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THE GENERALIZED MAXIMUM ANGLE CONDITION FOR THE $\mathcal{Q}_1$ ISOPARAMETRIC ELEMENT *[1)]

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Abstract

We consider the quadrilateral $\mathcal{Q}_1$ isoparametric element and establish an optimal error estimate in H^1 norm for the interpolation operator under a weaker mesh condition which admits anisotropic quadrilaterals and allows the quadrilateral to become a regular triangle in the sense of maximum angle condition [5, 11].

Mathematics subject classification: 65N30.
Key words: Quadrilateral mesh, $\mathcal{Q}_1$ isoparametric element, Generalized maximum angle condition

1. Introduction

We shall consider the quadrilateral $\mathcal{Q}_1$ element and establish an estimate for the interpolation error under a new mesh condition. This condition is weaker than the precede conditions proposed in [12] and [2] among others. Moreover, it allows the quadrilateral to degenerate into an anisotropic however regular triangle in the sense of maximum angle condition [5, 11, 2]. First we will review some known results and introduce some notations.

Let K be a convex quadrilateral with vertices M_1, M_2, M_3 and M_4. Let $\hat{K} = [-1,1]^2$ be the reference element. There exists a bijection mapping $\mathcal{F}_K : \hat{K} \to K$ that $K = \mathcal{F}_K(\hat{K})$.

Let $\hat{\mathcal{Q}}_1(\hat{K})$ be the bilinear polynomial space, and let $\mathcal{Q}_1 = \mathcal{Q}_1(K)$ be the corresponding space defined on K. Let Π_1 denote the usual bilinear interpolation operator.

Our aim is to obtain the following interpolation error estimate

$$\|u-\Pi_1 u\|_{0,K} + h \mid u-\Pi_1 u \mid_{1,K} \leq C_e h^2 \mid u \mid_{2,K} \tag{1.1}$$

under the condition we shall proposed, where h is the diameter of K. There are several conditions in the literature for (1.1) to hold, here we only review, among others, the J condition and RDP condition, proposed by Jamet [12] and Acosta and Duran [2] respectively, which can be expressed as follows

Definition 1.1 *K is regular with constant $\sigma > 0$, or shortly $J(\sigma)$, if it holds that*

$$h/\rho \leq \sigma,$$

where h denotes the diameter of K and ρ the maximum of the diameters of all circles contained in K.

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Definition 1.2 *K is regular with constant $N \in R$ and $0 < \psi < \pi$, or shortly RDP(ψ,N), if we can divide K into two triangles along one of its diagonals, which will always be called D_1, the other is D_2 in such a way that $\mid D_2 \mid / \mid D_1 \mid \leq N$ and both triangles satisfy the maximum angle condition, i.e., each interior angle of these two triangles is bounded from above by ψ.*

For other conditions, we referr to references [7, 8, 9, 14] and [3, 17]. A comprehensive review of quadrilateral meshes can be found in the introduction of [14], there the equivalency and the relation of some shape mesh conditions is also proved. The review of degenerate quadrilateral mesh conditions can also be found in [2].

Under the *J(σ)* condition, it was shown in [12] that the constant C_e in (1.1) depends only on σ. Under the constraint *RDP(ψ, N)*, Acosta and his colleague prove that C_e depends only on ψ and N. *RDP(ψ, N)* condition is so far the weakest mesh condition for (1.1) to hold. However, due to the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$, it does not allow a quadrilateral to become an anisotropic however regular triangle in the sense of maximum angle condition. As we will see below the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$ can be removed.

We introduce some notations and concepts. Let d_1 denote the longer diagonal of K, d_2 the shorter one. As illustrated in Fig.1, we denote by T_1 and T_2 the two triangles obtained by subdividing K along d_1, and t_1 and t_2 are the two triangles obtained by decomposing K along d_2.

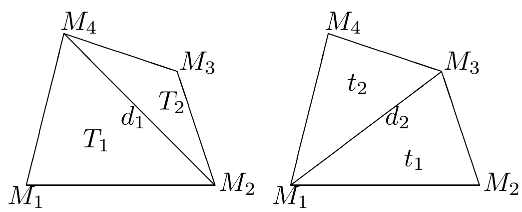

Fig.1. Quadrilateral K

We now give the definition of the maximum angle condition.

Definition 1.3 *[5, 11, 2] We say a triangle T(resp. a quadrilateral K) satisfies the maximum angle condition with a constant ψ, or shortly MAC(ψ), if the angles of T (resp. K) are less than or equal to ψ.*

In the sequel, the regularity of triangles is refered to as in this maximum angle sense. Our mesh condition can be stated as

Definition 1.4 *We say a convex quadrilateral K satisfies the generalized maximum angle condition, or shortly GMAC(ψ), if there exists a positive constant $\psi < \pi$ such that, among T_i, t_i, $i = 1, 2$, there are at least three regular triangles in the sense of MAC(ψ).*

Let us notice that the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$ in the *RDP(N, ψ)* condition is dropped in this condition. We shall prove the following result

Theorem 1.1 *Let K be a convex quadrilateral satisfying GMAC(ψ) with the constant $0 < \psi < \pi$ and $u \in H^2(K)$, then there exists a constant C_{err} only depending on ψ such that*

$$\mid u - \Pi_1 u \mid_{m,K} \leq C_{err}(\psi) h^{2-m} \mid u \mid_{2,K}, m = 0, 1 \tag{1.2}$$

Our analysis is based on three key points. First, we introduce an appropriate classification of quadrilaterals, which gives a close look on the geometry of quadrilaterals. Second, following the idea of [2], we adopt an appropriate affine change for the analysis, which is different from that used in [2]. Third, a sharper estimate for the integration $\int_{\hat{K}} \frac{1}{J_K}$ is given.

2. A Classification of Convex Quadrilaterals

In this section, we first introduce an appropriate method to classify convex quadrilaterals, and propose a mesh condition which is equivalent to *GMAC*(ψ), but is more convenient for our analysis.

Let $\mathcal{Q}$ denote the set of convex quadrilaterals. According to Definition 1.3, $\mathcal{Q}$ can be divided into the following two subsets:

$$\begin{aligned} \mathcal{R} &= \{K \in \mathcal{Q} \mid \quad K \text{ is regular } \}, \\ \mathcal{D} &= \{K \in \mathcal{Q} \mid \quad K \notin \mathcal{R}\}. \end{aligned}$$

$\mathcal{D}$ can be further divided into the following three subsets:

$$\begin{aligned} \mathcal{DB} &= \{K \in \mathcal{D} \mid \quad \text{both } T_1 \text{ and } T_2 \text{ are regular } \}, \\ \mathcal{DO} &= \{K \in \mathcal{D}/\mathcal{DB} \mid \quad \text{either } T_1 \text{ or } T_2 \text{ is regular } \}, \\ \mathcal{DN} &= \{K \in \mathcal{D} \mid \quad K \notin \mathcal{DB} \cup \mathcal{DO}\}. \end{aligned}$$

According to the regularity of t_1 and t_2, the set $\mathcal{DO}$ can be further divided as

$$\begin{aligned} \mathcal{DOB} &= \{K \in \mathcal{DO} \mid \quad \text{both of } t_1 \text{ and } t_2 \text{ are regular}\}, \\ \mathcal{DON} &= \{K \in \mathcal{DO} \mid \quad K \notin \mathcal{DOB}\}. \end{aligned}$$

Obviously we have

$$\mathcal{Q} = \mathcal{R} \cup \mathcal{DB} \cup \mathcal{DN} \cup \mathcal{DOB} \cup \mathcal{DON}. \tag{2.1}$$

With these preparations, we can state our equivalent mesh condition as following

Theorem 2.1 *Let K be a convex quadrilateral, then K is regular if and only if*

$$K \in \mathcal{RQ} = \mathcal{R} \cup \mathcal{DB} \cup \mathcal{DOB}. \tag{2.2}$$

Proof. The necessity of (2.2) is obvious, we only need to show it is sufficient. First, if $K \in \mathcal{R} \cup \mathcal{DOB}$, K is certainly regular in the sense of *GMAC*(ψ). Second, if $K \in \mathcal{DB}$, in this case, we can assert just one of $t_i, i = 1, 2$ is not regular in the sense of *MAC*(ψ). Otherwise, the fact that d_1 is the longer than d_2 will be violated, which completes the proof.

We note here that if K is regular in the sense of *RDP*(ψ, N), from Lemma 3.1 of [2], there exists a constant $\psi_1 = \psi_1(\psi, N) < \pi$ such that K is regular in the sense of *GMAC*(ψ_1). The converse is not true, because not all quadrilaterals in $\mathcal{DOB}$ are "regular" in that sense. For example, the quadrilateral on the left hand in Fig.2 is not regular in the sense of *RDP*(ψ, N), however is regular in the sense of *GMAC*(ψ) when the parameter a tends to zero.

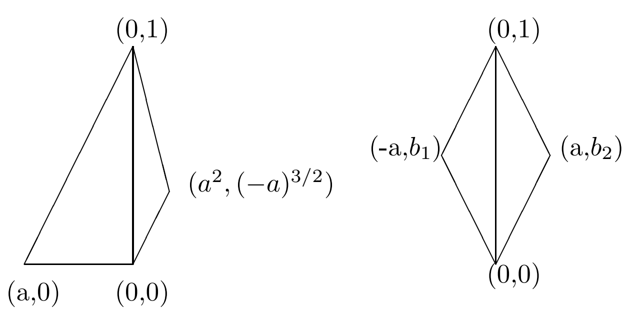

Fig.2. Examples of Quadrilaterals

3. The Affine Transformation

In this section, following the idea of [2], we introduce an affine change of variables, which is different from that defined in [2], however is more convenient for our analysis.

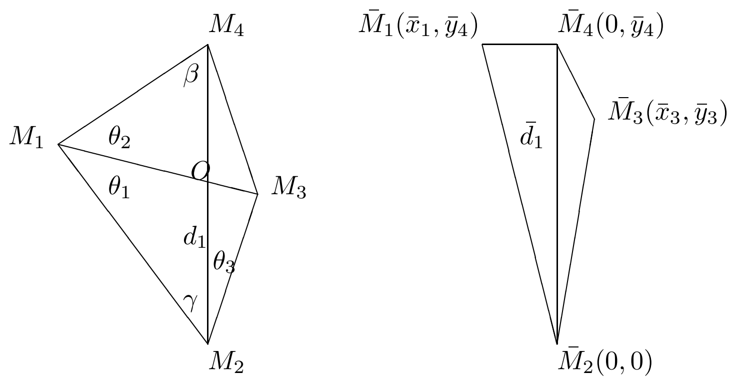

Fig.3. Quadrilateral K and the affine element $\bar{K}$

We fisrt quota a technical lemma from [1].

Lemma 3.1 *Let L be the linear transformation associated with a matrix B. Given two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, let α_1 be the angle between them and α_2 be the angle between $L(\boldsymbol{v}_1)$ and $L(\boldsymbol{v}_2)$, then it holds that*

$$\frac{2}{cond(B)\pi}\alpha_1 \le \alpha_2 \le \pi(1 - \frac{2}{cond(B)}) + \frac{2}{cond(B)\pi}\alpha_1. \tag{3.1}$$

We introducee an affine change by the following result.

Lemma 3.2 *Let K be a quadrilateral of diameter h satisfying GMAC(ψ). Then, there exist $\bar{x}_1$, $\bar{x}_3$, $\bar{y}_3$ and $\bar{y}_4$, an affine transformation $L\bar{x} = B\bar{x} + P$ such that $L(\bar{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)) = K$ and constants $C = C(\psi)$ such that*

$$\|B\| \le C, \quad \|B^{-1}\| \le C, \quad \textit{in particular}, cond(B) \le C^2. \tag{3.2}$$

Moreover, $\bar{K}$ is regular in the sense of GMAC($\bar{\psi}$) with constant $\bar{\psi} < \pi$ depending only on ψ. Hereinafter, $\bar{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ denotes the confirguration illustrated on the right-hand side of Fig.3.

Proof. We will construct $\bar{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ and L explicitly. Since K satisfies GMAC(ψ), we can assume without lose of generality that $\triangle M_1M_2M_4$ is regular, and that $\pi \ge \psi \ge \beta \ge \delta$ with $\delta = \frac{\pi}{2} - \frac{\psi}{2}$(See Fig.3).

Further, we assume that the diagonal d_1 lies along the y-axis and that the vertex M_2 is put at the origin (up to a rigid movement), y_4 is the length of the longer diagonal d_1, the vertex corresponding to the angle β is placed at vertex $M_4(0, y_4)$(See Fig.3).

At last we let $\bar{x}_1 = - \mid M_1M_4 \mid \sin\beta = x_1$, $\bar{y}_1 = y_4$, $\bar{y}_4 = y_4$ and

$$B = \begin{pmatrix} 1 & 0 \\ \cot\beta & 1 \end{pmatrix}, \qquad \begin{pmatrix} \bar{x}_3 \\ \bar{y}_3 \end{pmatrix} = B^{-1} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{3.3}$$

An elementary calculation finds $\|B\| \le \sqrt{2}/sin\beta$, $\|B^{-1}\| \le \sqrt{2}/sin\beta$, which imply that $cond(B) \le 2/sin^2\beta$. We let $\bar{K}(\bar{x_1}, \bar{x_3}, \bar{y_3}, \bar{y_4})$ be the convex quadrilateral with vertices $\bar{M}_1(\bar{x_1}, \bar{y_4})$, $\bar{M}_2(0, 0)$, $\bar{M}_3(\bar{x_3}, \bar{y_3})$ and $\bar{M}_4(0, \bar{y_4})$, note that $L(\bar{K}) = K$.

To prove the second part of the Lemma, we only show that the assertion is valid for $K \in \mathcal{R}$. In this case, any interior angle θ of K is not greater than ψ, in view of Lemma 3.1, we have

$$\bar{\theta} \leq \pi(1 - 2/\pi cond(B)) + (2/\pi cond(B))\psi = \bar{\psi} < \pi,$$

where the constant $\bar{\psi}$ obviously only depends on ψ. Therefore $\bar{K}$ is also a regular quadrilateral in the sense of *MAC($\bar{\psi}$)*, which ends the proof. For the other cases, the proof is similar because L and its inverse are bounded linear transformations with the norms in terms of ψ.

Proceeding along the same line of Lemma 3.4 in [2], we have

Lemma 3.3 *Given a quadrilateral K satisfying GMAC(ψ), let L and $\bar{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ be the affine transformation and the affine element given in Lemma 3.2, $\bar{u} = u \circ L$. Then there are two positive constants C_1 and C_2 depending only on ψ such that*

$$C_1 \mid \bar{u} - \overline{\Pi_1 u} \mid_{1,\bar{K}} \leq \mid u - \Pi_1 u \mid_{1,K} \leq C_2 \mid \bar{u} - \overline{\Pi_1 u} \mid_{1,\bar{K}} \tag{3.4}$$

$$C_1 \mid \bar{u} \mid_{2,\bar{K}} \leq \mid u \mid_{2,K} \leq C_2 \mid \bar{u} \mid_{2,\bar{K}} \tag{3.5}$$

Remark 3.1 Notice that in our transformation, the constants in Lemma 3.2 and Lemma 3.3 only depend on ψ, while the corresponding constants in [2] depend on both ψ and the ratio d_2/d_1.

4. Error Estimates

The purpose of this section is to derive the optimal interpolation error estimate for the quadrilateral element satisfying *GMAC(ψ)*. We shall follow the main ideas of [12] to decompose the $\mathcal{Q}_1$ interpolation error into two parts: one is the P_1 interpolation error and the other is the difference between Π_1 interpolation and P_1 interpolation, and then use the idea of function extension to estimate the sencond part. However, we shall use the idea of [2] to bound the first part.

We introduce some notations. Denote the angles of the two diagonals M_1M_3 and M_2M_4 by α and $\pi - \alpha$ with $0 \leq \alpha \leq \frac{\pi}{2}$, and let O be the point at which they intersect. Let $a_i = \mid OM_i \mid, i = 1, 2, 3, 4$, let T_1 denote triangle $\triangle M_1M_2M_4$ and T_2 denote $\triangle M_2M_3M_4$. Set $a = \max_{1\leq i\leq 4}(a_i a_{i+1})$, $d = \min\{\mid M_3M_1 \mid, \mid M_3M_2 \mid, \mid M_3M_4 \mid\}$. As in Lemma 3.2, we assume the longer diagonal d_1 lies on the y-axis, M_2 is placed at the fixed point $(0, 0)$, $x_1 \leq 0$ and $x_3 \geq 0$ (See. Fig 3), and that T_1 is regular.

To derive the estimate, following the idea of [12] and [2], we decompse the error in the following way:
Let Π be the P_1- Lagrange interpolation operator associated with the vertices M_1, M_2 and M_4, i.e, Πu is a linear function which admits the same values with u at these three nodes, then we have

$$\mid u - \Pi_1 u \mid_{1,K} \leq \mid u - \Pi u \mid_{1,K} + \mid \Pi u - \Pi_1 u \mid_{1,K} . \tag{4.1}$$

Because Πu is a linear function on quadrilateral K, $\Pi u - \Pi_1 u$ belongs to the isoparametric finite element space and vanishes at nodes $M_i, i = 1, 2, 4$, then we have

$$(\Pi u - \Pi_1 u)(x) = (\Pi u - \Pi_1 u)(M_3)\phi_3(x),$$

where ϕ_3 is the usual bilinear nodal basis at node M_3, therefore

$$\mid u - \Pi_1 u \mid_{1,K} \leq \mid u - \Pi u \mid_{1,K} + \mid (\Pi u - \Pi_1 u)(M_3) \mid\mid \phi_3 \mid_{1,K} . \tag{4.2}$$

We first take care of the term $\mid u - \Pi u \mid_{1,K}$. In view of Lemma 3.2 and Lemma 3.3 in the previous section, it suffices to analyze the case where the reference configuration $\bar{K}(\bar{x}_1, \bar{x}_3, \bar{y}_3, \bar{y}_4)$ is considered.

Let $\bar{u}$ be the function defined on $\bar{K}$ through u by the following canonical relation

$$\bar{u} = u(L(\bar{x})), \forall \bar{x} \in \bar{K}. \tag{4.3}$$

Denote $\bar{\Pi}$ the P_1-Lagrange interpolation of $\bar{u}$ on $\bar{K}$, which agrees with $\bar{u}$ at the vertices $\bar{M}_i, i = 1, 2, 4$, we have

$$\overline{\Pi u} = \bar{\Pi}\bar{u}. \tag{4.4}$$

Remark 4.1 Because d_1 is the longest diagonal of K, we can always assume, without lose of generality, that $\mid T_1 \mid \geq \mid T_2 \mid$. Owning to $\det(B) = 1$, we have $\mid \bar{T}_1 \mid = \mid T_1 \mid \geq \mid T_2 \mid = \mid \bar{T}_2 \mid$, therefore $\frac{|\bar{T}_1|}{|\bar{K}|} \geq \frac{1}{2}$. Where $\mid T_1 \mid$ is the volume of the triangle T_1.

We need to give a brief verification for such an assumpation. First, if $K \in \mathcal{R} \cup \mathcal{DB}$, by the symmetry, it is reasonable for us to say so. Second, if $K \in \mathcal{DOB}$, since d_1 is the longer diagonal by the definition, we assert the interior angle at the vertex M_3 is greater than ψ, otherwise the fact that t_1 and t_2 are regular in the sense of *MAC(ψ)* will be violated. Then $\angle M_4M_3M_2 > \angle M_4M_1M_2$, which implies that $\mid x_1 \mid \geq \mid x_3 \mid$

Under this assumpation, we can obtain the following error estimate for the P_1-Lagrange interpolation operator $\bar{\Pi}$ and Π,

Lemma 4.1 *Let $\bar{\Pi}\bar{u}$ and Πu be defined as above, then*

$$\mid \bar{u} - \bar{\Pi}\bar{u} \mid_{m,\bar{K}} \leq Ch^{2-m} \mid \bar{u} \mid_{2,\bar{K}}, m = 0, 1. \tag{4.5}$$

$$\mid u - \Pi u \mid_{m,K} \leq C(\psi)h^{2-m} \mid u \mid_{2,K}, m = 0, 1. \tag{4.6}$$

Proof. For $m = 1$, the inequality (4.5) can be proved in a similar way as Lemma 4.3 of [2] by using the fact that $\frac{|\bar{T}_1|}{|\bar{K}|} \geq \frac{1}{2}$. For $m = 0$, using the affine transformation $\tilde{L}$ befined in Theorem 4.1 of [2], we can obtain another quadrilateral $\tilde{K}$, which obviously satisfies the condition *J(σ)* with $\sigma = 1$. Therefore, the estimate on $\tilde{K}$ follows from [12] and the estimate on $\bar{K}$ is obtained by changing variables. At last,(4.6) is the immediate consequence of (4.5), Lemma 3.2 and Lemma 3.3.

We now turn to the second term on the right hand of the inequality (4.2). This time, following the idea of Lemma 3.2 of [12], we shall apply the theory of function extension from [13], which first prove the following lemma.

Lemma 4.2 *There exists a constant $\beta_0 > 0$ only depending on ψ, such that, let G_0 be a fixed isosceles triangle with two angles equal to β_0, there exists an isosceles triangle G contained in K which is similar to G_0, and admits the segment M_3M_i as its base, where M_3M_i denotes the edge which satisfies $d = M_3M_i$ among $M_3M_l, l = 1, 2, 4$.*

Proof. In view of the equivalent result Lemma 2.1, there are three cases of which we have to take care.

Case I $K \in \mathcal{R}$.

First, we assume $d = \mid M_3M_1 \mid$, this is to say, $\mid M_3M_1 \mid \leq \mid M_3M_2 \mid$ and $\mid M_3M_1 \mid \leq \mid M_3M_4 \mid$. Owning to Remark 4.1, namely $\mid T_1 \mid \geq \mid T_2 \mid$, it is easy to see $\max(\mid M_1M_2 \mid, \mid M_1M_4 \mid) \geq \min(\mid M_3M_2 \mid, \mid M_3M_4 \mid) \geq \mid M_3M_1 \mid$, therefore, there exists at least one triangle bewteen t_1 and t_2 such that d_2 is its shortest edge, due to the regularity of t_1 and t_2, we conclude the assertion is valid for this case.

Second, if the shortest edge is not $\mid M_3M_1 \mid$, without lose of generality, we assume $d = \mid M_2M_3 \mid$, consequently $\mid d_1 \mid \geq \mid M_3M_1 \mid > \mid M_2M_3 \mid$, therefore M_2M_3 is the shortest edge of T_2, by virtue of the regularity of T_2, we conclude the assertion is also valid for this case.

Case II $K \in \mathcal{DB}$.

Note that in this case M_3M_1 is not the shortest one among $M_3M_i, i = 1, 2, 4$. Following the line of the second part of **Case I**, we can achieve the desired result.

Case III $K \in \mathcal{DOB}$.

If M_3M_1 is the shortest edge among $M_3M_i, i = 1, 2, 4$, taking into account the regularity of t_1 and t_2, following the same procedure of the first part of **case I**, we obtain the result. On the contrary, without lose of generality, we assume again $d =| M_2M_3 |$. Since $\triangle M_1M_2M_3$ is regular and d_2 is not the shortest edge of $\triangle M_1M_2M_3$, we obtain

$$\psi \geq \angle M_1M_2M_3 \geq \delta.$$

Moreover T_2 is not regular and d_1 is the longer diagonal, then

$$\angle M_2M_3M_4 \geq \psi,$$

which implies the desired result.

Using Lemma 4.2 and Lemma 4.1, we have

Lemma 4.3 *Let K be a regular quadrilateral in the sense of GMAC(ψ) with constant ψ and $u \in H^2(K)$, then, for any real $0 < \gamma < 1$, it holds that*

$$| (u - \Pi u)(M_3) | \leq C(\gamma, \psi) d^\gamma h^{1-\gamma} | u |_{2,K} \tag{4.7}$$

Proof. Owning to the Holder-continuous property with real $0 < \gamma < 1$ of functions in $H^2(K)$, using Lemma 4.2 and Lemma 4.1, proceeding along the same line of Lemma 3.2 of [12], we can obtain the asserted result.

We remain to bound the term $| \phi_3 |_{1,K}$, which forces us to estimate the integration $\int_{\hat{K}} | J_K^{-1} | d\xi d\eta$, where J_K is the Jacobian determinant of the bilinear transformation. Here we adopt the method of [12]. Replacing the inequality $\log(1 + t) \leq t^{1/2}, t > 0$ in the proof of Lemma 2.1 of [12] by the inequality $\log(1 + t) \leq t, t > 0$, we get the following sharper estimate

$$\int_{\hat{K}} | J_K^{-1} | d\xi d\eta \leq 8/(sin\alpha \cdot a), \tag{4.8}$$

with $a = \max_{1 \leq i \leq 4}(a_i a_{i+1})$. We now bound the term $\frac{1}{\sin\alpha}$ in terms of ψ and $\frac{|d_2|}{d}$. Let s_1 be the shortest edge of T_1 and s_2 be the shortest edge of T_2. If $K \in \mathcal{DOB}$, as illustrated in Fig.3, θ_3 will go to zero, $\theta_1 + \gamma \geq \frac{\delta}{2}$ because $\triangle M_1M_2M_3$ is regular in the sense of *MAC(ψ)*, moreover $\beta \geq \delta$, thus it is easy to see $\frac{\delta}{2} \leq \alpha$, i.e,

$$\frac{1}{\sin\alpha} \leq C(\psi)\frac{| d_2 |}{d} \tag{4.9}$$

If $K \in \mathcal{R} \cup \mathcal{DB}$, without lose of generality, we assume $s_1 = M_1M_4$. If s_1 is not the shortest edge of $\triangle M_1OM_4$, we have $\delta \leq \alpha$ because $\triangle M_1M_2M_4$ is regular. If s_1 is the shortest edge $\triangle M_1OM_4$, in this case, if $s_2 = M_3M_4$, by the regularity of $\triangle M_2M_3M_4$, it is easy to see $\alpha \geq \delta$, therefore without lose of generality, we assume $s_2 = M_2M_3$. Because $\psi \geq \beta \geq \delta$, we get $\psi \geq \theta_2 \geq \delta$. Moreover,

$$\begin{aligned}
\frac{| M_1M_4 |}{\sin\alpha} &= \frac{| M_1O |}{\sin\beta}, \\
\frac{| M_2M_3 |}{\sin\alpha} &= \frac{| OM_3 |}{\sin\angle M_4M_2M_3}.
\end{aligned}$$

Since $\triangle M_2M_3M_4$ is regular and M_3M_4 is not its shortest edge,

$$\begin{aligned}
\frac{| s_1 |}{| s_2 |} &= \frac{| M_1O | \sin\angle M_4M_2M_3}{| OM_3 | \sin\beta} \\
&= \frac{| x_1 | \sin\angle M_4M_2M_3}{| x_3 | \sin\beta} \\
&\geq C(\psi).
\end{aligned}$$

therefore,

$$\frac{1}{\sin\alpha} = \frac{| OM_1 |}{| s_1 | \sin\beta} \leq C(\psi)\frac{| OM_1 |}{| s_2 |} \leq C(\psi)\frac{| d_2 |}{d}. \tag{4.10}$$

By virtue of Lemma 2.2 in [12], (4.9) and (4.10), we derive as

$$| \phi_3 |_{1,K} \leq \frac{8\sqrt{2}h}{\sqrt{sin\alpha \cdot a}} | \hat{\phi}_3 |_{1,\infty,\hat{K}} \leq \frac{16\sqrt{2}h}{\sqrt{sin\alpha \mid d_1 \mid\mid d_2 \mid}} | \hat{\phi}_3 |_{1,\infty,\hat{K}} \leq C(\psi) \frac{h}{\sqrt{d \mid d_1 \mid}},$$

which, together with Lemma 4.3 with $\gamma = \frac{1}{2}$, implies

$$| (u - \Pi u)(M_3) || \phi |_{1,K} \leq C(\psi) h \mid u |_{2,K} \ . \tag{4.11}$$

Now we prove our main result Theorem 1.1.

Proof of Theorem 1.1.

Proof. Because

$$| u - \Pi_1 u |_{1,K} \leq | u - \Pi u |_{1,K} + | (u - \Pi u)(M_3) || \phi_3 |_{1,K} \ .$$

The first term is bounded in Lemma 4.1, and the second term is bounded in (4.11). As for the optimal L^2 error estimate, it can be obtained by using our transformation, the idea of Theorem 4.1 of [2] and the estimates given in Theorem 1 of [12], for brevity, we skip the details.

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