delta A\|)\|\widetilde{Q}^{-1}\|^{\frac{3}{2}} < \frac{2}{3\sqrt{3}} \tag{3.8}$$

and

$$\|A\|\|\widetilde{Q}^{-1}\|\|Q^{-1}\|^{\frac{1}{2}} = \left(\|A\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(\|A\|\|Q^{-1}\|^{\frac{3}{2}}\right)^{\frac{1}{3}} < \frac{2}{3\sqrt{3}}.$$

Let

$$S = \widetilde{X}_L - X_L.$$

By noticing that X_L and $\widetilde{X}_L$ satisfy (3.4), i.e.,

$$X_L + A^*X_L^{-2}A = Q \quad \text{and} \quad \widetilde{X}_L + \widetilde{A}^*\widetilde{X}_L^{-2}\widetilde{A} = \widetilde{Q},$$

respectively, we can get

$$S + A^*(\widetilde{X}_L^{-2} - X_L^{-2})A + \Delta A^*\ \widetilde{X}_L^{-2}\widetilde{A} + A^*\widetilde{X}_L^{-2}\ \Delta A = \Delta Q.$$

It then follows from

$$\widetilde{X}_L^{-2} - X_L^{-2} = -\widetilde{X}_L^{-2}SX_L^{-1} - \widetilde{X}_L^{-1}SX_L^{-2}$$

that

$$S - A^*\widetilde{X}_L^{-2}SX_L^{-1}A - A^*\widetilde{X}_L^{-1}SX_L^{-2}A + \Delta A^*\ \widetilde{X}_L^{-2}\widetilde{A} + A^*\widetilde{X}_L^{-2}\ \Delta A = \Delta Q.$$

Now, by making use of (3.1), (3.6) and (3.8), we obtain

$$\begin{aligned}
&\|S - A^*\widetilde{X}_L^{-2}SX_L^{-1}A - A^*\widetilde{X}_L^{-1}SX_L^{-2}A\| \\
&\geq \|S\| - \|A\|^2\|\widetilde{X}_L^{-2}\|\|X_L^{-1}\|\|S\| - \|A\|^2\|\widetilde{X}_L^{-1}\|\|X_L^{-2}\|\|S\| \\
&\geq \|S\| - \frac{27}{8}\|A\|^2\|\widetilde{Q}^{-1}\|^2\|Q^{-1}\|\|S\| - \frac{27}{8}\|A\|^2\|\widetilde{Q}^{-1}\|\|Q^{-1}\|^2\|S\| \\
&\geq \|S\| - \frac{27}{8}\|A\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}}\|A\|\|Q^{-1}\|\|\widetilde{Q}^{-1}\|^{\frac{1}{2}}\|S\| \\
&\qquad - \frac{27}{8}\|A\|\|Q^{-1}\|^{\frac{3}{2}}\|A\|\|\widetilde{Q}^{-1}\|\|Q^{-1}\|^{\frac{1}{2}}\|S\| \\
&\geq \|S\| - \frac{3\sqrt{3}}{4}\cdot\frac{\|A\|\|Q^{-1}\|^{\frac{3}{2}}}{\left[\frac{1}{3}(1+2\theta_2)\right]^{\frac{1}{3}}}\cdot\|S\| - \frac{3\sqrt{3}}{4}\cdot\|A\|\|Q^{-1}\|^{\frac{3}{2}}\|S\| \\
&\geq \|S\| - \frac{3\sqrt{3}}{2}\cdot\frac{\|A\|\|Q^{-1}\|^{\frac{3}{2}}}{\left[\frac{1}{3}(1+2\theta_2)\right]^{\frac{1}{3}}}\cdot\|S\| \\
&\geq \|S\| - \frac{\theta_2}{\theta_2^{\frac{1}{3}}}\cdot\|S\| \\
&= \left(1-\theta_2^{\frac{2}{3}}\right)\|S\|.
\end{aligned}$$

This estimate implies that

$$\begin{aligned}
\left(1-\theta_2^{\frac{2}{3}}\right)\|S\| &\leq \|S - A^*\widetilde{X}_L^{-2}SX_L^{-1}A - A^*\widetilde{X}_L^{-1}SX_L^{-2}A\| \\
&= \|\Delta Q - \Delta A^*\ \widetilde{X}_L^{-2}\widetilde{A} - A^*\widetilde{X}_L^{-2}\ \Delta A\| \\
&\leq \|\Delta Q\| + \|\Delta A\|\cdot\frac{9}{4}\|\widetilde{Q}^{-1}\|^2\|\widetilde{A}\| + \|\Delta A\|\cdot\frac{9}{4}\|\widetilde{Q}^{-1}\|^2\|A\| \\
&\leq \|\Delta Q\| + \frac{\sqrt{3}\|Q^{-1}\|^{\frac{1}{2}}}{\theta_2^{\frac{1}{3}}}\cdot\|\Delta A\| \\
&\leq \|\Delta Q\| + \frac{\sqrt{3}}{\left(\frac{3\sqrt{3}}{2}\|A\|\right)^{\frac{1}{3}}}\cdot\|\Delta A\|. \qquad (3.9)
\end{aligned}$$

Here, the last inequality follows from

$$\|A\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}} < \frac{2}{3\sqrt{3}} \quad \text{and} \quad \|\widetilde{A}\|\|\widetilde{Q}^{-1}\|^{\frac{3}{2}} < \frac{2}{3\sqrt{3}}.$$

Because

$$\|X_L\| \geq \frac{2}{3}\|Q\| \geq \frac{2}{3}\|Q^{-1}\|^{-1} \geq \frac{2}{3}\left(\frac{3\sqrt{3}}{2}\|A\|\right)^{\frac{2}{3}},$$

it holds that

$$\frac{\|Q\|}{\|X_L\|} \leq \frac{3}{2} \quad \text{and} \quad \frac{\|A\|^{\frac{2}{3}}}{\|X_L\|} \leq \frac{1}{\frac{2}{3}\left(\frac{3\sqrt{3}}{2}\right)^{\frac{2}{3}}}. \qquad (3.10)$$

Now, by combining (3.9) and (3.10), we immediately obtain

$$\begin{aligned}
\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} &\leq \frac{1}{1-\theta_2^{\frac{2}{3}}}\left(\frac{\|\Delta Q\|}{\|X_L\|} + \frac{\sqrt{3}}{\left(\frac{3\sqrt{3}}{2}\|A\|\right)^{\frac{1}{3}}}\cdot\frac{\|\Delta A\|}{\|X_L\|}\right) \\
&\leq \frac{1}{1-\theta_2^{\frac{2}{3}}}\left(\frac{\|\Delta Q\|}{\|Q\|}\cdot\frac{\|Q\|}{\|X_L\|} + \frac{\sqrt{3}}{\left(\frac{3\sqrt{3}}{2}\right)^{\frac{1}{3}}}\cdot\frac{\|\Delta A\|}{\|A\|}\cdot\frac{\|A\|^{\frac{2}{3}}}{\|X_L\|}\right) \\
&\leq \frac{1}{1-\theta_2^{\frac{2}{3}}}\left(\frac{3}{2}\frac{\|\Delta Q\|}{\|Q\|} + \frac{\|\Delta A\|}{\|A\|}\right).
\end{aligned}$$

From Theorem 3.1 we clearly see that the sensitivity of the maximal HPD solution X_L of the NME(1.1) is strongly dependent on the quantity

$$\kappa(A,Q) := \frac{1}{1-\left(\frac{3\sqrt{3}}{2}\|A\|\|Q^{-1}\|^{\frac{3}{2}}\right)^{\frac{2}{3}}}. \tag{3.11}$$

When $\kappa(A,Q)$ is small, X_L is less sensitive to the perturbations of the matrices A and Q. However, when $\kappa(A,Q)$ is relatively large, X_L may be very sensitive to the perturbations of the matrices A and Q. Hence, it is reasonable for us to refer $\kappa(A,Q)$ as the *condition number* of the NME(1.1) with respect to its maximal HPD solution X_L.

In particular, when $Q = I$, Theorem 3.1 immediately recovers the known perturbation theory about the maximal HPD solution of the nonlinear matrix equation

$$X + A^*X^{-2}A = I.$$

Corollary 3.1. *Let $A, \widetilde{A} \in \mathbb{C}^{n\times n}$. Assume*

$$\|A\| \leq \frac{2}{3\sqrt{3}} \quad \textit{and} \quad \|\widetilde{A} - A\| < \frac{1}{3}\left(\frac{2}{3\sqrt{3}} - \|A\|\right).$$

If X_L and $\widetilde{X}_L$ are the maximal HPD solutions of the nonlinear matrix equations

$$X + A^*X^{-2}A = I \quad \textit{and} \quad \widetilde{X} + \widetilde{A}^*\widetilde{X}^{-2}\widetilde{A} = I,$$

respectively, then it holds that

$$\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} \leq \frac{1}{1-\left(\frac{3\sqrt{3}}{2}\|A\|\right)^{\frac{2}{3}}}\cdot\frac{\|\widetilde{A} - A\|}{\|A\|}.$$

Besides, if $\overline{X}$ is an approximation to the maximal HPD solution of the NME(1.1), then we can demonstrate the following computable error bound about $\overline{X}$, which shows how to determine the accuracy of such an $\overline{X}$.

Theorem 3.2. *Let $\overline{X}_L$ be an approximation to the maximal HPD solution X_L of the NME(1.1). Assume*

$$\theta_2 := \frac{3\sqrt{3}}{2}\|A\|\|Q^{-1}\|^{\frac{3}{2}} < 1,$$

$$\|\overline{X}_L^{-1}\| \leq \frac{3}{2}\|(\overline{X}_L + A\overline{X}_L^{-2}A)^{-1}\|$$

and the residual

$$\mathcal{R}(\overline{X}_L) = \overline{X}_L + A\overline{X}_L^{-2}A - Q$$

satisfy

$$\|\mathcal{R}(\overline{X}_L)\| \le \left(1 - \left[\frac{1}{3}(1+2\theta_2)\right]^{\frac{2}{3}}\right)\|Q^{-1}\|^{-1}.$$

Then

$$\frac{\|\overline{X}_L - X_L\|}{\|X_L\|} \le \frac{3}{2}\cdot\frac{1}{1-\theta_2^{\frac{2}{3}}}\cdot\frac{\|\mathcal{R}(\overline{X}_L)\|}{\|Q\|}. \tag{3.12}$$

Proof. Noticing that $\overline{X}_L$ is an HPD solution of the nonlinear matrix equation

$$X + A^*X^{-2}A = \overline{Q}, \quad \text{with} \quad \overline{Q} = Q + \mathcal{R}(\overline{X}_L),$$

and $\overline{X}_L$ satisfies

$$\|\overline{X}_L^{-1}\| \le \frac{3}{2}\|(\overline{X}_L + A\overline{X}_L^{-2}A)^{-1}\| = \frac{3}{2}\|\overline{Q}^{-1}\|,$$

we can obtain the estimate (3.12) in an analogous manner to the proof of (3.5).

4. Generalization to the NME $X + A^*X^{-n}A = Q$

More generally, we consider the nonlinear matrix equation

$$X + A^*X^{-n}A = Q, \tag{4.1}$$

where n is an arbitrary positive integer, $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix, and $Q \in \mathbb{C}^{n\times n}$ is an HPD matrix.

Analogously to the demonstrations in Section 2, we can obtain the following properties about the maximal HPD solution of the NME(4.1).

Theorem 4.1. *Let the matrices A and Q satisfy*

$$\|A\|\|Q^{-1}\|^{\frac{n+1}{2}} < \left(\frac{n}{n+1}\right)^{\frac{n}{2}}\frac{1}{\sqrt{n+1}}.$$

Then the maximal HPD solution X_L of the NME(4.1) exists and satisfies

$$\frac{n}{n+1}\|Q\| \le \|X_L\| \le \|Q\| \quad \textit{and} \quad \|X_L^{-1}\| \le \frac{n+1}{n}\|Q^{-1}\|.$$

Moreover, any other HPD solution X of the NME(4.1) must obey

$$\|X^{-1}\| > \frac{n+1}{n}\|Q\|^{-1}.$$

For the sensitivity of the maximal HPD solution of the NME(4.1), analogously to the demonstrations in Section 3 we have the following result.

Theorem 4.2. *Let $A, \widetilde{A} \in \mathbb{C}^{n\times n}$, and $Q, \widetilde{Q} \in \mathbb{C}^{n\times n}$ be HPD matrices. If*

$$\theta_n := \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}}}\cdot\|A\|\|Q^{-1}\|^{\frac{n+1}{2}} < 1,$$

$$\|\widetilde{A} - A\| < \frac{1}{n^{\frac{n}{2}} + 1} \cdot \frac{n^{\frac{n}{2}}}{(n+1)^{\frac{n+1}{2}}} \cdot (1 - \theta_n)\|Q^{-1}\|^{-\frac{n+1}{2}}$$

and

$$\|\widetilde{Q} - Q\| \le \left\{1 - \left[\frac{1}{n^{\frac{n}{2}} + 1}\left(1 + n^{\frac{n}{2}}\theta_n\right)\right]^{\frac{2}{n+1}}\right\}\|Q^{-1}\|^{-1},$$

then the maximal HPD solutions X_L *and* $\widetilde{X}_L$ *of the nonlinear matrix equations*

$$X + A^*X^{-n}A = Q \quad \text{and} \quad \widetilde{X} + \widetilde{A}^*\widetilde{X}^{-n}\widetilde{A} = \widetilde{Q}$$

exist and satisfy

$$\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} \le \frac{1}{1 - \theta_n^{\frac{2}{n+1}}} \cdot \left(\frac{n+1}{n} \cdot \frac{\|\widetilde{Q} - Q\|}{\|Q\|} + \frac{2}{n} \cdot \frac{\|\widetilde{A} - A\|}{\|A\|}\right).$$

Similarly to (3.11), the quantity

$$\kappa(A, Q) := \frac{1}{1 - \left(\frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}}} \cdot \|A\|\|Q^{-1}\|^{\frac{n+1}{2}}\right)^{\frac{2}{n+1}}}$$

can be called as the *condition number* of the NME(4.1) with respect to its maximal HPD solution X_L.

We remark that when $n = 1$, Theorem 4.2 automatically becomes Theorem 3.1 in [23] for the nonlinear matrix equation

$$X + A^*X^{-1}A = Q,$$

and when $n = 2$, it naturally turns to Theorem 3.1 in this paper.

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