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A CLASS OF TWO-STEP CONTINUITY RUNGE-KUTTA METHODS FOR SOLVING SINGULAR DELAY DIFFERENTIAL EQUATIONS AND ITS STABILITY ANALYSIS *

Xin Leng   De-gui Liu
(*Beijing Institute of Computer Application and Simulation Technology, 100854, China; Beijing Institute of Applied Physics and Computational Mathematics, 100088, China*)

Xiao-qiu Song   Li-rong Chen
(*Beijing Institute of Computer Application and Simulation Technology, 100854, China*)

Abstract

In this paper, a class of two-step continuity Runge-Kutta(TSCRK) methods for solving singular delay differential equations(DDEs) is presented. Analysis of numerical stability of this methods is given. We consider the two distinct cases: $(i)\tau \geq h$, $\quad (ii)\tau < h$, where the delay τ and step size h of the two-step continuity Runge-Kutta methods are both constant. The absolute stability regions of some methods are plotted and numerical examples show the efficiency of the method.

Mathematics subject classification: 34k20
Key words: Analysis of numerical stability, Singular delay differential equations, Two-step continuity Runge-Kutta methods

1. Introduction

Consider the following delay differential equations (DDEs):

$$\begin{cases} y'(x) = f(x, y(x), y(x - \tau(x))) & a \leq x \leq b \\ y(x) = \varphi(x) & x_{min} \leq x \leq a \end{cases} \tag{1.1}$$

where y, f, φ are n-vector functions, $\varphi(x)$ is initial value function, $\tau(x) \geq 0$ is delay function.

Definition 1.1. *DDEs (1.1) is singular at the point x_α if the delay function satisfies $\tau(x_\alpha) = 0$. If there is no such point $x_\alpha \in [a\ b]$, then the DDEs (1.1) is non-singular.*

In the numerical solution of DDEs (1.1) by a continuous explicit Runge-Kutta method, we suppose that we have an approximation y_n to $y(x)$ at x_n and wish to compute an approximation at $x_{n+1} = x_n + h$. For $i = 1, 2, \cdots, s$, the stages $f_{ni} = f(x_{ni}, y_{ni}, \tilde{y}(x_{ni} - \tau(x_{ni})))$ are defined in terms of $x_{ni} = x_n + c_i h$ and $0 \leq c_i \leq 1$. Continuous explicit Runge-Kutta method is

$$\begin{cases} y_{ni} = y_n + h \sum\limits_{j=1}^{i-1} a_{ij} f_{ni} \\ y_{n+\sigma} = \tilde{y}(x_n + \sigma h) = y_n + h \sum\limits_{i=1}^{s} b_i(\sigma) f_{ni} \end{cases} \tag{1.2}$$

When DDE is singular or has a vanishing delay, a delay may fall in the span of the current step where there is no available approximation for the solution value at the delayed argument. This situation can also arise particularly at relaxed tolerance when the delay does not vanishing but actual optimal step size is larger than the size of the delay. In such case, $x_{ni} - \tau(x_{ni}) > x_n$ for some x_{ni}. Since no approximation for $\tilde{y}(x_{ni} - \tau(x_{ni}))$ is available, the explicit Runge-Kutta

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formulae (1.2) become implicit. For solving singular delay differential equations several authors have adopt the iteration scheme [1], [3]. However iteration is a computationally expensive approach. In this paper we relax the effect of delay when computing the Runge-Kutta stages in the span of the current step and construct a class of two-step continuity Runge-Kutta (TSCRK) methods. This class of methods keeps the explicit process of computing the Runge-Kutta stages and avoids iteration, then reduces the computing workload. Numerical stability analysis of the methods is given, and the regions of absolute stability for this class of methods are plotted. The numerical results show the efficiency of the TSCRK methods. The vanishing delay can be handled automatically by the methods so the users do not need to know where the delay vanishes.

2. TSCRK Methods

For the numerical solution of DDEs (1.1) we construct the TSCRK methods. These methods have the form

$$\begin{cases} y_{ni} = \alpha_i y_{n-1} + (1-\alpha_i)y_n + h\sum_{j=1}^{s}(a_{ij}f_{n-1j} + b_{ij}f_{nj}) \\ y_{n+\sigma} = Q(x_n+\sigma h) = \tilde{\theta}(\sigma)y_{n-1} + (1-\tilde{\theta}(\sigma))y_n \\ +h\sum_{i=1}^{s}(v_i(\sigma)f_{n-1i} + w_i(\sigma)f_{ni}) \end{cases} \tag{2.1}$$

where $f_{ni} = f(x_{ni}, y_{ni}, Q(x_{ni} - \tau(x_{ni})))$, $Q(x_n + \sigma h)$ is an approximation to $y(x_n + \sigma h)$, $0 \le \sigma \le 1$, $Q(x_n + h) = y_{n+1}$, $Q(x_n) = y_n$, $x_{ni} = x_n + c_i h$, $b_{ij} = 0$, for $j \ge i$. Assume that $c_1 \equiv 0$. $\tilde{\theta}(\sigma)$, $v_i(\sigma)$, $w_i(\sigma)$ are polynomials in σ of degree p^*, $p^* \ge p$, p is the order of the methods. $\tilde{\theta}(0) = v_i(0) = w_i(0) = 0$, $\tilde{\theta}(1) = \theta$, $v_i(1) = v_i$, $w_i(1) = w_i$, $i = 1, \cdots, s$. We define $Q(x) = \varphi(x)$ when $x \le a$. Methods (2.1) are not self-starting and we use the continuous RK method of the same order as the TSCRK methods (2.1) to compute the required approximations, y_1, y_{01}, y_{02}, $\cdots$, y_{0s} and y_σ, $0 \le \sigma \le 1$. When delay falls in the first interval $[x_0,\ x_1]$, we use the iteration scheme constructed in [1] to compute the approximations. When DDEs is singular or has a vanishing delay, a delay may fall in the current step. We relax the effect of delay when computing the Runge-Kutta stages in the span of the current step and assume that $w_2(\sigma) \equiv w_3(\sigma) \equiv \cdots \equiv w_s(\sigma) \equiv 0$, for all $0 \le \sigma \le 1$, we have

$$\begin{cases} y_{ni} = \alpha_i y_{n-1} + (1-\alpha_i)y_n + h\sum_{j=1}^{s}(a_{ij}f_{n-1j} + b_{ij}f_{nj}) \\ y_{n+\sigma} = Q(x_n+\sigma h) = \tilde{\theta}(\sigma)y_{n-1} + (1-\tilde{\theta}(\sigma))y_n \\ +h\sum_{i=1}^{s}v_i(\sigma)f_{n-1i} + hw_1(\sigma)f_{n1} \end{cases} \tag{2.2}$$

Methods (2.2) keep the explicit process when solving singular delay differential equations and vanishing delay differential equations. Introducing the vectors

$z_{n,\sigma} = z(x_n, \sigma h) = (y(x_n + \sigma h), y(x_n), B(\Phi_1, y(x_n)), \cdots, B(\Phi_s, y(x_n)))^T$
$z_n = z(x_n, h) = (y(x_n + h), y(x_n), B(\Phi_1, y(x_n)), \cdots, B(\Phi_s, y(x_n)))^T$
$u_{n,\sigma} = (y_{n+\sigma}, y_n, y_{n,1}, \cdots, y_{n,s})^T$
$u_n = (y_{n+1}, y_n, y_{n,1}, \cdots, y_{n,s})^T$

Then the methods (2.1) can be written in the form

$$\begin{cases} u_{0,\sigma} = \Psi(\sigma h) \\ u_{n+1,\sigma} = Ru_n + h\bar{\Phi}(x_{n+1}, u_n, \sigma; Q(x)) \end{cases} \tag{2.3}$$

Here $\Psi(\sigma h)$ specifies the "starting procedure", the matrix R is given by

$R = \begin{pmatrix} 1-\tilde{\theta}(\sigma) & \tilde{\theta}(\sigma) & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \mathbf{1}-\alpha & \alpha & \mathbf{0} \end{pmatrix}$, $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{pmatrix}$, and the increment function

$\bar{\Phi}(x, u, \sigma; Q(x))$ is implicitly defined by (2.1).

The method (2.3) is stable if $\|R^n\|$ is uniformly bounded. Namely, the methods (2.3) is stable if $-1 < \tilde{\theta}(\sigma) \leq 1$, for all $0 \leq \sigma \leq 1$[5]. This will be assumed throughout the paper.

Definition 2.1. *The local truncation error of the method (2.3) is defined by*

$$\begin{cases} d_{0,\sigma} = z_{0,\sigma} - \Psi(\sigma h) \\ d_{n+1,\sigma} = z_{n+1,\sigma} - Rz_n - h\bar{\Phi}(x_{n+1}, z_n, \sigma; y(x)) \quad 0 \leq \sigma \leq 1 \end{cases} \tag{2.4}$$

Denoting $F = (1 + \tilde{\theta}(\sigma))^{-1} \begin{pmatrix} 1 \\ 1 \\ \mathbf{1} \end{pmatrix} (1, \tilde{\theta}(\sigma), 0, \cdots, 0)$, we give the definition of order[3] of the method (2.3).

Definition 2.2. *The method (2.3) is of order p (consistent of order p), if for all problems (1.1) with p times continuously differentiable f, the local truncation error satisfies*

$$d_{n,\sigma} = O(h^p), \quad Fd_{n+1,\sigma} = O(h^{p+1}) \quad \textit{for all} \quad 0 \leq \sigma \leq 1 \tag{2.5}$$

If we assume f is p-times continuously differentiable then, in general, $\Psi(\sigma h)$, $\bar{\Phi}(x, z, \sigma; y(x))$, $z(x, \sigma h)$ and $z(x, h)$ are also smooth, so that the local truncation error (2.4) can be expended into a Taylor series in h:

$$\begin{aligned} d_{0,\sigma} &= \gamma_0(\sigma) + \gamma_1(\sigma)h + \cdots + \gamma_{p-1}(\sigma)h^{p-1} + O(h^p) \\ d_{n+1,\sigma} &= \delta_0(x_{n+1}, \sigma) + \delta_1(x_{n+1}, \sigma)h + \cdots + \delta_p(x_{n+1}, \sigma)h^p + O(h^{p+1}) \end{aligned} \tag{2.6}$$

So if the TSCRK methods (2.3) are consistent of order p then $F\delta_p(x, \sigma) = 0$, for all $0 \leq \sigma \leq 1$.

Theorem 2.1. *[6] The method (2.1) is consistent of order p if*

$$\begin{cases} \sigma^{\rho(t)} = \tilde{\theta}(\sigma)(-1)^{\rho(t)} + \sum\limits_{i=1}^{s}(v_i(\sigma)(e^{-1}\Phi_i)'(t) + w_i(\sigma)(\Phi_i)'(t)) \\ for \quad \rho(t) \leq p \\ \Phi_i(t) = \alpha_i(-1)^{\rho(t)} + \sum\limits_{j=1}^{s}(a_{ij}(e^{-1}\Phi_i)'(t) + b_{ij}(\Phi_i)'(t)) \\ for \quad \rho(t) \leq p-1, i = 1, \cdots, s \end{cases} \tag{2.7}$$

where $e^{-1}(t) = (-1)^{\rho(t)}$, for all trees $t \in T$.

When solving singular delay differential equations or vanishing delay differential equations we assume $w_2(\sigma) \equiv w_3(\sigma) \equiv \cdots \equiv w_s(\sigma) \equiv 0$, for all $0 \leq \sigma \leq 1$. We can construct s-stage TSCRK methods of order p and stage-order q by using the order condition equations (2.7). After satisfying the appropriate order and stage conditions we choose the remaining free parameters trying to minimize the norm of the coefficients of the principal part of the local truncation error. Then we choose the other free parameters to maximize the area of absolute stability. When solving nonvanishing delay differential equations we construct continuous methods based on the TSRK methods constructed in [2]. The coefficients of TSCRK methods for singular delay differential equations of $s = 2, p = 2, q = 1$; $s = 2, p = 2, q = 2$; $s = 3, p = 3, q = 2$ and $s = 4, p = 4, q = 4$ are listed in Table 2.1, Table 2.2, Table 2.3 and Table 2.4 respectively. We assume $\tilde{\theta}(\sigma) \equiv 0$, $0 \leq \sigma \leq 1$, when construct the TSCRK methods.

The coefficients of the TSCRK methods for nonvanishing delay differential equations with $s = 2, p = 2, q = 1$; $s = 3, p = 3, q = 2$ are listed in Table 2.5 and Table 2.6 respectively.

We assume that there is no derivative discontinuities in DDEs (1.1), and f satisfies a Lipschitz condition

$$\|f(x, \tilde{y}, \tilde{z}) - f(x, y, z)\| \leq L \max(\|\tilde{y} - y\|, \|\tilde{z} - z\|)$$

We use the continuous RK method with the order same as the TSCRK methods (2.1) or methods (2.2) to start the integration, i.e., $\|d_{0,\sigma}\| = O(h^p)$, for all $0 \leq \sigma \leq 1$.

Theorem 2.2. *[6] Consider a stable method (2.1) or (2.2) and assume that the local truncation error satisfies (2.6) with $\delta_p(x)$ continuously differentiable. If method (2.1) or (2.2) is consistent of order p and the step size h used satisfies (2.8) then $\|y(x)-Q(x)\| = O(h^p)$, for all $a \le x \le b$.*

$$hL\max_i\left(\sum_{j=1}^{s}|a_{ij}| + \sum_{j=1}^{i-1}|b_{ij}|\right) \le 1 \tag{2.8}$$

Table 2.1: The TSCRK method with $s=2, p=2, q=1$ (method a)

$a_{11}=0.12$	$a_{12}=0.28$	$a_{21}=0.465$
$a_{22}=0.21$	$b_{21}=0.725$	$\alpha_1=0.4$
$\alpha_2=0.4$	$w_1=1.405325$	$w_2=0$
$w_1(\sigma)=153\sigma/169+\sigma^2/2$	$w_2(\sigma)=0$	$v_1=-0.5$
$v_1(\sigma)=-\sigma^2/2$	$v_2=0.094674556$	$v_2(\sigma)=16\sigma/169$

Table 2.2: The TSCRK method with $s=2, p=2, q=2$ (method b)

$a_{11}=0.2$	$a_{12}=0.2$	$a_{21}=-0.55$
$a_{22}=-0.11$	$b_{21}=1.56$	$\alpha_1=0.4$
$\alpha_2=-0.1$	$w_1=1.61$	$w_2=0$
$w_1(\sigma)=61\sigma/100+\sigma^2$	$w_2(\sigma)=0$	$v_1=-0.5$
$v_1(\sigma)=-\sigma^2/2$	$v_2=-0.11$	$v_2(\sigma)=39\sigma/100-\sigma^2/2$

Table 2.3: The TSCRK method with $s=3, p=3, q=2$ (method c)

$a_{11}=0.22$	$a_{12}=-0.14$	$a_{13}=0.22$
$a_{21}=0.43$	$a_{22}=-0.97$	$a_{23}=0.62$
$a_{31}=0.66$	$a_{32}=-1.23$	$a_{33}=0.64$
$b_{21}=0.56$	$b_{31}=0.14$	$b_{32}=0.14$
$\alpha_1=0.3$	$\alpha_2=0.14$	$v_2(\sigma)=-2\sigma^2-4\sigma^3/3$
$w_1=1.48$	$w_2=0$	$v_3(\sigma)=\sigma+7133\sigma^2/10000-799\sigma^3/30000$
$v_1=1.1667$	$v_2=-3.3333$	$v_1(\sigma)=\sigma^2/2+2\sigma^3/3$
$v_3=1.6867$	$\alpha_3=0.15$	$w_3=0$
$w_2(\sigma)=0$	$w_3(\sigma)=0$	$w_1(\sigma)=7867\sigma^2/10000+6933\sigma^3/10000$

Table 2.4: The TSCRK method with $s=4, p=4, q=4$ (method d)

$a_{11}=0.058833$	$a_{12}=0.23533$	$a_{13}=0$
$a_{14}=0.058833$	$a_{21}=-0.10717$	$a_{22}=1.82133$
$a_{23}=-3$	$a_{24}=1.871533$	$a_{31}=-0.334271$
$a_{32}=3.609792$	$a_{33}=-5.9875$	$a_{34}=3.121979$
$a_{41}=-0.67667$	$a_{42}=6.12$	$a_{43}=-9.85333$
$a_{44}=4.48$	$b_{21}=0.2713$	$b_{31}=0.45$
$b_{32}=0.2$	$b_{41}=0.71$	$b_{42}=0.28$
$b_{43}=0.2$	$\alpha_1=0.353$	$\alpha_2=0.357$
$\alpha_3=0.310$	$\alpha_4=0.26$	$v_1(\sigma)=-\sigma^2/6-2\sigma^3/3-2\sigma^4/3$
$w_1=1.81$	$w_2=0$	$v_2(\sigma)=2\sigma^2+20\sigma^3/3+4\sigma^4$
$w_3=0$	$w_4=0$	$v_3(\sigma)=-16\sigma^2/3-32\sigma^3/3-16\sigma^4/3$
$v_1=-1.5$	$v_2=12.6667$	$v_4(\sigma)=44\sigma/25+93\sigma^2/100+17\sigma^3/3+\sigma^4$
$v_3=-21.33333$	$v_4=9.35667$	$w_1(\sigma)=-19\sigma/25+257\sigma^2/100-\sigma^3+\sigma^4$
$w_2(\sigma)=0$	$w_3(\sigma)=0$	$w_4(\sigma)=0$

Table 2.5: The TSCRK method with $s = 2, p = 2, q = 1$ (method e)

$a_{11} = 0.692385$	$a_{12} = 0.219172$	$v_1(\sigma) = 219\sigma/2000 - 1363\sigma^2/5000$
$a_{22} = 0.235132$	$b_{21} = 0.730872$	$\alpha_1 = 0.911557$
$\alpha_2 = 0.601892$	$w_1 = 0.892098$	$w_2(\sigma) = 219\sigma/2000 + 1137\sigma^2/5000$
$v_1 = -0.163152$	$w_2 = 0.336848$	$w_1(\sigma) = 737\sigma/1000 + 1551\sigma^2/10000$
$a_{21} = 0.63588$	$v_2 = -0.065794$	$v_2(\sigma) = 11\sigma/250 - 1099\sigma^2/10000$

Table 2.6: The TSCRK method with $s = 3, p = 3, q = 2$ (method f)

$a_{11} = 0.26445$	$a_{12} = -0.50751$	$a_{13} = 0.264446$
$a_{21} = 0.24029$	$a_{22} = -0.63147$	$a_{23} = 0.597963$
$a_{31} = 0.09693$	$a_{32} = -0.57815$	$w_3(\sigma) = 0.0000219\sigma - 0.5\sigma^2 + 0.61586\sigma^3$
$b_{21} = 0.39233$	$b_{31} = 0.582927$	$w_2(\sigma) = 1.926159\sigma^2 - 1.305112\sigma^3$
$\alpha_1 = 0.02138$	$\alpha_2 = 0.099$	$v_2(\sigma) = 0.000175\sigma - 0.22152\sigma^2 - 0.32178\sigma^3$
$w_1 = 0.0867$	$w_2 = 0.621047$	$v_3(\sigma) = -0.00013\sigma + 0.60393\sigma^2 - 0.08228\sigma^3$
$v_1 = 0.19797$	$v_2 = -0.54313$	$v_1(\sigma) = 0.07384\sigma^2 + 0.124195\sigma^3$
$v_3 = 0.52151$	$\alpha_3 = 0.353437$	$w_3 = 0.115883$
$b_{32} = 0.26228$	$a_{33} = 0.98944$	$w_1(\sigma) = \sigma - 1.882405\sigma^2 + 0.9691199\sigma^3$

3. Stability Analysis of the TSCRK Methods

We apply the methods (2.1) to the linear delay differential test equation

$$y'(x) = \lambda y(x) + \mu y(x - \tau), \quad x \geq a \tag{3.1}$$

where λ, μ are complex numbers, $\tau = (N + \eta)h$, $0 \leq \eta < 1$, N is a natural number. When delay falls in the span of the current step, $N = 0$. If $\eta \leq c_i$, we define $\delta_i = 1, \gamma_i = 0, \varepsilon_i = c_i - \eta$. Alternatively, if $\eta > c_i$, we define $\delta_i = 0, \gamma_i = 1, \varepsilon_i = c_i - \eta + 1$, $i = 1, \cdots, s$. Let $\delta = [\delta_1, \cdots, \delta_s]^T$, $\gamma = [\gamma_1, \cdots, \gamma_s]^T$, $\Delta = diag(\delta_1, \cdots, \delta_s)$, $\Gamma = diag(\gamma_1, \cdots, \gamma_s)$, $W(\zeta) = [w_j(\varepsilon_i)]$, $V(\zeta) = [v_j(\varepsilon_i)]$, $i, j = 1, \cdots, s$. $\tilde{\theta}(\zeta) = diag(\tilde{\theta}(\varepsilon_1), \cdots, \tilde{\theta}(\varepsilon_s))$, $v^T = [v_1, \cdots, v_s]$, $w^T = [w_1, \cdots, w_s]$, $A = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ b_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ b_{s1} & \cdots & b_{ss-1} & 0 \end{pmatrix}$, $I = diag(1, \cdots, 1)_{s\times s}$, $I_1 = diag(1, \cdots, 1)_{(2s+2)\times(2s+2)}$, $\alpha = [\alpha_1, \cdots, \alpha_s]^T$, $e = [1, \cdots, 1]^T$, $S = (I - h\lambda B)^{-1}$, $z = h\lambda$, $z_1 = h\mu$, $T_1 = z_1 v^T + z z_1 w^T SA$, $T_2 = z_1 w^T S$, $T_3 = z z_1 SA$, $T_4 = z_1 S$, $T_5 = z_1 SA$, $T_6 = z_1 SB$.

If we let $\Psi_{n+1} = [y_{n+1}, y_n, Y_{n1}, \cdots, Y_{ns}, hY'_{n1}, \cdots, hY'_{ns}]^T$ then we have

$$\Psi_{n+1} = A_1\Psi_n + A_2\Psi_{n-N+1} + A_3\Psi_{n-N} + A_4\Psi_{n-N-1} + A_5\Psi_{n-N-2} \tag{3.2}$$

where $A_1 = \begin{pmatrix} 1 - \theta + zw^T S(e - u) & \theta + zw^T Su & zv^T + z^2 w^T SA & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} \\ S(e - u) & Su & zSA & \mathbf{0} \\ zS(e - u) & zSu & z^2 SA & \mathbf{0} \end{pmatrix}$,

$A_2 = \begin{pmatrix} 0 & T_2(I - \tilde{\theta}(\zeta))\delta & \mathbf{0} & T_2\Delta W(\zeta) \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_6(I - \tilde{\theta}(\zeta))\delta & \mathbf{0} & T_6\Delta W(\zeta) \\ \mathbf{0} & T_4(I - \tilde{\theta}(\zeta))\delta & \mathbf{0} & T_4\Delta W(\zeta) \end{pmatrix}$,

$$A_3 = \begin{pmatrix} 0 & T_2\tilde{\theta}(\zeta)\delta + T_2(I-\tilde{\theta}(\zeta))\gamma + T_1(I-\tilde{\theta}(\zeta))\delta & \mathbf{0} & T_2\Delta V(\zeta) + (T_2\Gamma + T_1\Delta)W(\zeta) \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_6\tilde{\theta}(\zeta)\delta + T_6(I-\tilde{\theta}(\zeta))\gamma + T_5(I-\tilde{\theta}(\zeta))\delta & \mathbf{0} & T_6\Delta V(\zeta) + (T_6\Gamma + T_5\Delta)W(\zeta) \\ \mathbf{0} & T_4\tilde{\theta}(\zeta)\delta + T_4(I-\tilde{\theta}(\zeta))\gamma + T_3(I-\tilde{\theta}(\zeta))\delta & \mathbf{0} & T_4\Delta V(\zeta) + (T_4\Gamma + T_3\Delta)W(\zeta) \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & T_1\tilde{\theta}(\zeta)\delta + T_1(I-\tilde{\theta}(\zeta))\gamma + T_2\tilde{\theta}(\zeta)\gamma & \mathbf{0} & (T_1\Delta + T_2\Gamma)V(\zeta) + T_1\Gamma W(\zeta) \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_5\tilde{\theta}(\zeta)\delta + T_5(I-\tilde{\theta}(\zeta))\gamma + T_6\tilde{\theta}(\zeta)\gamma & \mathbf{0} & (T_5\Delta + T_6\Gamma)V(\zeta) + T_5\Gamma W(\zeta) \\ \mathbf{0} & T_3\tilde{\theta}(\zeta)\delta + T_3(I-\tilde{\theta}(\zeta))\gamma + T_4\tilde{\theta}(\zeta)\gamma & \mathbf{0} & (T_3\Delta + T_4\Gamma)V(\zeta) + T_3\Gamma W(\zeta) \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & T_1\tilde{\theta}(\zeta)\gamma & \mathbf{0} & T_1\Gamma V(\zeta) \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_5\tilde{\theta}(\zeta)\gamma & \mathbf{0} & T_5\Gamma V(\zeta) \\ \mathbf{0} & T_3\tilde{\theta}(\zeta)\gamma & \mathbf{0} & T_3\Gamma V(\zeta) \end{pmatrix}.$$

Theorem 3.1. *If we let $\xi^n = \Psi_n$, then TSCRK methods (2.1) is stable if and only if all the zeros of*

$$S_h(\lambda, \mu, \xi) = det[\xi^{N+3}I_1 - \xi^{N+2}A_1 - \xi^3 A_2 - \xi^2 A_1 - \xi A_4 - A_5]$$

satisfy root condition.

We plot the stability regions of the TSCRK methods we have given in last section. $z = h\lambda$, $z1 = h\mu$ in the figures.

The stability regions of the TSCRK method (a) (dashed line) and RK method with order 2(solid line) are plotted in Figure 3.1 respectively, where the test equation is

$$y'(x) = \lambda y(x), \qquad x \geq a. \tag{3.3}$$

The stability regions of TSCRK method (a) are plotted in Figure 3.2, where the test equation is

$$y'(x) = \lambda y(x) + \mu y(x - 0.25h). \tag{3.4}$$

The stability regions of the TSCRK method (b) are plotted in Figure 3.3 where the test equation is (3.4) and the stability regions of the improved Euler method (the smaller regions in the middle) are plotted in Figure 3.3 where the test equation is

$$y'(x) = \lambda y(x) + \mu y(x - h). \tag{3.5}$$

The stability regions of the TSCRK method (c) (the left one) and RK method with order 3(the right one) are plotted in Figure 3.4 respectively, where the test equation is (3.3).
The stability regions of the TSCRK method (c) are plotted in Figure 3.5, where the test equation is

$$y'(x) = \lambda y(x) + \mu y(x - 0.5h). \tag{3.6}$$

The stability regions of the TSCRK method (e) (the larger one) and the stability regions of the improved Euler method (the smaller one) are plotted in Figure 3.6 respectively, where the test equation is

$$y'(x) = \lambda y(x) + \mu y(x - 5.25h). \tag{3.7}$$

The stability regions of the TSCRK method (f) (the larger one) and the stability regions of the 4-stage continuous RK method with order 3 (the smaller one) are plotted in Figure 3.7 respectively, where the test equation is equation (3.7).
The stability regions of the TSCRK method (f) (the larger one) and the stability regions of the 4-stage continuous RK method with order 3 (the smaller one) are plotted in Figure 3.8 respectively, where the test equation is equation (3.5).

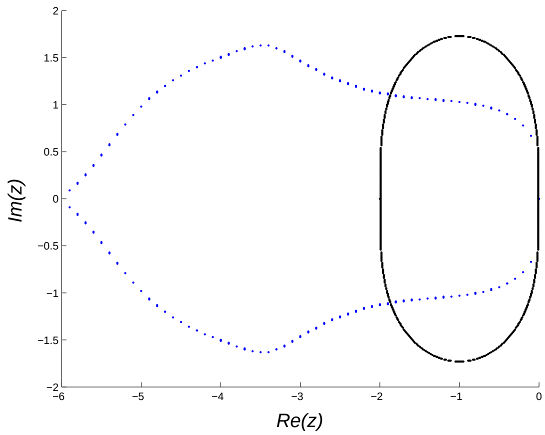

Figure 3.1: The stability regions of the TSCRK method (a) (dashed line) and RK method with order 2(solid line), where the test equation is (3.3).

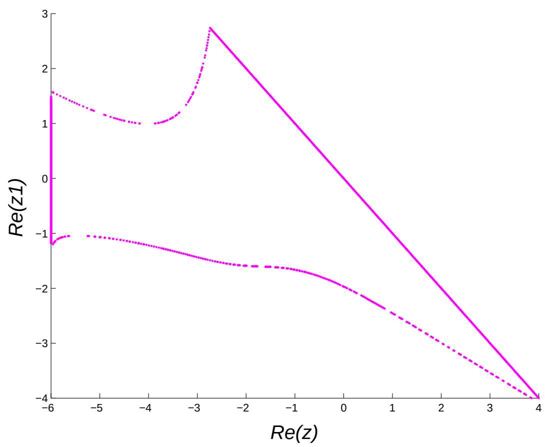

Figure 3.2: The stability regions of TSCRK method (a), where the test equation is (3.4).

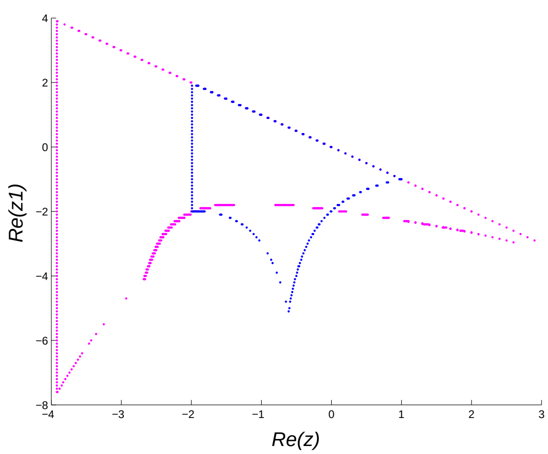

Figure 3.3: The stability regions of the TSCRK method (b) where the test equation is (3.4) and the stability regions of the improved Euler method (the smaller regions in the middle) where the test equation is (3.5).

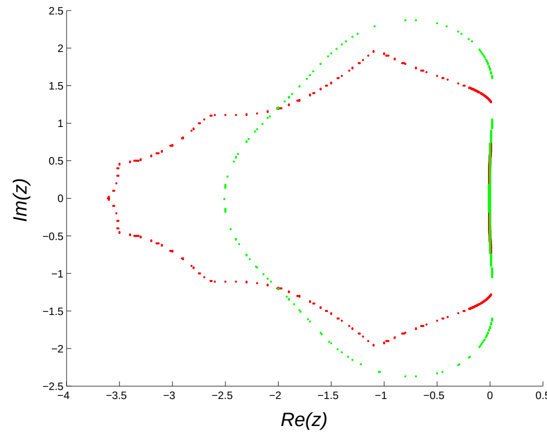

Figure 3.4: The stability regions of the TSCRK method (c) (the left one) and RK method with order 3(the right one), where the test equation is (3.3).

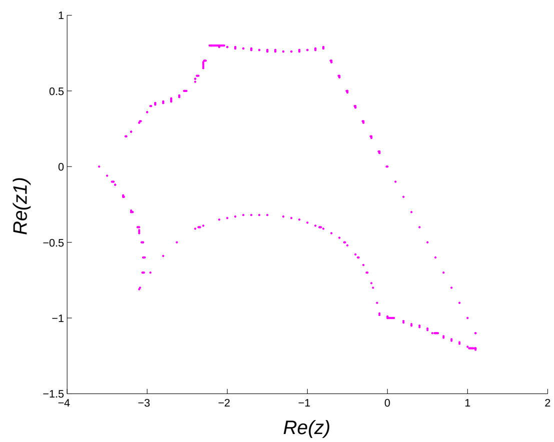

Figure 3.5: The stability regions of the TSCRK method (c), where the test equation is (3.6).

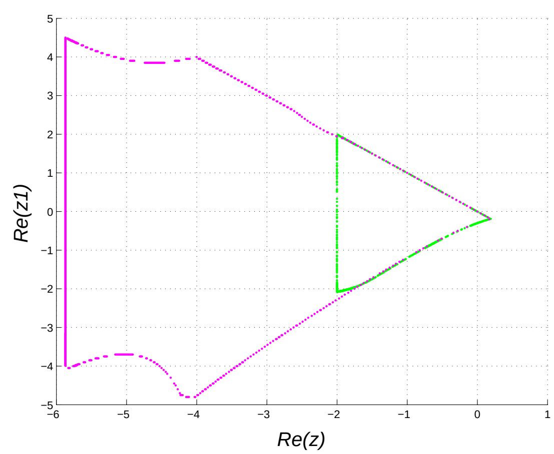

Figure 3.6: The stability regions of the TSCRK method (e) (the larger one) and the stability regions of the improved Euler method (the smaller one), where the test equation is (3.7).

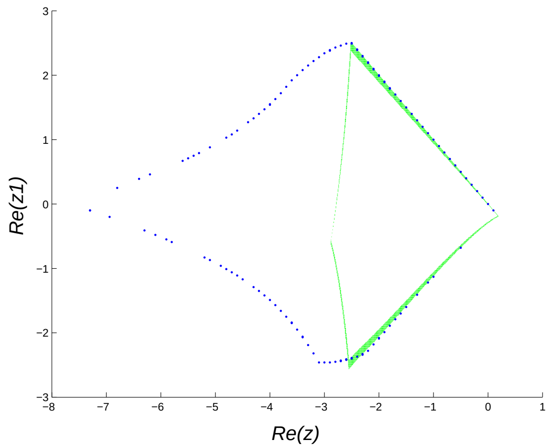

Figure 3.7: The stability regions of the TSCRK method (f) (the larger one) and the stability regions of the 4-stage continuous RK method with order 3 (the smaller one), where the test equation is equation (3.7).

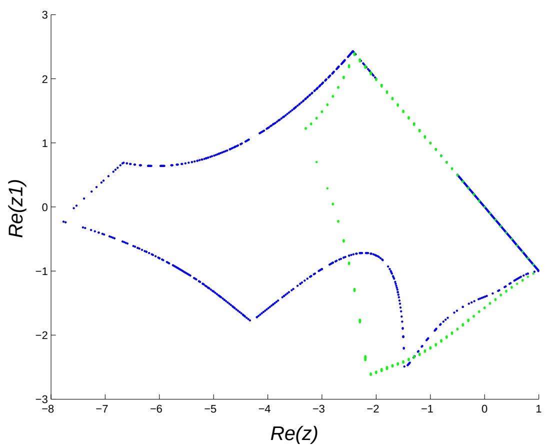

Figure 3.8: The stability regions of the TSCRK method (f) (the larger one) and the stability regions of the 4-stage continuous RK method with order 3 (the smaller one), where the test equation is equation (3.5).

We can find in Figure 3.6, Figure 3.7, and Figure 3.8 that the stability regions of the TSCRK methods are obviously larger than the stability region of the continuous RK methods of the same order when solving nonvanishing delay differential equations.

4. Numerical Results

We tested the methods we have given in Table 2.1 and Table 2.4 on two examples. The first test problem is an asymptotically vanishing delay differential equation, where $h > \tau$ on almost every step when $x > 3$ and the solutions increase significantly. We apply the TSCRK methods with $s = 4, p = 4, q = 4$ to the first example. We apply the TSCRK methods with $s = 2, p = 2, q = 1$ to the second example. We list the results in Table 4.1 and Table 4.2 respectively. The error is the maximum absolute error over mesh points, $\|y_n - y(x_n)\|$, on the whole integration interval.

Example 1.[3] (Asymptotically vanishing delay)

$y'(x) = (1 + e^{-x})y(x - e^{-x})exp(e^{-x+e^{-x}}), \qquad x \in [0.6, 4]$

$y(x) = e^{x-e^{-x}}, \qquad x \in [0, 0.6]$

The exact solution is $y(x) = e^{x-e^{-x}}$.

Example 2.[1] (constant delay)

$y'(x) = -y(x) - y(x - \pi) + 3cos(x) + 5sin(x), \quad x \in [0, 10]$

$y(x) = 3sin(x) - 5cos(x), \quad x \leq 0$

The exact solution is $y(x) = 3sin(x) - 5cos(x)$.

Table 4.1: The maximum absolute error on the interval $[0, 4]$ for example 1

Stepsize	$h = 0.1$	$h = 0.05$
error	$7.141310195351025e - 004$	$4.455799361124946e - 005$

Table 4.2: The maximum absolute error on the interval $[0, 10]$ for example 2

Stepsize	$h = 0.01$	$h = 0.005$
error	$3.521952101568360e - 004$	$8.776590240078264e - 005$

The numerical results confirm our analysis of the order of our methods and show that the TSCRK methods is efficient in solving singular delay differential equations and vanishing delay differential equations.

References

[1] C.T.H. Baker, C.A.H. Paul, Parallel continuous Runge-Kutta methods and vanishing lag delay differential equations, *Adv. Comput. Math.* **1** (1993), 367-394.

[2] J.Chollom, Z.Jackiewicz, Construction of two-step Runge-Kutta methods with large regions of absolute stability, *J. comput. and Appl. Math.*, **157** (2003), 125-137.

[3] W.H.Enright and Min Hu, Interpolating Runge-Kutta methods for vanishing delay differential equations, *computing*, **55** (1995), 223-236.

[4] E.Hairer, S.P.Nørsett, and G.Wanner, Solving Ordinary Differential Equations I. Nonstiff problems, Springer Series in Computational Mathematics 8, Springer-Verlag Berlin Heidelberg, 1987, second Revised Edition, 1993.

[5] E.Hairer, G.Wanner, Order conditions for general Two-Step Runge-Kutta Methods, *SIAM J. Num. Anal.* **34** (1997), 2087-2089.

[6] Leng xin, Liu Degui, Song Xiaoqiu, Chen Lirong, The Convergence of a Class of Two-step Continuity Runge-Kutta Methods for Solving Singular Delay Differential Equations, to appear.