THE GENERALIZED MAXIMUM ANGLE CONDITION FOR THE $Q_1$ ISOPARAMETRIC ELEMENT

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Abstract

We consider the quadrilateral $Q_1$ isoparametric element and establish an optimal error estimate in $H^1$ norm for the interpolation operator under a weaker mesh condition which admits anisotropic quadrilaterals and allows the quadrilateral to become a regular triangle in the sense of maximum angle condition [5, 11].

Key words: Quadrilateral mesh, $Q_1$ isoparametric element, Generalized maximum angle condition

1. Introduction

We shall consider the quadrilateral $Q_1$ element and establish an estimate for the interpolation error under a new mesh condition. This condition is weaker than the preceding conditions proposed in [12] and [2] among others. Moreover, it allows the quadrilateral to degenerate into an anisotropic however regular triangle in the sense of maximum angle condition [5, 11, 2]. First we will review some known results and introduce some notations.

Let $K$ be a convex quadrilateral with vertices $M_1, M_2, M_3$ and $M_4$. Let $\hat{K} = [-1, 1]^2$ be the reference element. There exists a bijection mapping $F_K : \hat{K} \rightarrow K$ that $K = F_K(\hat{K})$.

Let $\hat{Q}_1(\hat{K})$ be the bilinear polynomial space, and let $Q_1 = Q_1(K)$ be the corresponding space defined on $K$. Let $\Pi_1$ denote the usual bilinear interpolation operator.

Our aim is to obtain the following interpolation error estimate

$$\|u - \Pi_1 u\|_{0,K} + h \|u - \Pi_1 u\|_{1,K} \leq C \epsilon h^2 \|u\|_{2,K}$$ (1.1)

under the condition we shall proposed, where $h$ is the diameter of $K$. There are several conditions in the literature for (1.1) to hold, here we only review, among others, the $J$ condition and $RDP$ condition, proposed by Jamet [12] and Acosta and Duran [2] respectively, which can be expressed as follows

Definition 1.1 $K$ is regular with constant $\sigma > 0$, or shortly $J(\sigma)$, if it holds that

$$h/\rho \leq \sigma,$$

where $h$ denotes the diameter of $K$ and $\rho$ the maximum of the diameters of all circles contained in $K$.

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**Definition 1.2** \( K \) is regular with constant \( N \in \mathbb{R} \) and \( 0 < \psi < \pi \), or shortly \( \text{RDP}(\psi, N) \), if we can divide \( K \) into two triangles along one of its diagonals, which will always be called \( D_1 \), the other is \( D_2 \) in such a way that \( |D_2| / |D_1| \leq N \) and both triangles satisfy the maximum angle condition, i.e., each interior angle of these two triangles is bounded from above by \( \psi \).

For other conditions, we refer to references \([7, 8, 9, 14]\) and \([3, 17]\). A comprehensive review of quadrilateral meshes can be found in the introduction of \([14]\), there the equivalency and the relation of some shape mesh conditions is also proved. The review of degenerate quadrilateral mesh conditions can also be found in \([2]\).

Under the \( J(\sigma) \) condition, it was shown in \([12]\) that the constant \( C_e \) in (1.1) depends only on \( \sigma \). Under the constraint \( \text{RDP}(\psi, N) \), Acosta and his colleague prove that \( C_e \) depends only on \( \psi \) and \( N \). \( \text{RDP}(\psi, N) \) condition is so far the weakest mesh condition for (1.1) to hold. However, due to the constraint \( |D_2| / |D_1| \leq N \), it does not allow a quadrilateral to become an anisotropic however regular triangle in the sense of maximum angle condition. As we will see below the constraint \( |D_2| / |D_1| \leq N \) can be removed.

We introduce some notations and concepts. Let \( d_1 \) denote the longer diagonal of \( K \), \( d_2 \) the shorter one. As illustrated in Fig.1, we denote by \( T_1 \) and \( T_2 \) the two triangles obtained by subdividing \( K \) along \( d_1 \), and \( t_1 \) and \( t_2 \) are the two triangles obtained by decomposing \( K \) along \( d_2 \).

![Fig.1. Quadrilateral K](image)

**Definition 1.3** \([5, 11, 2]\) We say a triangle \( T \) (resp. a quadrilateral \( K \)) satisfies the maximum angle condition with a constant \( \psi \), or shortly \( \text{MAC}(\psi) \), if the angles of \( T \) (resp. \( K \)) are less than or equal to \( \psi \).

In the sequel, the regularity of triangles is referred to as in this maximum angle sense. Our mesh condition can be stated as

**Definition 1.4** We say a convex quadrilateral \( K \) satisfies the generalized maximum angle condition, or shortly \( \text{GMAC}(\psi) \), if there exists a positive constant \( \psi < \pi \) such that, among \( T_i, t_i, i = 1, 2 \), there are at least three regular triangles in the sense of \( \text{MAC}(\psi) \).

Let us notice that the constraint \( |D_2| / |D_1| \leq N \) in the \( \text{RDP}(N, \psi) \) condition is dropped in this condition. We shall prove the following result

**Theorem 1.1** Let \( K \) be a convex quadrilateral satisfying \( \text{GMAC}(\psi) \) with the constant \( 0 < \psi < \pi \) and \( u \in H^2(K) \), then there exists a constant \( C_{\text{err}} \) only depending on \( \psi \) such that

\[
|u - \Pi_1 u|_{m,K} \leq C_{\text{err}}(\psi) h^{2-m} \quad |u|_{2,K,m} = 0, 1
\]
Our analysis is based on three key points. First, we introduce an appropriate classification of quadrilaterals, which gives a close look on the geometry of quadrilaterals. Second, following the idea of [2], we adopt an appropriate affine change for the analysis, which is different from that used in [2]. Third, a sharper estimate for the integration $\int_K \frac{1}{\lambda_K}$ is given.

### 2. A Classification of Convex Quadrilaterals

In this section, we first introduce an appropriate method to classify convex quadrilaterals, and propose a mesh condition which is equivalent to $GMAC(\psi)$, but is more convenient for our analysis.

Let $Q$ denote the set of convex quadrilaterals. According to Definition 1.3, $Q$ can be divided into the following two subsets:

$$
\mathcal{R} = \{ K \in Q \mid K \text{ is regular} \}, \\
\mathcal{D} = \{ K \in Q \mid K \notin \mathcal{R} \}.
$$

$\mathcal{D}$ can be further divided into the following three subsets:

$$
\mathcal{DB} = \{ K \in \mathcal{D} \mid \text{both } t_1 \text{ and } t_2 \text{ are regular} \}, \\
\mathcal{DO} = \{ K \in \mathcal{D} \setminus \mathcal{DB} \mid \text{either } t_1 \text{ or } t_2 \text{ is regular} \}, \\
\mathcal{DN} = \{ K \in \mathcal{D} \mid K \notin \mathcal{DB} \cup \mathcal{DO} \}.
$$

According to the regularity of $t_1$ and $t_2$, the set $\mathcal{DO}$ can be further divided as

$$
\mathcal{DOB} = \{ K \in \mathcal{DO} \mid \text{both of } t_1 \text{ and } t_2 \text{ are regular} \}, \\
\mathcal{DON} = \{ K \in \mathcal{DO} \mid K \notin \mathcal{DOB} \}.
$$

Obviously we have

$$
Q = \mathcal{R} \cup \mathcal{DB} \cup \mathcal{DN} \cup \mathcal{DOB} \cup \mathcal{DON}, \quad (2.1)
$$

With these preparations, we can state our equivalent mesh condition as following

**Theorem 2.1** Let $K$ be a convex quadrilateral, then $K$ is regular if and only if $K \in RQ = \mathcal{R} \cup \mathcal{DB} \cup \mathcal{DOB}$. \hspace{1cm} (2.2)

**Proof.** The necessity of (2.2) is obvious, we only need to show it is sufficient. First, if $K \in \mathcal{R} \cup \mathcal{DOB}$, $K$ is certainly regular in the sense of $GMAC(\psi)$. Second, if $K \in \mathcal{DB}$, in this case, we can assert just one of $t_i, i = 1, 2$ is not regular in the sense of $MAC(\psi)$. Otherwise, the fact that $d_1$ is the longer than $d_2$ will be violated, which completes the proof.

We note here that if $K$ is regular in the sense of $RDP(\psi, N)$, from Lemma 3.1 of [2], there exists a constant $\psi_1 = \psi_1(\psi, N) < \pi$ such that $K$ is regular in the sense of $GMAC(\psi_1)$. The converse is not true, because not all quadrilaterals in $\mathcal{DOB}$ are "regular" in that sense. For example, the quadrilateral on the left hand in Fig.2 is not regular in the sense of $RDP(\psi, N)$, however is regular in the sense of $GMAC(\psi)$ when the parameter $a$ tends to zero.

![Fig.2. Examples of Quadrilaterals](image-url)
3. The Affine Transformation

In this section, following the idea of [2], we introduce an affine change of variables, which is different from that defined in [2], however is more convenient for our analysis.

![Fig.3. Quadrilateral K and the affine element $\tilde{K}$](image)

We first quote a technical lemma from [1].

**Lemma 3.1** Let $L$ be the linear transformation associated with a matrix $B$. Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, let $\alpha_1$ be the angle between them and $\alpha_2$ be the angle between $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$, then it holds that

$$\frac{2}{\text{cond}(B)\pi} \alpha_1 \leq \alpha_2 \leq \pi \left(1 - \frac{2}{\text{cond}(B)}\right) + \frac{2}{\text{cond}(B)\pi} \alpha_1. \quad (3.1)$$

We introduce an affine change by the following result.

**Lemma 3.2** Let $K$ be a quadrilateral of diameter $h$ satisfying $\text{GMAC}(\psi)$. Then, there exist $\bar{x}_1$, $\bar{x}_3$, $\bar{y}_3$ and $\bar{y}_4$, an affine transformation $L\bar{x} = B\bar{x} + P$ such that $L(\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)) = K$ and constants $C = C(\psi)$ such that

$$\|B\| \leq C, \quad \|B^{-1}\| \leq C, \quad \text{in particular, } \text{cond}(B) \leq C^2. \quad (3.2)$$

Moreover, $\tilde{K}$ is regular in the sense of $\text{GMAC}(\tilde{\psi})$ with constant $\tilde{\psi} < \pi$ depending only on $\psi$. Hereinafter, $\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ denotes the configuration illustrated on the right-hand side of Fig.3.

**Proof.** We will construct $\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ and $L$ explicitly. Since $K$ satisfies $\text{GMAC}(\psi)$, we can assume without lose of generality that $\triangle M_1 M_2 M_4$ is regular, and that $\pi \geq \psi \geq \beta \geq \delta$ with $\delta = \frac{\pi}{2} - \frac{\psi}{2}$ (See Fig.3).

Further, we assume that the diagonal $d_1$ lies along the $y$-axis and that the vertex $M_2$ is put at the origin (up to a rigid movement), $y_4$ is the length of the longer diagonal $d_1$, the vertex corresponding to the angle $\beta$ is placed at vertex $M_4(0, y_4)$ (See Fig.3).

At last we let $\bar{x}_1 = -|M_1 M_4|\sin \beta = x_1$, $\bar{y}_1 = y_4$, $\bar{y}_4 = y_4$ and

$$B = \begin{pmatrix} 1 & 0 \\ \cot \beta & 1 \end{pmatrix}, \quad \begin{pmatrix} \bar{x}_3 \\ \bar{y}_3 \end{pmatrix} = B^{-1} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \quad (3.3)$$

An elementary calculation finds $\|B\| \leq \sqrt{2}/\sin \beta$, $\|B^{-1}\| \leq \sqrt{2}/\sin \beta$, which imply that $\text{cond}(B) \leq 2/\sin^2 \beta$. We let $\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ be the convex quadrilateral with vertices $\tilde{M}_1(\bar{x}_1, \bar{y}_4)$, $\tilde{M}_2(0, 0)$, $\tilde{M}_3(\bar{x}_3, \bar{y}_3)$ and $\tilde{M}_4(0, \bar{y}_4)$, note that $L(\tilde{K}) = K$. 

To prove the second part of the Lemma, we only show that the assertion is valid for $K \in \mathcal{R}$. In this case, any interior angle $\theta$ of $K$ is not greater than $\psi$, in view of Lemma 3.1, we have

$$\theta \leq \pi(1 - 2/\pi\text{cond}(B)) + (2/\pi\text{cond}(B))\psi < \pi,$$

where the constant $\tilde{\psi}$ obviously only depends on $\psi$. Therefore $\tilde{K}$ is also a regular quadrilateral in the sense of $\text{MAC}(\tilde{\psi})$, which ends the proof. For the other cases, the proof is similar because $L$ and its inverse are bounded linear transformations with the norms in terms of $\psi$.

Proceeding along the same line of Lemma 3.4 in [2], we have

**Lemma 3.3** Given a quadrilateral $K$ satisfying $\text{GMAC}(\psi)$, let $L$ and $\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ be the affine transformation and the affine element given in Lemma 3.2, $\bar{u} = u \circ L$. Then there are two positive constants $C_1$ and $C_2$ depending only on $\psi$ such that

$$C_1 | \bar{u} - \Pi_1 \bar{u} |_{1,\tilde{K}} \leq | u - \Pi_1 u |_{1,K} \leq C_2 | \bar{u} - \Pi_1 \bar{u} |_{1,\tilde{K}},$$

$$C_1 | \bar{u} |_{2,\tilde{K}} \leq | u |_{2,K} \leq C_2 | \bar{u} |_{2,\tilde{K}}.$$  

**Remark 3.1** Notice that in our transformation, the constants in Lemma 3.2 and Lemma 3.3 only depend on $\psi$, while the corresponding constants in [2] depend on both $\psi$ and the ratio $d_2/d_1$.

### 4. Error Estimates

The purpose of this section is to derive the optimal interpolation error estimate for the quadrilateral element satisfying $\text{GMAC}(\psi)$. We shall follow the main ideas of [12] to decompose the $Q_1$ interpolation error into two parts: one is the $P_1$ interpolation error and the other is the difference between $P_1$ interpolation and $P_1$ interpolation, and then use the idea of function extension to estimate the second part. However, we shall use the idea of [2] to bound the first part.

We introduce some notations. Denote the angles of the two diagonals $M_1M_3$ and $M_2M_4$ by $\alpha$ and $\pi - \alpha$ with $0 \leq \alpha \leq \frac{\pi}{2}$, and let $O$ be the point at which they intersect. Let $a_i = |OM_i|$, $i = 1, 2, 3, 4$, let $T_1$ denote triangle $\triangle M_1M_2M_3$, and $T_2$ denote $\triangle M_2M_3M_4$. Set $a = \max(a_i, a_{i+1})$, $d = \min(|M_3M_1|, |M_3M_2|, |M_3M_4|)$. As in Lemma 3.2, we assume the longer diagonal $d_1$ lies on the y-axis, $M_2$ is placed at the fixed point $(0,0)$, $x_1 \leq 0$ and $x_3 \geq 0$ (Sec. Fig 3), and that $T_1$ is regular.

To derive the estimate, following the idea of [12] and [2], we decompose the error in the following way:

Let $\Pi$ be the $P_1$-Lagrange interpolation operator associated with the vertices $M_1, M_2$ and $M_4$, i.e., $\Pi u$ is a linear function which admits the same values with $u$ at these three nodes, then we have

$$| u - \Pi_1 u |_{1,K} \leq | u - \Pi u |_{1,K} + | \Pi u - \Pi_1 u |_{1,K}. 
$$

(4.1)

Because $\Pi u$ is a linear function on quadrilateral $K$, $\Pi u - \Pi_1 u$ belongs to the isoparametric finite element space and vanishes at nodes $M_i$, $i = 1, 2, 4$, then we have

$$(\Pi u - \Pi_1 u)(x) = (\Pi u - \Pi_1 u)(M_3) \phi_3(x),$$

where $\phi_3$ is the usual bilinear nodal basis at node $M_3$, therefore

$$| u - \Pi_1 u |_{1,K} \leq | u - \Pi u |_{1,K} + | (\Pi u - \Pi_1 u)(M_3) | \phi_3 |_{1,K}. 
$$

(4.2)

We first take care of the term $| u - \Pi u |_{1,K}$. In view of Lemma 3.2 and Lemma 3.3 in the previous section, it suffices to analyze the case where the reference configuration $\tilde{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ is considered.

Let $\bar{u}$ be the function defined on $\tilde{K}$ through $u$ by the following canonical relation

$$\bar{u} = u(L(\bar{x})), \forall \bar{x} \in \tilde{K}.$$  

(4.3)
Denote $\bar{\Pi}$ the $P_1$-Lagrange interpolation of $\bar{u}$ on $\bar{K}$, which agrees with $\bar{u}$ at the vertices $\bar{M}_i, i = 1, 2, 4$, we have
$$\bar{\Pi}u = \bar{\Pi}\bar{u}. \tag{4.4}$$

**Remark 4.1** Because $d_1$ is the longest diagonal of $K$, we can always assume, without lose of generality, that $|T_1| \geq |T_2|$. Owning to det($B$) = 1, we have $|\bar{T}_1| = |T_1| \geq |T_2| = |\bar{T}_2|$, therefore $\frac{|\bar{T}_1|}{|T_1|} \geq \frac{1}{2}$. Where $|T_1|$ is the volume of the triangle $T_1$.

We need to give a brief verification for such an assumption. First, if $K \in \mathcal{R} \cup \mathcal{DB}$, by the symmetry, it is reasonable for us to say so. Second, if $K \in \mathcal{DOB}$, since $d_1$ is the longer diagonal by the definition, we assert the interior angle at the vertex $M_3$ is greater than $\psi$, otherwise the fact that $t_1$ and $t_2$ are regular in the sense of $MAC(\psi)$ will be violated. Then $\angle M_1M_3M_2 > \angle M_1M_2M_3$, which implies that $|x_1| \geq |x_3|$

Under this assumption, we can obtain the following error estimate for the $P_1$-Lagrange interpolation operator $\bar{\Pi}$ and $\Pi$,

**Lemma 4.1** Let $\bar{\Pi}\bar{u}$ and $\Pi u$ be defined as above, then
$$|\bar{u} - \bar{\Pi}\bar{u}| \leq Ch^{2-m}|\bar{u}|_{2,\bar{\bar{K}}}, m = 0, 1. \tag{4.5}$$
$$|u - \Pi u| \leq C(\psi)h^{2-m}|u|_{2,\bar{K}}, m = 0, 1. \tag{4.6}$$

**Proof.** For $m = 1$, the inequality (4.5) can be proved in a similar way as Lemma 4.3 of [2] by using the fact that $\frac{|\bar{T}_1|}{|T_1|} \geq \frac{1}{2}$. For $m = 0$, using the affine transformation $\bar{L}$ defined in Theorem 4.1 of [2], we can obtain another quadrilateral $\bar{K}$, which obviously satisfies the condition $J(\sigma)$ with $\sigma = 1$. Therefore, the estimate on $\bar{K}$ follows from [12] and the estimate on $K$ is obtained by changing variables. At last, (4.6) is the immediate consequence of (4.5), Lemma 3.2 and Lemma 3.3.

We now turn to the second term on the right hand of the inequality (4.2). This time, following the idea of Lemma 3.2 of [12], we shall apply the theory of function extension from [13], which first prove the following lemma.

**Lemma 4.2** There exists a constant $\beta_0 > 0$ only depending on $\psi$, such that, let $G_0$ be a fixed isosceles triangle with two angles equal to $\beta_0$, there exists an isosceles triangle $\hat{G}$ contained in $K$ which is similar to $G_0$, and admits the segment $M_3M_i$ as its base, where $M_3M_i$ denotes the edge which satisfies $d = M_3M_i$ among $M_3M_i, i = 1, 2, 4$.

**Proof.** In view of the equivalent result Lemma 2.1, there are three cases of which we have to take care.

**Case I** $K \in \mathcal{R}$.

First, we assume $d = |M_3M_1|$, this is to say, $|M_3M_1| \leq |M_3M_2|$ and $|M_3M_1| \leq |M_3M_4|$. Owning to Remark 4.1, namely $|T_1| \geq |T_2|$, it is easy to see max($|M_1M_2|, |M_1M_4|$) $\geq$ min($|M_3M_2|, |M_3M_4|$) $\geq |M_3M_1|$, therefore, there exists at least one triangle between $t_1$ and $t_2$ such that $d_2$ is its shortest edge, due to the regularity of $t_1$ and $t_2$, we conclude the assertion is valid for this case.

Second, if the shortest edge is not $|M_3M_1|$, without lose of generality, we assume $d = |M_2M_3|$, consequently $|d_1| \geq |M_2M_3|$, therefore $M_2M_3$ is the shortest edge of $T_2$, by virtue of the regularity of $T_2$, we conclude the assertion is also valid for this case.

**Case II** $K \in \mathcal{DB}$. Note that in this case $M_3M_1$ is not the shortest one among $M_3M_i, i = 1, 2, 4$. Following the line of the second part of **Case I**, we can achieve the desired result.

**Case III** $K \in \mathcal{DOB}$. 

If \( M_2M_4 \) is the shortest edge among \( M_2M_i, i = 1, 2, 4, \) taking into account the regularity of \( t_1 \) and \( t_2, \) following the same procedure of the first part of case I, we obtain the result. On the contrary, without lose of generality, we assume again \( d = |M_2M_3| \). Since \( \Delta M_1M_2M_3 \) is regular and \( d_2 \) is not the shortest edge of \( \Delta M_1M_2M_3, \) we obtain
\[
\psi \geq \angle M_1M_2M_3 \geq \delta.
\]
Moreover \( T_2 \) is not regular and \( d_1 \) is the longer diagonal, then
\[
\angle M_2M_3M_4 \geq \psi,
\]
which implies the desired result.

Using Lemma 4.2 and Lemma 4.1, we have

**Lemma 4.3** Let \( K \) be a regular quadrilateral in the sense of \( \text{GMAC}(\psi) \) with constant \( \psi \) and \( u \in H^2(K), \) then, for any real \( 0 < \gamma < 1, \) it holds that
\[
| (u - \Pi u)(M_3) | \leq C(\gamma, \psi)d^\gamma h^{1-\gamma} | u |_{2,K}
\]
(4.7)

**Proof.** Owning to the Holder-continuous property with real \( 0 < \gamma < 1 \) of functions in \( H^2(K), \) using Lemma 4.2 and Lemma 4.1, proceeding along the same line of Lemma 3.2 of [12], we can obtain the asserted result.

We remain to bound the term \( | \phi_3 |_{1,K}, \) which forces us to estimate the integration
\[
\int_K | J_K^{-1} | d\xi d\eta, \quad \text{where } J_K \text{ is the Jacobian determinant of the bilinear transformation. Here we adopt the method of [12].}
\]
Replacing the inequality \( \log(1 + t) \leq t^{\gamma/2}, t > 0 \) in the proof of Lemma 2.1 of [12] by the inequality \( \log(1 + t) \leq t, t > 0, \) we get the following sharper estimate
\[
\int_K | J_K^{-1} | d\xi d\eta \leq \frac{8}{(\sin \alpha \cdot a)},
\]
(4.8)
with \( a = \max(\alpha_i, \alpha_{i+1}) \). We now bound the term \( \frac{1}{\sin \alpha} \) in terms of \( \psi \) and \( \frac{|d_2|}{d}. \) Let \( s_1 \) be the shortest edge of \( T_1 \) and \( s_2 \) be the shortest edge of \( T_2. \) If \( K \in \text{DOB}, \) as illustrated in Fig.3, \( \theta_4 \) will go to zero, \( \theta_1 + \gamma \geq \frac{\delta}{2} \) because \( \Delta M_1M_2M_3 \) is regular in the sense of \( \text{MAC}(\psi), \) moreover \( \beta \geq \delta, \) thus it is easy to see \( \frac{\delta}{2} \leq \alpha, \) i.e.
\[
\frac{1}{\sin \alpha} \leq C(\psi) \frac{|d_2|}{d}
\]
(4.9)

If \( K \in \mathcal{R} \cup \mathcal{DB}, \) without lose of generality, we assume \( s_1 = M_1M_4. \) If \( s_1 \) is not the shortest edge of \( \Delta M_1OM_4, \) we have \( \delta \leq \alpha \) because \( \Delta M_1M_2M_4 \) is regular. If \( s_1 \) is the shortest edge \( \Delta M_1OM_4, \) in this case, if \( s_2 = M_3M_4, \) by the regularity of \( \Delta M_2M_3M_4, \) it is easy to see \( \alpha \geq \delta, \) therefore without lose of generality, we assume \( s_2 = M_2M_3. \) Because \( \psi \geq \beta \geq \delta, \) we get \( \psi \geq \theta_2 \geq \delta. \) Moreover,
\[
\frac{|M_1M_4|}{\sin \alpha} = \frac{|M_1O|}{\sin \beta},
\]
\[
\frac{|M_2M_3|}{\sin \alpha} = \frac{|OM_3|}{\sin \angle M_4M_2M_3}
\]
Since \( \Delta M_2M_3M_4 \) is regular and \( M_3M_4 \) is not its shortest edge,
\[
\frac{|s_1|}{|s_2|} = \frac{|M_1O| \sin \angle M_4M_2M_3}{|OM_3| \sin \beta} \geq \frac{x_1 \sin \angle M_4M_2M_3}{x_3 \sin \beta} \geq C(\psi),
\]
therefore,
\[
\frac{1}{\sin \alpha} = \frac{|OM_1|}{|s_1| \sin \beta} \leq C(\psi) \frac{|OM_1|}{|s_2|} \leq C(\psi) \frac{|d_2|}{d}.
\]
(4.10)
By virtue of Lemma 2.2 in [12], (4.9) and (4.10), we derive as
\[
|\phi_3|_{1,K} \leq \frac{8\sqrt{2}h}{\sqrt{\sin \alpha \cdot a}} |\hat{\phi}_3|_{1,\infty,K} \leq \frac{16\sqrt{2}h}{\sqrt{\sin \alpha}} \frac{\sqrt{d}}{d_1} |\hat{\phi}_3|_{1,\infty,K} \leq C(\psi) \frac{h}{\sqrt{d_1} d_2},
\]
which, together with Lemma 4.3 with \(\gamma = \frac{1}{2}\), implies
\[
|\phi_3|_{1,K} \leq \frac{8\sqrt{2}h}{\sqrt{\sin \alpha}} |\hat{\phi}_3|_{1,\infty,K} \leq \frac{16\sqrt{2}h}{d_1} |\hat{\phi}_3|_{1,\infty,K} \leq C(\psi) h |u|_{2,K}.
\]
Now we prove our main result Theorem 1.1.

**Proof of Theorem 1.1.**

**Proof.** Because
\[
|u - \Pi_1 u|_{1,K} \leq |u - \Pi u|_{1,K} + |(u - \Pi_1 u)(M_3)| \leq C(\psi) h |u|_{2,K}.
\]
The first term is bounded in Lemma 4.1, and the second term is bounded in (4.11). As for the optimal \(L^2\) error estimate, it can be obtained by using our transformation, the idea of Theorem 4.1 of [2] and the estimates given in Theorem 1 of [12], for brevity, we skip the details.

**References**