

P_1 -NONCONFORMING QUADRILATERAL FINITE VOLUME ELEMENT METHOD AND ITS CASCADE MULTIGRID ALGORITHM FOR ELLIPTIC PROBLEMS *

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Abstract

In this paper, we discuss the finite volume element method of P_1 -nonconforming quadrilateral element for elliptic problems and obtain optimal error estimates for general quadrilateral partition. An optimal cascadic multigrid algorithm is proposed to solve the non-symmetric large-scale system resulting from such discretization. Numerical experiments are reported to support our theoretical results.

Mathematics subject classification: 65N30, 65N55.

Key words: finite volume element method, cascadic multigrid, Elliptic problems.

1. Introduction

Finite volume method(FVM) is a discretization technique widely used in the approximation of conservation laws, in computational fluid dynamics, and in convection-diffusion problems. Apart from an approximation of the solution at discrete points, we can seek a discrete solution in a finite element space. This version of approximation is often called the finite volume element method(FVEM). On the one hand, it has a simplicity for implementation comparable to the finite difference method and can be viewed as a generalization of the finite difference method; on the other hand, it has a flexibility similar to that of the finite element method(FEM) for handling complicated geometries and boundary conditions and preserves more mathematical structures of the original continuous problem, which makes systematic error analysis possible. Another important advantage of this method is that such generated numerical solutions usually have certain conservation property locally, thus it can be expected to capture shocks, to produce simple stencils, or to study other physical phenomena more effectively. About its recent developments, we refer to the monographs [2, 9, 10, 11, 14, 18, 22] for details.

Nonconforming elements have been used effectively especially in the computation of fluid and solid mechanics due to their stability nature. Recently increasing attentions have been paid to these elements for their potential application in parallel computing. Driven by these reasons, many nonconforming elements have been proposed in the triangular and quadrilateral cases from 1970s [13, 15, 16, 20, 21]. Observing the fact that any P_1 function on a quadrilateral can be uniquely determined by its values on any three of the four midpoints on the edges, [20] and [16] introduced the P_1 -nonconforming quadrilateral element from different points of view and this element has the least degrees of freedom among all the low order nonconforming quadrilateral elements. The quadrilateral finite element spaces are generally constructed starting from a given

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finite dimensional polynomials space \hat{V} on a reference element \hat{K} by a bilinear isomorphism. Recent observation made by [1] implies that for such defined finite element spaces, a necessary and sufficient condition for approximation of order $r + 1$ in L^p and r in $W^{1,p}$ is that \hat{v} contains the space Q_r . Thus for the truly quadrilateral element, the P_1 -nonconforming finite element space obtained from the standard reference element will not guarantee the optimal convergence rate anymore. But the nonparametric scheme proposed in [20] provides an efficient way of computing without losing the order of convergence.

In this paper, we are interested in using the nonparametric P_1 -nonconforming quadrilateral element to solve elliptic problems by FVEM. Considering the particular characteristic of this element, we propose its finite volume element discretization scheme corresponding to a dual partition of overlapping type. Numerical analysis shows optimal convergence rate under H^1 -norm, but in order to obtain optimal error estimate under L^2 -norm, additional assumptions on the source term and the partition are needed. A counterexample is given to show that more regular assumption on the source term is necessary. But numerical experiments demonstrate that the assumption on the partition is unnecessary, which means the L^2 -norm error estimate may can be improved.

In the field of scientific computing, designing effective algorithm to solve the systems resulting from the discretization of PDEs is always the concern of many researchers. Cascadic multigrid method, which requires no coarse grid corrections and can be viewed as a "one-way" multigrid method, is proven to be effective for solving large-scale finite element discretization problem, see [3, 4, 5, 6, 7, 17, 24, 25, 27] for details. But for the finite volume element discretization, the algebraic systems of self-adjoint elliptic problems are nonsymmetric in general, which brings many difficulties for designing optimal cascadic multigrid algorithms. Based on the observation that the nonsymmetric equations are a small perturbation of the usual finite element discretization equations, we propose a new cascadic multigrid algorithm in [26] to solve the finite volume element discretization problem for P_1 -conforming triangular element. The aim of this paper is to apply this algorithm to the P_1 -nonconforming quadrilateral element. The nonconformity of this element is conquered by defining a new inter-grid transfer operator. Theoretical analysis and numerical experiments show that this algorithm is optimal in both accuracy and computational complexity.

The rest of our paper is organized as follows: In Section 2, we give some notations used in this paper and formulate the FVE scheme for the nonparametric P_1 -nonconforming quadrilateral element for self-adjoint elliptic problems; then in Section 3, we obtain optimal H^1 - and L^2 -norm error estimates for it, and a counterexample is given to show that the L^2 -norm error estimate cannot be optimal in regularity; Section 4 is devoted to analyze the cascadic multigrid algorithm for the discretization problem; then in the last section, we give some numerical experiments to support our theoretical results.

2. Notations and the Finite Volume Element Scheme for the P_1 -nonconforming Quadrilateral Element

In this paper, we consider the following self-adjoint elliptic problem

$$\begin{aligned} -\nabla \cdot (\mathbf{A}\nabla u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a convex polygonal domain in R^2 , and $\mathbf{A} = (a_{i,j})_{2 \times 2} \in (W^{1,\infty}(\Omega))^4$ is a given real matrix function satisfying

$$0 < \alpha_* |\xi|^2 \leq \xi^t \mathbf{A}(x) \xi \leq \alpha^* |\xi|^2 < \infty, \quad \forall \xi \in R^2. \tag{2.2}$$

In what follows we shall adopt the standard definitions of Sobolev spaces, the notations of their norms and semi-norms as presented in [12].

For convenience, we give a brief introduction of the nonparametric P_1 -nonconforming quadrilateral element proposed in [20]. Suppose \mathcal{T}_h is a regular and quasi-uniform quadrilateral decomposition of the domain Ω , h the maximum meshsize of the partition, let N_K, N_P , and N_E denote the number of quadrilaterals, vertices, and edges respectively. Set

$$\begin{aligned} \mathcal{T}_h &= \{K_1, K_2, \dots, K_{N_K}\} : \bigcup_{j=1}^{N_K} K_j = \overline{\Omega}; \\ \mathcal{P} &= \{P_1, P_2, \dots, P_{N_P}\} : \text{the set of all vertices of } K \in \mathcal{T}_h; \\ \mathcal{E}_h &= \{e_1, e_2, \dots, e_{N_E}\} : \text{the set of all edges of } K \in \mathcal{T}_h; \\ \mathcal{M} &= \{m_1, m_2, \dots, m_{N_E}\} : \text{the set of all midpoints of } e \in \mathcal{E}_h. \end{aligned} \tag{2.3}$$

In particular, let N_P^i, N_E^i denote the number of interior vertices, edges and midpoints of the partition \mathcal{T}_h .

Remark 2.1 For convenience, we let $\{P_1, P_2, \dots, P_{N_P^i}\}$ denote the set of interior points and $\{P_{N_P^i+1}, P_{N_P^i+2}, \dots, P_{N_P}\}$ the set of boundary points.

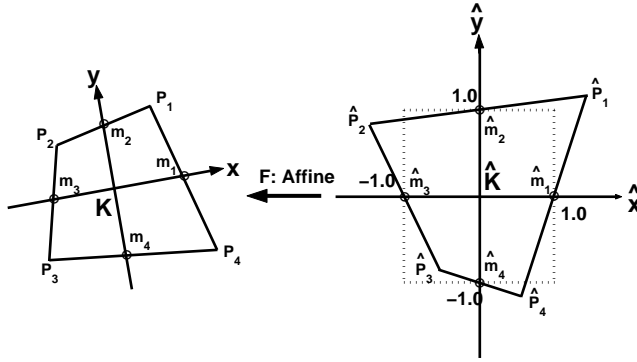


Figure 1.

For given $K \in \mathcal{T}_h$ with vertices $P_j, 1 \leq j \leq 4$, and midpoints of edges $m_j, 1 \leq j \leq 4$, as in Figure 1., there is a unique affine transformation $F : R^2 \rightarrow R^2$ such that

$$F(\hat{m}_1) = m_1, \quad F(\hat{m}_2) = m_2, \quad F(\hat{m}_3) = m_3, \quad F(\hat{m}_4) = m_4, \tag{2.4}$$

since the four midpoints of any quadrilateral form a parallelogram. Denote $\hat{K} = F^{-1}(K)$ and define $\hat{\varphi}_j \in \text{Span}\{1, \hat{x}, \hat{y}\}, 1 \leq j \leq 4$, such that

$$\hat{\varphi}_j(\hat{m}_k) = \begin{cases} 1, & k = j, j + 1 \text{ mod } 4, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Then the basis function space can be constructed by using the fixed reference basis function $\{\hat{\varphi}_j\}_{j=1}^4$, although \hat{K} may vary. Now the P_1 -nonconforming quadrilateral finite element space can be defined as

$$\begin{aligned} \mathcal{NC}^h &= \{v_h : \Omega \rightarrow R \mid v_h|_K \in P_1(K) \text{ for any } K \in \mathcal{T}_h, v_h \text{ is continuous at each } m \in \mathcal{M} \setminus \partial\Omega\}, \\ \mathcal{NC}_0^h &= \{v_h \in \mathcal{NC}^h \mid v_h(m) = 0 \text{ for any } m \in \mathcal{M} \cap \partial\Omega\}. \end{aligned} \tag{2.6}$$

To each vertex $P_j \in \mathcal{P}$, denote by $\mathcal{E}(j)$ the set of all edges $e \in \mathcal{E}$ such that one of endpoints on each edge is P_j , and by $\mathcal{M}(j)$ the set of all midpoints m on edges in $\mathcal{E}(j)$. Let $\varphi_j \in \mathcal{NC}^h$ be defined as

$$\varphi_j(m) = \begin{cases} 1, & \text{if } m \in \mathcal{M}_j, \\ 0, & \text{if } m \in \mathcal{M} \setminus \mathcal{M}(j). \end{cases} \tag{2.7}$$

In fact φ_j can be obtained from the basis defined on \hat{K} by the affine transformation F . Under the assumption that each interior edge has at least one interior vertex as its endpoint, all

functions associated with the interior vertex $P_j \in \mathcal{P} \setminus \partial\Omega$, $j = 1, 2, \dots, N_P^i$ form the basis of \mathcal{NC}_0^h

Remark 2.2 For the rectangular partition, the nonparametric element is equivalent to the standard reference element and all the following results still hold.

Now we come to discretize problem (2.1). Its variational form is to find $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in V, \quad (2.8)$$

where

$$a(u, v) = \int_K \mathbf{A} \nabla u \cdot \nabla v \, dx. \quad (2.9)$$

Then the standard Galerkin finite element approximation of (2.8) is to find $\hat{u}_h \in V_h = \mathcal{NC}_0^h$ such that

$$a_h(\hat{u}_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (2.10)$$

where

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in V_h. \quad (2.11)$$

In the finite volume discretization, to obtain a unique approximate solution, we require the number of dual element should equal to the number of unknowns. For P_1 -nonconforming quadrilateral element, the basis functions are corresponding to the vertices while continuous at the midpoints of the partition. Considering this particular characteristic, we choose the following defined dual partition: for any $K \in \mathcal{T}_h$, the diagonals of each element split itself into four triangles, let $K_{P_i, K}^*$ be the union of all triangles with P_i as their common vertex on K and define

$$\mathcal{T}_h^* = \{K_{P_i}^*, K_{P_i}^* = \bigcup_K K_{P_i, K}^*, i = 1, 2, \dots, N_P\}, \quad (2.12)$$

then \mathcal{T}_h^* constitutes a dual partition of \mathcal{T}_h and each $K_{P_i}^*$ is called a control volume[see Figure 2.] It is obvious that the control volumes are overlapped.

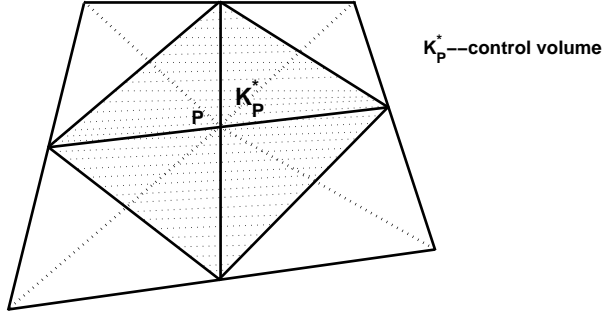


Figure 2.

Then the finite volume element method of (2.1) is to find $u_h \in V_h$ such that

$$-\int_{\partial K_{P_i}^*} (\mathbf{A} \nabla_h u_h) \cdot \mathbf{n} \, ds = \int_{K_{P_i}^*} f \, dx, \quad i = 1, 2, \dots, N_P^i, \quad (2.13)$$

where \mathbf{n} is the unit outward normal to $\partial K_{P_i}^*$. It should be noted that the formulation is a discretization form of stating that we have an integral conservation locally on the control

volume. Let χ_i be the characteristic function of $K_{P_i}^*$, then problem (2.13) is equivalent to find $u_h \in V_h$ such that

$$-\sum_{i=1}^{N_P^i} V^i \int_{\partial K_{P_i}^*} (\mathbf{A} \nabla_h u_h) \cdot \mathbf{n} \, ds = \int_{\Omega} f \sum_{i=1}^{N_P^i} V^i \chi_i \, dx, \quad \forall \{V^i\}_{i=1}^{N_P^i} \in \mathbf{R}^{N_P^i}. \quad (2.14)$$

Let

$$W_h = \{w_h \in L^2(\Omega) : w_h|_{K_{i,j}^*} = \text{const and } w_h|_{\partial\Omega} = 0\}, \quad (2.15)$$

where $K_{i,j}^* = K_{P_i}^* \cap K_{P_j}^*$. Then we can define a one-to-one operator $r_h : V_h \rightarrow W_h$ such that

for any $v_h = \sum_{i=1}^{N_P^i} V^i \varphi_i \in V_h$,

$$r_h v_h = \sum_{i=1}^{N_P^i} V^i \chi_i. \quad (2.16)$$

It is easy to check that such defined operator has the following approximation property

$$\|r_h v_h - v_h\|_{0,q} \leq ch |v_h|_{1,q,h}, \quad \forall v_h \in V_h, \quad 1 < q < \infty. \quad (2.17)$$

Employing the operator, problem (2.14) can be rewritten as: seek $u_h \in V_h$ such that

$$a_h^*(u_h, v_h) = (f, r_h v_h), \quad \forall v_h \in V_h, \quad (2.18)$$

where

$$a_h^*(u_h, v_h) = -\sum_{i=1}^{N_P^i} V^i \int_{\partial K_{P_i}^*} (\mathbf{A} \nabla_h u_h) \cdot \mathbf{n} \, ds, \quad \forall v_h = \sum_{i=1}^{N_P^i} V^i \varphi_i. \quad (2.19)$$

3. Numerical Analysis

In order to get optimal H^1 and L^2 -norm error estimates for (2.13), we first give several important lemmas which are crucial to the numerical analysis in section 3.1, while solvability of the problem is presented in section 3.2.

3.1 Lemmas

Lemma 3.1 ^[20] *The semi-norm $|v|_{1,h}^2 = \sum_{K \in T_h} \|\nabla v\|_{0,K}^2$ is also a norm on the space \mathcal{NC}_0^h .*

Define

$$\begin{aligned} \omega_{i,K} &= \{j : P_i \text{ and } P_j \text{ are diagonal points of } K\}, \\ w_i &= \bigcup_{K \in T_h} \omega_{i,K}, \\ w &= \{(i, j) : 1 \leq i < j \leq N_P, j \in \omega_i\}, \end{aligned} \quad (3.1)$$

then

Lemma 3.2 *There exists a positive constant $c_0 > 0$ independent of h such that for any $v_h =$*

$$\sum_{i=1}^{N_P^i} V^i \varphi_i \in V_h,$$

$$\sum_{(i,j) \in \omega} |V^i - V^j|^2 \leq c_0 \|\nabla_h v_h\|_{0,\Omega}^2, \quad (3.2)$$

where if $k > N_P^i$, $V^k = 0$.

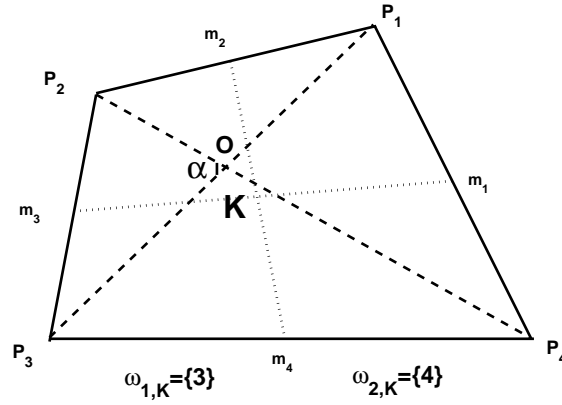


Figure 3.

Proof. Since v_h is piecewise linear on \mathcal{T}_h and continuous at the midpoint of each edge, using the notations in Figure 3., on element K we have

$$V^1 + V^4 - V^2 - V^3 = \nabla v_{h,K} \cdot \left(\frac{a_1 + a_4 - a_2 - a_3}{2}, \frac{b_1 + b_4 - b_2 - b_3}{2} \right), \quad (3.3)$$

$$V^1 + V^2 - V^3 - V^4 = \nabla v_{h,K} \cdot \left(\frac{a_1 + a_2 - a_3 - a_4}{2}, \frac{b_1 + b_2 - b_3 - b_4}{2} \right), \quad (3.4)$$

where (a_i, b_i) , $1 \leq i \leq 4$, is the coordinate of the vertex P_i . After simple manipulation, we get

$$V^1 - V^3 = \frac{1}{2} \nabla v_{h,K} \cdot (a_1 - a_3, b_1 - b_3), \quad (3.5)$$

$$V^2 - V^4 = \frac{1}{2} \nabla v_{h,K} \cdot (a_2 - a_4, b_2 - b_4). \quad (3.6)$$

So

$$(V^1 - V^3)^2 + (V^2 - V^4)^2 \leq \frac{1}{4} |\nabla v_{h,K}|^2 (|P_1P_3|^2 + |P_2P_4|^2). \quad (3.7)$$

By [19], since \mathcal{T}_h is regular, there exists a constant $\sigma_1 > 0$, such that for any element $K \in \mathcal{T}_h$,

$$\frac{|P_1P_3|}{|P_3P_4|} \leq \sigma_1, \quad \frac{|P_3P_4|}{|P_2P_4|} \leq \sigma_1, \quad (3.8)$$

thus

$$|P_1P_3| \leq \sigma_1^2 |P_2P_4|. \quad (3.9)$$

Similarly, there exists $\sigma_2 > 0$ such that

$$|P_2P_4| \leq \sigma_2^2 |P_1P_3|. \quad (3.10)$$

Substituting (3.9). (3.10) into (3.7) to obtain

$$(V^1 - V^3)^2 + (V^2 - V^4)^2 \leq \frac{1}{4} (\sigma_1^2 + \sigma_2^2) |\nabla v_{h,K}|^2 |P_1P_3| |P_2P_4|. \quad (3.11)$$

On the other hand, the regularity of \mathcal{T}_h implies that the four subtriangles contained in each K is also regular, i.e., there exists $\theta_0 > 0$, such that any inner angle of the subtriangles is bigger than θ_0 . Let α denote the acute angle between the lines P_1P_2 and P_3P_4 , then

$$2\theta_0 < \alpha < \pi - 2\theta_0. \quad (3.12)$$

Since

$$meas(K) = \frac{1}{2} |P_1P_3| |P_2P_4| \sin \alpha > \frac{1}{2} \sin(2\theta_0) |P_1P_3| |P_2P_4|, \quad (3.13)$$

combining this with (3.11), we have

$$(V^1 - V^3)^2 + (V^2 - V^4)^2 \leq \frac{\sigma_1^2 + \sigma_2^2}{2 \sin(2\theta_0)} |\nabla v_{h,K}|^2 \text{meas}(K) = \frac{\sigma_1^2 + \sigma_2^2}{2 \sin(2\theta_0)} |v_h|_{1,K}^2. \quad (3.14)$$

Summing this inequality over all $K \in \mathcal{T}_h$ gives (3.2).

Define

$$Q_h = \{q_h \in L^2(\Omega) : q_h|_K = \text{const}, K \in \mathcal{T}_h\}, \quad (3.15)$$

then we have that

Lemma 3.3 For any matrix-valued function $\bar{\mathbf{A}} \in (Q_h)^4$, there holds

$$-\sum_{i=1}^{N_P^i} V^i \int_{\partial K_{P_i}^*} (\bar{\mathbf{A}} \nabla_h u_h) \cdot \mathbf{n} ds = \int_{\Omega} \bar{\mathbf{A}} \nabla_h u_h \cdot \nabla_h v_h dx, \quad \forall u_h, v_h \in V_h, \quad (3.16)$$

where v_h can be expressed as $\sum_{i=1}^{N_P^i} V^i \varphi_i$.

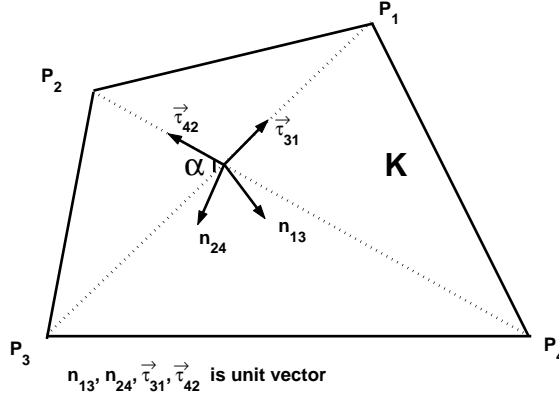


Figure 4.

Proof. Using the notations in Figure 4., the left-hand of (3.16) can be rewritten as

$$-\sum_{i=1}^{N_P^i} V^i \int_{\partial K_{P_i}^*} (\bar{\mathbf{A}} \nabla_h u_h) \cdot \mathbf{n} ds = -\sum_{K \in \mathcal{T}_h} A_K, \quad (3.17)$$

where

$$A_K = (V^1 - V^3) \int_{P_2 P_4} (\bar{\mathbf{A}}|_K \nabla_h u_h) \cdot \mathbf{n}_{42} ds + (V^2 - V^4) \int_{P_1 P_3} (\bar{\mathbf{A}}|_K \nabla_h u_h) \cdot \mathbf{n}_{31} ds. \quad (3.18)$$

Since u_h is linear on each K , we have

$$A_K = (V^1 - V^3) |P_2 P_4| \bar{\mathbf{A}}_K \nabla u_{h,K} \cdot \mathbf{n}_{42} + (V^2 - V^4) |P_1 P_3| \bar{\mathbf{A}}_K \nabla u_{h,K} \cdot \mathbf{n}_{31}. \quad (3.19)$$

Noting (3.5) and (3.6),

$$A_K = \frac{1}{2} |P_1 P_3| |P_2 P_4| (\nabla v_{h,K} \cdot \vec{\tau}_{31} \bar{\mathbf{A}}_K \nabla u_{h,K} \cdot \mathbf{n}_{42} + \nabla v_{h,K} \cdot \vec{\tau}_{42} \bar{\mathbf{A}}_K \nabla u_{h,K} \cdot \mathbf{n}_{31}). \quad (3.20)$$

On the other hand,

$$\vec{\tau}_{31} = -\sin \alpha \mathbf{n}_{42} - \cos \alpha \vec{\tau}_{42}, \quad \mathbf{n}_{31} = \cos \alpha \mathbf{n}_{42} - \sin \alpha \vec{\tau}_{42}, \quad (3.21)$$

so

$$\begin{aligned} A_K &= -\frac{1}{2}|P_1P_3||P_2P_4|\sin \alpha (\nabla v_{h,K} \cdot \mathbf{n}_{42} \bar{\mathbf{A}}|_K \nabla u_{h,K} \cdot \mathbf{n}_{42} + \nabla v_{h,K} \cdot \vec{\tau}_{42} \bar{\mathbf{A}}|_K \nabla u_{h,K} \cdot \vec{\tau}_{42}) \\ &= -\text{meas}(K)\bar{\mathbf{A}}|_K \nabla u_{h,K} \nabla v_{h,K} = -\int_K \bar{\mathbf{A}} \nabla u_h \nabla v_h dx. \end{aligned} \quad (3.22)$$

Substituting this equality into (3.17), we get the desired result.

3.2 The solvability of the discrete problem

We first give a lemma which shows that the finite volume element bilinear form $a_h^*(\cdot, \cdot)$ is only a perturbation of the finite element bilinear form $a_h(\cdot, \cdot)$.

Lemma 3.4 *If $\mathbf{A} \in (W^{1,\infty}(\Omega))^4$, then there exists a constant C independent of the meshsize h , such that for any $u_h, v_h \in V_h$,*

$$|a_h^*(u_h, v_h) - a_h(u_h, v_h)| \leq Ch \|u_h\|_{1,h} \|v_h\|_{1,h}. \quad (3.23)$$

proof. Let $\bar{\mathbf{A}}$ be the L^2 orthogonal projection of \mathbf{A} onto space $(Q_h)^4$, i.e.,

$$\bar{\mathbf{A}}_{ij}|_K = \frac{1}{\text{meas}(K)} \int_K a_{ij}(x) dx, \quad 1 \leq i, j \leq 2, \quad K \in \mathcal{T}_h. \quad (3.24)$$

Define

$$\bar{a}_h^*(u_h, v_h) = -\sum_{i=1}^{N_P^i} V^i \int_{\partial K_{P_i}^*} \bar{\mathbf{A}} \nabla_h u_h \cdot \mathbf{n} ds, \quad (3.25)$$

by lemma 3.3,

$$\bar{a}_h^*(u_h, v_h) = \int_{\Omega} \bar{\mathbf{A}} \nabla_h u_h \cdot \nabla_h v_h dx, \quad (3.26)$$

so

$$|\bar{a}_h^*(u_h, v_h) - a_h(u_h, v_h)| \leq Ch \|u_h\|_{1,h} \|v_h\|_{1,h}. \quad (3.27)$$

On the other hand,

$$\begin{aligned} & a_h^*(u_h, v_h) - \bar{a}_h^*(u_h, v_h) \\ &= -\sum_{K \in \mathcal{T}_h} ((V^1 - V^3) \int_{P_2P_4} (\mathbf{A} - \bar{\mathbf{A}})|_K \nabla_h u_h \cdot \mathbf{n}_{42} ds \\ & \quad + (V^2 - V^4) \int_{P_1P_3} (\mathbf{A} - \bar{\mathbf{A}})|_K \nabla_h u_h \cdot \mathbf{n}_{31} ds), \end{aligned} \quad (3.28)$$

so

$$\begin{aligned} & |a_h^*(u_h, v_h) - \bar{a}_h^*(u_h, v_h)| \\ & \leq Ch \|\mathbf{A}\|_{1,\infty,\Omega} \|u_h\|_{1,h} \left(\sum_{(i,j) \in \omega} |V^i - V^j|^2 \right)^{\frac{1}{2}} \\ & \leq Ch \|\mathbf{A}\|_{1,\infty,\Omega} \|u_h\|_{1,h} \|v_h\|_{1,h}. \end{aligned} \quad (3.29)$$

Combining it with (3.27), we complete the proof of this lemma.

By Lemma 3.4,

$$\begin{aligned} a_h^*(v_h, v_h) &= a_h(v_h, v_h) + a_h^*(v_h, v_h) - a_h(v_h, v_h) \\ &\geq (\alpha_* - Ch) \|v_h\|_{1,h}^2, \end{aligned} \quad (3.30)$$

where the uniform ellipticity of $a_h(\cdot, \cdot)$ is used. If we choose properly small $h_0 = \frac{\alpha_*}{2C}$, then we have the uniform ellipticity of $a_h^*(\cdot, \cdot)$ stated in the following theorem:

Theorem 3.1 *There exists a constant C independent of the meshsize h , such that for any $h \in (0, h_0)$,*

$$a_h^*(v_h, v_h) \geq C \|v_h\|_{1,h}^2, \quad \forall v_h \in V_h. \quad (3.31)$$

Now an application of Lax-Milgram lemma gives the existence and uniqueness of the solution of the discrete problem (2.13).

3.3 H^1 and L^2 -norm error estimates

Theorem 3.2 *Let u, u_h be the solution of (2.1) and (2.13) respectively, then there exists a constant C independent of h , such that*

$$\|u - u_h\|_{1,h} \leq Ch \|u\|_2. \quad (3.32)$$

Proof. Define the interpolation operator $\Pi_h : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow NC_h^0$ as in [20] such that

$$\Pi_h u(m) = \frac{1}{2}(u(P_1) + u(P_2)), \quad \forall m \in \mathcal{M}, \quad (3.33)$$

where P_1 and P_2 are the endpoints of an edge with m as its midpoint. For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$, its interpolation can be expressed as

$$\Pi_h u = \frac{1}{2} \sum_{i=1}^{N_P^i} u^i \varphi_i, \quad (3.34)$$

where $u^i = u(P_i)$. It has the following approximation property

$$\|u - \Pi_h u\| + h \|u - \Pi_h u\|_{1,h} \leq Ch^2 \|u\|_2, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega). \quad (3.35)$$

Let $e_h = \Pi_h u - u_h$, by Theorem 3.1,

$$c_0 \|\nabla e_h\|^2 \leq - \sum_{i=1}^{N_P^i} \left(\frac{u^i}{2} - U^i \right) \int_{\partial K_{P_i}^*} \mathbf{A} \nabla_h (\Pi_h u - u_h) \cdot \mathbf{n} \, ds. \quad (3.36)$$

Noting (2.13) and the fact that

$$- \int_{\partial K_{P_i}^*} \mathbf{A} \nabla u \cdot \mathbf{n} \, ds = \int_{K_{P_i}^*} f \, dx, \quad (3.37)$$

(3.36) can be rewritten as

$$\begin{aligned} c_0 \|\nabla_h e_h\|^2 &\leq - \sum_{i=1}^{N_P^i} \left(\frac{u^i}{2} - U^i \right) \int_{\partial K_{P_i}^*} \mathbf{A} \nabla_h (\Pi_h u - u) \cdot \mathbf{n} \, ds \\ &\leq C_0 \|\nabla_h e_h\|_{0,\Omega} \left\{ \sum_{(i,j) \in \omega} \left(\int_{P_i P_j} \mathbf{A} \nabla_h (\Pi_h u - u) \cdot \mathbf{n}_{ij} \, ds \right)^2 \right\}^{\frac{1}{2}}, \end{aligned} \quad (3.38)$$

where Lemma 3.2 is used.

On the other hand,

$$\left| \int_{P_i P_j} \mathbf{A} \nabla (\Pi_h u - u) \cdot \mathbf{n}_{ij} \, ds \right| \leq Ch \|u\|_{2,K}, \quad (3.39)$$

where K is the element with P_i, P_j as its two vertices.

So

$$\|\nabla_h e_h\|_0 \leq Ch \|u\|_2. \quad (3.40)$$

Combining it with (3.35) completes the proof of this theorem.

To obtain optimal convergence rate for the L^2 -norm error estimates, we need the following assumption that each quadrilateral in \mathcal{T}_h is almost a parallelogram, i.e.,

Assumption 3.1 *The distant d_K between the midpoints of the diagonals is of order $O(h^2)$ for all elements K as $h \rightarrow 0$.*

As stated in [23] all quadrilaterals produced by the bi-section scheme of mesh subdivision satisfy this assumption. Under this assumption we have

Theorem 3.3 *Let u, u_h be the solution of (2.1) and (2.13) respectively. If $\mathbf{A} \in W^{2,\infty}(\Omega)$ and $f \in W^{1,p}(\Omega)$, $p > 1$, then there exists a constant $C > 0$ independent of h , such that*

$$\|u - u_h\|_0 \leq Ch^2 \|f\|_{1,p}. \quad (3.41)$$

Proof. Let $\psi \in H_0^1(\Omega)$ be the solution of

$$-\nabla \cdot (\mathbf{A}\nabla\psi) = u - u_h \quad \text{in } \Omega, \quad \text{and } \psi = 0 \quad \text{on } \partial\Omega, \quad (3.42)$$

then we have

$$\|\psi\|_2 \leq C\|u - u_h\|_0. \quad (3.43)$$

An application of Green's formula shows that

$$\begin{aligned} \|u - u_h\|_0^2 &= \sum_{K \in \mathcal{T}_h} (\mathbf{A}\nabla(u - u_h), \nabla(\psi - \Pi_h\psi))_K + \sum_{K \in \mathcal{T}_h} \mathbf{A}\nabla(u - u_h), \nabla\Pi_h\psi)_K \\ &\quad - \sum_{K \in \mathcal{T}_h} \langle \mathbf{A}\nabla\psi \cdot \mathbf{n}, u - u_h \rangle_{\partial K} \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.44)$$

By (3.35) and Theorem 3.2,

$$|I_1| \leq Ch^2 \|u\|_2 \|\psi\|_2. \quad (3.45)$$

For the consistency error I_3 , as in [20], it can be bounded by

$$|I_3| \leq Ch \|u - u_h\|_{1,h} \|\psi\|_2 \leq Ch^2 \|u\|_2 \|\psi\|_2. \quad (3.46)$$

About I_2 , it can be rewritten as

$$\begin{aligned} I_2 &= (f, \Pi_h\psi) - (f, r_h\Pi_h\psi) - \sum_{i=1}^{N_P^i} \frac{\psi^i}{2} \int_{\partial K_{P_i}^*} \mathbf{A}\nabla_h u_h \cdot \mathbf{n} \, ds \\ &\quad + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{A}\nabla u \cdot \mathbf{n} \Pi_h\psi \, ds - \sum_{K \in \mathcal{T}_h} (\mathbf{A}\nabla u_h, \nabla\Pi_h\psi)_K. \end{aligned} \quad (3.47)$$

As in [10], let $C_K(f) = \int_B f w_K dx$ on K , where B is the biggest ball in R^2 satisfying $B \subset\subset K$ and w_K a cut-off function supported in \bar{B} as defined by Def.4.1.3 in [8], then the first two terms in the right-hand side of (3.47) can be bounded by

$$(f, \Pi_h\psi - r_h\Pi_h\psi) \leq ch^2 \|f\|_{1,p} \|\psi\|_2 + \sum_{K \in \mathcal{T}_h} \int_K C_K(f) (\Pi_h\psi - r_h\Pi_h\psi) dx. \quad (3.48)$$

For the parallelogram, $\int_K (\Pi_h\psi - r_h\Pi_h\psi) dx = 0$, the last term in (3.48) disappears. But this does not hold for the general quadrilateral mesh. Using notations in Figure 3.,

$$(\Pi_h\psi - r_h\Pi_h\psi)|_K = \sum_{i=1}^4 \frac{\psi^i}{2} (\varphi_i - \chi_i). \quad (3.49)$$

A careful calculation leads to

$$\int_K (\varphi_1 - \chi_1) dx = \frac{\text{meas}(K)}{3} \left(\frac{|P_3O| - |P_1O|}{|P_1P_3|} \right), \quad (3.50)$$

$$\int_K (\varphi_2 - \chi_2) dx = \frac{\text{meas}(K)}{3} \left(\frac{|P_4O| - |P_2O|}{|P_2P_4|} \right), \quad (3.51)$$

$$\int_K (\varphi_3 - \chi_3) dx = \frac{\text{meas}(K)}{3} \left(\frac{|P_1O| - |P_3O|}{|P_1P_3|} \right), \quad (3.52)$$

$$\int_K (\varphi_4 - \chi_4) dx = \frac{\text{meas}(K)}{3} \left(\frac{|P_2O| - |P_4O|}{|P_2P_4|} \right). \quad (3.53)$$

So

$$\begin{aligned} & \int_K (\Pi_h \psi - r_h \Pi_h \psi) dx \\ &= \frac{\text{meas}(K)}{3} \left\{ \frac{\psi^1 - \psi^3}{2} \frac{|P_3O| - |P_1O|}{|P_1P_3|} + \frac{\psi^2 - \psi^4}{2} \frac{|P_4O| - |P_2O|}{|P_2P_4|} \right\}. \end{aligned} \quad (3.54)$$

By Assumption 3.1 and the regularity of \mathcal{T}_h ,

$$\begin{aligned} & \left| \sum_{K \in \mathcal{T}_h} \int_K C_K(f) (\Pi_h \psi - r_h \Pi_h \psi) dx \right| \\ & \leq Ch^2 \left(\sum_{K \in \mathcal{T}_h} |C_K(f)|^2 \cdot \text{meas}(K) \right)^{\frac{1}{2}} \left(\sum_{(i,j) \in \omega} \left(\frac{\psi^i}{2} - \frac{\psi^j}{2} \right)^2 \right)^{\frac{1}{2}} \\ & \leq Ch^2 \left(\sum_{K \in \mathcal{T}_h} \|C_K(f)\|_{0,K}^2 \right)^{\frac{1}{2}} \|\Pi_h \psi\|_{1,h} \\ & \leq Ch^2 \|f\|_0 \|\psi\|_1. \end{aligned} \quad (3.55)$$

Substituting it into (3.48), we get

$$(f, \Pi_h \psi - r_h \Pi_h \psi) \leq Ch^2 \|f\|_{1,p} \|\psi\|_2, \quad p > 1. \quad (3.56)$$

Since

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot (\mathbf{A} \nabla u_h) r_h \Pi_h \psi dx \\ &= \sum_{i=1}^{N_P} \frac{\psi^i}{2} \int_{\partial K_{P_i}^*} \mathbf{A} \nabla u_h \cdot \mathbf{n} ds + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{A} \nabla u_h \cdot \mathbf{n} r_h \Pi_h \psi ds, \end{aligned} \quad (3.57)$$

and

$$\sum_{K \in \mathcal{T}_h} (\mathbf{A} \nabla u_h, \nabla \Pi_h \psi)_K = - \sum_{K \in \mathcal{T}_h} (\nabla \cdot (\mathbf{A} \nabla u_h), \Pi_h \psi) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{A} \nabla u_h \cdot \mathbf{n} \Pi_h \psi ds, \quad (3.58)$$

the last three terms of (3.47) can be rewritten as

$$\begin{aligned} s &= \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot (\mathbf{A} \nabla u_h) (\Pi_h \psi - r_h \Pi_h \psi) dx \\ &+ \sum_{K \in \mathcal{T}_h} \left\{ - \int_{\partial K} \mathbf{A} \nabla u_h \cdot \mathbf{n} (\Pi_h \psi - r_h \Pi_h \psi) ds + \int_{\partial K} \mathbf{A} \nabla u \cdot \mathbf{n} \Pi_h \psi ds \right\} \\ &\doteq A_1 + A_2, \end{aligned} \quad (3.59)$$

where

$$|A_1| \leq Ch^2 \|\mathbf{A}\|_{2,\infty} \|u\|_2 \|\psi\|_2. \quad (3.60)$$

Note the fact that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{A} \nabla u \cdot \mathbf{n} r_h v_h ds = 0, \quad \forall v_h \in V_h, \quad (3.61)$$

and

$$\int_e (v_h - r_h v_h) ds = 0, \quad \forall e \in \mathcal{E}_h, \quad (3.62)$$

let $\bar{\mathbf{A}}_e = \mathbf{A}(m_e)$, A_2 can be rewritten as

$$\begin{aligned} A_2 &= \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_h(K)} \int_e (\mathbf{A} - \bar{\mathbf{A}}_e) \nabla(u - u_h) \cdot \mathbf{n} (\Pi_h \psi - r_h \Pi_h \psi) ds \\ &+ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_h(K)} \int_e \bar{\mathbf{A}}_e \nabla u \cdot \mathbf{n} \Pi_h \psi ds \\ &\doteq B_1 + B_2, \end{aligned} \quad (3.63)$$

where

$$|B_1| \leq Ch^2 \|u\|_2 \|\psi\|_2. \quad (3.64)$$

For B_2 , we introduce the standard Q_1 -conforming finite element space X_h^0 defined on \mathcal{T}_h with zero boundary and let Π_{Q_1} be its standard interpolation operator. Since

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \bar{\mathbf{A}}_e \nabla u \cdot \mathbf{n} w_h = 0, \quad \forall w_h \in X_h^0, \quad (3.65)$$

$$\int_e (\Pi_h \psi - \Pi_{Q_1} \psi) ds = 0, \quad (3.66)$$

$$B_2 = \sum_{e \in \mathcal{E}_h} \int_e (\bar{\mathbf{A}}_e \nabla u \cdot \mathbf{n} - c_1) [\Pi_h \psi - \Pi_{Q_1} \psi] ds, \quad \forall c_1 \in \mathbb{R}, \quad (3.67)$$

where $[\cdot]$ denotes the jump of a function over an edge. Then we have

$$\begin{aligned} |B_2| &\leq Ch \|u\|_2 (h^{-1} \|\Pi_h \psi - \Pi_{Q_1} \psi\|_0 + \|\Pi_h \psi - \Pi_{Q_1} \psi\|_{1,h}) \\ &\leq Ch^2 \|u\|_2 \|\psi\|_2. \end{aligned} \quad (3.68)$$

Noting (3.43), we complete the proof of this theorem.

In the following, we will use the counterexample given in [18] to show that the optimal L^2 -norm error estimate

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_2 \leq Ch^2 \|f\|_0, \quad (3.69)$$

does not hold for the P_1 -nonconforming quadrilateral finite volume element discretization either.

Counterexample: Consider the model problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.70)$$

where $\Omega = [-2, 2] \times [-2, 2]$. Let \mathcal{T}_h be a decomposition of Ω into $4n^2$ equal-size squares with edge $h = \frac{1}{n}$, \mathcal{T}_h^* be the dual partition of Ω defined in section 2.

If we assume (3.69) is true, by the definition of L^2 -norm, we have

$$\left| \int_{\Omega} (u - u_h) \phi dx \right| \leq Ch^2 \|\phi\|_0 \|f\|_0, \quad \forall \phi \in L^2(\Omega). \quad (3.71)$$

Introduce the auxiliary function $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ defined by

$$\begin{cases} -\Delta \psi = \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.72)$$

then as in [18], it can be deduced that

$$\|\Pi_h \psi - r_h \Pi_h \psi\|_0 \leq Ch^2 \|\phi\|_0, \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega). \quad (3.73)$$

Choosing a special function $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\psi(x_1, x_2) = x_1(1 - x_1), \quad \text{in } \Omega_1 = [0, 1] \times [0, 1]. \quad (3.74)$$

By (3.73), the following estimate should hold for this ψ :

$$\|\psi - r_h \Pi_h \psi\|_{0, \Omega_1} \leq Ch^2. \quad (3.75)$$

However, a direct calculation shows that

$$\begin{aligned} & \|\psi - r_h \Pi_h \psi\|_{0, \Omega_1}^2 \\ &= \int_{\Omega_1} \psi^2 dx + \int_{\Omega_1} (r_h \Pi_h \psi)^2 dx - 2 \int_{\Omega_1} \psi r_h \Pi_h \psi dx \\ &= \frac{1}{30} + \frac{h^5}{24} \sum_{i=1}^N \left\{ -12i^4 + 24(N+1)i^3 - (12N^2 + 36N + 19)i^2 \right. \\ & \quad \left. + (N+1)(12N+7)i - (N+1)(3N + \frac{1}{2}) \right\} \\ & \quad + \frac{h^5}{6} \sum_{i=1}^{N-1} \left\{ -3i^4 + 6Ni^3 - (3N^2 + \frac{1}{4})i^2 + \frac{1}{4}Ni \right\} \\ &= \frac{1}{72}h^2 + o(h^2), \end{aligned} \quad (3.76)$$

i.e.,

$$\|\psi - r_h \Pi_h \psi\|_{0, \Omega_1} = \frac{\sqrt{2}}{12}h + o(h). \quad (3.77)$$

Comparing it with (3.75), we get a contradiction. That means (3.69) is not true in fact.

4. Cascadic Multigrid Algorithm

In this section, we will apply the cascadic multigrid algorithm proposed in [26] to solve the discrete problem (2.13). We construct a sequence of nested quadrilateral partition of Ω as follows: suppose a coarse partition \mathcal{T}_0 is given, we define the finer partition \mathcal{T}_l for $l \geq 1$ by subdividing every quadrilateral in \mathcal{T}_{l-1} into four sub-quadrilaterals by the bi-section technique. For such defined nested partitions,

(1) If the coarse partition \mathcal{T}_0 is regular, quasi-uniform, and satisfies Assumption 3.1, then all these properties still hold for every \mathcal{T}_l , $l \geq 1$;

(2) Let h_l denote the maximum mesh size of \mathcal{T}_l , then $h_l = \frac{h_{l-1}}{2}$;

(3) For every partition \mathcal{T}_l , the notations V_h , N_P^i , P_i , φ_i , \mathcal{P} , \mathcal{E}_h , \mathcal{M} defined in Section 2 are rewritten as V_l , $N_{P,l}^i$, P_i^l , φ_i^l , \mathcal{P}_l , \mathcal{E}_l , \mathcal{M}_l ;

(4) For \mathcal{T}_l , denote its dual partition as \mathcal{T}_l^* .

Using these partitions, the problem (2.10) defined on \mathcal{T}_l can be written as: find $\hat{u}_l \in V_l$ such that

$$a_l(\hat{u}_l, v_l) = (f, v_l), \quad \forall v_l \in V_l, \quad (4.1)$$

and the problem (2.18) can be written as: find $u_l \in V_l$ such that

$$a_l^*(u_l, v_l) = (f, r_l v_l), \quad \forall v_l \in V_l, \quad (4.2)$$

where $a_l(\cdot, \cdot)$, $a_l^*(\cdot, \cdot)$, r_l are the restriction of $a_h(\cdot, \cdot)$, $a_h^*(\cdot, \cdot)$ and r_h on \mathcal{T}_l respectively.

Define the energy norm on level l as

$$\|v\|_l^2 = a_l(v, v), \quad \forall v \in V_l. \quad (4.3)$$

It is easy to see that $\|\cdot\|_l$ is equivalent to the norm $\|\cdot\|_{1, h_l}$, i.e.,

$$C_* \|v\|_{1, h_l} \leq \|v\|_l \leq C^* \|v\|_{1, h_l}, \quad \forall v \in V_l. \quad (4.4)$$

Since $V_{l-1} \subseteq V_l$ does not hold in this case, in order to get a cascadic multigrid algorithm for problem (4.2), we define an inter-grid transfer operator $I_l : V_{l-1} \rightarrow V_l$ as follows: for any

$v_{l-1} \in V_{l-1}$,

$$I_l v_{l-1} = \sum_{i=1}^{N_{P,l}^i} W^i \varphi_i^l, \quad (4.5)$$

where

(1) If $P_i^l \in \mathcal{P}_{l-1}$

$$W^i = \frac{1}{2N_C} \sum_{j=1}^{N_C} v_{l-1}|_{K_j}(P_i^l), \quad (4.6)$$

where K_j , $j = 1, 2, \dots, N_C$ are elements in \mathcal{T}_l with common vertex P_i^l ;

(2) else

$$W^i = \frac{1}{2} V_{l-1}(P_i^l). \quad (4.7)$$

For such defined operator, it has the following approximation property:

Lemma 4.1 *There exists a constant $C > 0$, such that for any $v \in V_{l-1}$,*

$$\|I_l v\|_0 \leq C \|v\|_0, \quad (4.8)$$

$$\|I_l v\|_l \leq C \|v\|_{l-1}, \quad (4.9)$$

$$\|v - I_l v\|_0 \leq Ch_l \|v\|_{1,h_{l-1}}, \quad (4.10)$$

$$\|u_l - I_l u_{l-1}\|_0 \leq Ch_l^2 \|f\|_{1,p}, \quad (4.11)$$

where u_l, u_{l-1} are the solution of (4.2) on level $l, l-1$ respectively.

proof. Since (4.8) and (4.9) can be easily obtained from (4.10) by inverse inequality and the equivalence of the norms, we need only to prove (4.10) and (4.11). Without loss of generality, we assume that $N_C = 4$ for convenience of analysis.

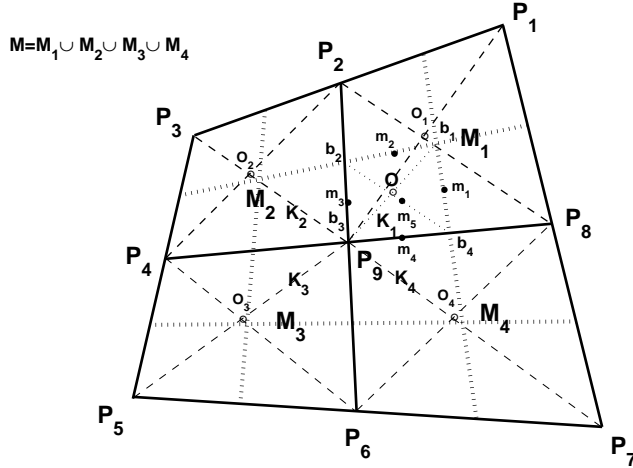


Figure 5.

For any $v = \sum_{i=1}^{N_{P,l}^i} V^i \varphi_i^{l-1}$, using the notations in Figure 5., $v|_M = \sum_{i=1}^9 V^i \varphi_i^{l-1}$. By the definition of I_l , $I_l v|_{K_1}$ can be expressed as

$$I_l v|_{K_1} = \sum_{i=1}^4 W^i \varphi_i^l, \quad (4.12)$$

where

$$\begin{aligned} W^1 &= \frac{1}{4}(V^1 + V^2 + V^8 + V^9), \quad W^2 = \frac{1}{2}(V^2 + V^9), \quad W^4 = \frac{1}{2}(V^8 + V^9), \\ W^3 &= \frac{1}{8} \left\{ (4V^9 + 2V^2 + 2V^6) + (V^9 - V^1) \frac{|P_9 O_1|}{|P_1 P_9|} + (V^8 - V^2) \frac{|P_2 O_1|}{|P_2 P_8|} \right. \\ &\quad + (V^9 - V^3) \frac{|P_9 O_2|}{|P_3 P_9|} + (V^4 - V^2) \frac{|P_2 O_2|}{|P_2 P_4|} + (V^9 - V^5) \frac{|P_9 O_3|}{|P_5 P_9|} \\ &\quad \left. + (V^4 - V^6) \frac{|P_6 O_3|}{|P_4 P_6|} + (V^9 - V^7) \frac{|P_9 O_4|}{|P_9 P_7|} + (V^8 - V^6) \frac{|P_6 O_4|}{|P_6 P_8|} \right\}. \end{aligned} \quad (4.13)$$

After careful manipulations, we obtain

$$\begin{aligned} &\int_{K_1} (v - I_l v)^2 dx \\ &= \left\{ \frac{|b_1 o|}{|b_1 b_3|} ((v(m_1) - I_l v(m_1))^2 + (v(m_2) - I_l v(m_2))^2 + (v(o) - I_l v(o))^2) \right. \\ &\quad \left. + \frac{|b_3 o|}{|b_1 b_3|} ((v(m_3) - I_l v(m_3))^2 + (v(m_4) - I_l v(m_4))^2 + (v(o) - I_l v(o))^2) \right\} \frac{|K_1|}{3} \\ &\leq Ch_l^2 \left\{ (V^1 - V^9)^2 + (V^2 - V^8)^2 + (V^2 - V^4)^2 + (V^3 - V^9)^2 \right. \\ &\quad \left. + (V^5 - V^9)^2 + (V^4 - V^6)^2 + (V^6 - V^8)^2 + (V^7 - V^9)^2 \right\}. \end{aligned} \quad (4.14)$$

So

$$\|v - I_l v\|_0 \leq Ch_l \left(\sum_{(i,j) \in \omega^{l-1}} (V^i - V^j)^2 \right)^{\frac{1}{2}} \leq Ch_l \|v\|_{1, h_{l-1}}. \quad (4.15)$$

On the other hand, since

$$\begin{aligned} &\|u_l - I_l u_{l-1}\|_0 \\ &\leq \|u_l - u\|_0 + \|u - \Pi_l u\|_0 + \|\Pi_l u - I_l \Pi_{l-1} u\|_0 + \|I_l \Pi_{l-1} u - I_l u_{l-1}\|_0 \\ &\leq Ch_l^2 \|f\|_{1,p} + \|\Pi_l u - I_l \Pi_{l-1} u\|_0, \end{aligned} \quad (4.16)$$

we need only to estimate the last term in (4.16). Note that

$$\Pi_l u|_{K_1} = \frac{u(b_1)}{2} \varphi_1^l + \frac{u(b_2)}{2} \varphi_2^l + \frac{u(b_3)}{2} \varphi_3^l + \frac{u(b_4)}{2} \varphi_4^l, \quad (4.17)$$

$$\begin{aligned} I_l \Pi_{l-1} u|_{K_1} &= \frac{\Pi_{l-1} u(b_1)}{2} \varphi_1^l + \frac{\Pi_{l-1} u(b_2)}{2} \varphi_2^l + \frac{\Pi_{l-1} u(b_4)}{2} \varphi_4^l \\ &\quad + \frac{1}{8} \left\{ \Pi_{l-1} u|_{M_1}(P_9) + \Pi_{l-1} u|_{M_2}(P_9) \right. \\ &\quad \left. + \Pi_{l-1} u|_{M_3}(P_9) + \Pi_{l-1} u|_{M_4}(P_9) \right\} \varphi_3^l, \end{aligned} \quad (4.18)$$

we have

$$\begin{aligned} &\|\Pi_l u - I_l \Pi_{l-1} u\|_{0, K_1}^2 \\ &\leq C \left\{ [u(b_1) - \Pi_{l-1} u(b_1)]^2 + [u(b_2) - \Pi_{l-1} u(b_2)]^2 + [u(b_4) - \Pi_{l-1} u(b_4)]^2 \right. \\ &\quad \left. + [u(b_3) - \Pi_{l-1} u|_{M_1}(b_3)]^2 + [u(b_3) - \Pi_{l-1} u|_{M_2}(b_3)]^2 \right. \\ &\quad \left. + [u(b_3) - \Pi_{l-1} u|_{M_3}(b_3)]^2 + [u(b_3) - \Pi_{l-1} u|_{M_4}(b_3)]^2 \right\} |K_1|. \end{aligned} \quad (4.19)$$

Let $\tilde{\mathcal{T}}_l$ be the subdivision of \mathcal{T}_l with each quadrilateral divided into two triangles and $\tilde{\Pi}_l$ be the standard interpolation operator of P_1 -conforming triangular element defined on $\tilde{\mathcal{T}}_l$, then

$$\begin{aligned} &\int_{K_1} (\tilde{\Pi}_l u - \Pi_{l-1} u)^2 dx \\ &\geq \frac{|K_1|}{20} \frac{|b_1 o|}{|b_1 b_3|} \left\{ [u(b_1) - \Pi_{l-1} u(b_1)]^2 + [u(b_2) - \Pi_{l-1} u(b_2)]^2 + [u(b_4) - \Pi_{l-1} u(b_4)]^2 \right\} \\ &\quad + \frac{|K_1|}{20} \frac{|b_3 o|}{|b_1 b_3|} \left\{ [u(b_2) - \Pi_{l-1} u(b_2)]^2 + [u(b_3) - \Pi_{l-1} u|_{M_1}(b_3)]^2 + [u(b_4) - \Pi_{l-1} u(b_4)]^2 \right\}. \end{aligned} \quad (4.20)$$

By Assumption 3.1, for properly small h_0 , it holds that

$$\frac{|b_1 o|}{|b_1 b_3|} > \frac{1}{3}, \quad \frac{|b_3 o|}{|b_1 b_3|} > \frac{1}{3}. \quad (4.21)$$

So

$$\|\tilde{\Pi}_l u - \Pi_{l-1} u\|_{0, K_1}^2 \geq \frac{|K_1|}{60} \{ [u(b_1) - \Pi_{l-1} u(b_1)]^2 + [u(b_2) - \Pi_{l-1} u(b_2)]^2 + [u(b_3) - \Pi_{l-1} u(b_3)]^2 + [u(b_4) - \Pi_{l-1} u(b_4)]^2 \}. \quad (4.22)$$

Combining it with (4.19), we obtain

$$\|\Pi_l u - I_l \Pi_{l-1} u\|_{0, K_1}^2 \leq C \sum_{i=1}^4 \|\tilde{\Pi}_l u - \Pi_{l-1} u\|_{0, K_i}^2. \quad (4.23)$$

That is to say

$$\|\Pi_l u - I_l \Pi_{l-1} u\|_0 \leq C \|\tilde{\Pi}_l u - \Pi_{l-1} u\|_0 \leq C h_l^2 \|f\|_0. \quad (4.24)$$

Substituting it into (4.16), we get (4.11).

Considering the fact that the finite volume quadratic form is a small perturbation of the finite element quadratic form, as in [26], we propose a cascadic multigrid algorithm to solve the system (4.2) as follows:

Algorithm I.

- (1) Let $u_0^0 = u_0^* \hat{=} u_0$ be the exact solution of (4.2) for $l = 0$;
- (2) for $l = 1, 2, \dots, L$, let \tilde{u}_l be the solution of the following problem

$$a_l(\tilde{u}_l, v) = (f, r_l v) - N_l(I_l u_{l-1}^*, v), \quad \forall v \in V_l, \quad (4.25)$$

where

$$N_l(u, v) = a_l^*(u, v) - a_l(u, v), \quad \forall u, v \in V_l, \quad (4.26)$$

let $u_l^0 = I_l u_{l-1}^*$, for (4.25) do iterations

$$u_l^{m_l} = G_l^{m_l} u_l^0; \quad (4.27)$$

- (3) Set $u_l^* \hat{=} u_l^{m_l}$;

where $G_l : V_l \rightarrow V_l$ is the iteration operator on level l , such as the Richardson, Jacobi, Gauss-seidel or CG iterations, m_l is the number of iteration steps and it is the smallest integer satisfying

$$m_l \geq \beta^{L-l} m_L \quad (4.28)$$

for some fixed $\beta > 1$, $m_L > 0$.

It is well known that for the smoothing operator mentioned above, there exists a linear operator $S_l : V_l \rightarrow V_l$ such that

$$\tilde{u}_l - G_l^{m_l} u_l^0 = S_l^{m_l} (\tilde{u}_l - u_l^0), \quad (4.29)$$

and it holds that

$$\| \| S_l^{m_l} v \| \|_l \leq C \frac{h_l^{-1}}{m_l^\gamma} \|v\|_0, \quad \forall v \in V_l, \quad (4.30)$$

$$\| \| S_l^{m_l} v \| \|_l \leq \| \| v \| \|_l, \quad \forall v \in V_l, \quad (4.31)$$

where γ is a positive number depending on the given iteration, $\gamma = 1$ for the CG iteration and $\gamma = \frac{1}{2}$ for the other three iterations mentioned above.

By the equivalence of the norm $\| \| \cdot \| \|$ and $\| \cdot \|_{1, h_l}$, (3.23), (3.32) can be rewritten as

$$|a_l^*(w_l, v_l) - a_l(w_l, v_l)| \leq c_1 h_l \| \| w_l \| \|_l \| \| v_l \| \|_l, \quad \forall w_l, v_l \in V_l, \quad (4.32)$$

and

$$\|u - u_l\|_l \leq c_2 h_l \|u\|_2. \quad (4.33)$$

To avoid ambiguity in the following analysis, we rewrite (3.41) as

$$\|u - u_l\|_0 \leq c_3 h_l^2 \|f\|_{1,p} \quad (4.34)$$

and denote the constants in (4.8)-(4.11), (4.30) as c_4, c_5 respectively. The inverse inequality that will be used is written as

$$\|v\|_l \leq c_6 h_l^{-1} \|v\|_0, \quad \forall v_l \in V_l, \quad l = 0, 1, \dots, L. \quad (4.35)$$

For convenience of theoretical analysis, we introduce a projection operator $P_l : V_{l-1} + V_l \rightarrow V_l$ defined by

$$a_l(P_l u, v) = a_l(u, v), \quad \forall v \in V_l. \quad (4.36)$$

From the definition, it is easily seen that

$$\|P_l v\|_l \leq \|v\|_{l-1}, \quad \forall v \in V_{l-1}. \quad (4.37)$$

Similar argument as Lemma 2.4 in [24] leads to

Lemma 4.2 *For the above defined operator P_l , there exists a constant c_7 such that*

$$\|v - P_l v\|_0 \leq c_7 h_l \|v\|_{l-1}, \quad \forall v \in V_{l-1}. \quad (4.38)$$

Lemma 4.3 *If u_l, \tilde{u}_l are the solutions of (4.2) and (4.25) respectively, then there exists a constant C_1 such that*

$$\|u_l - \tilde{u}_l\|_l \leq C_1 h_l^2 \|f\|_{1,p} + C_1 h_l \{ \|u_{l-1} - \tilde{u}_{l-1}\|_{l-1} + \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1} \}. \quad (4.39)$$

proof. Since for any $v \in V_l$,

$$\begin{aligned} a_l(u_l - \tilde{u}_l, v) &= a_l(u_l, v) - (f, r_l v) + a_l^*(I_l u_{l-1}^*, v) - a_l(I_l u_{l-1}^*, v) \\ &= a_l(u_l - I_l u_{l-1}^*, v) - a_l^*(u_l - I_l u_{l-1}^*, v) \\ &\leq c_1 h_l \|u_l - I_l u_{l-1}^*\|_l \|v\|_l, \end{aligned} \quad (4.40)$$

we obtain

$$\begin{aligned} \|u_l - \tilde{u}_l\|_l &\leq c_1 h_l \{ \|u_l - I_l u_{l-1}^*\|_l + \|I_l u_{l-1} - I_l u_{l-1}^*\|_l \} \\ &\leq c_1 c_4 c_6 h_l^2 \|f\|_{1,p} + c_1 c_6 h_l \{ \|u_{l-1} - \tilde{u}_{l-1}\|_{l-1} + \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1} \}. \end{aligned} \quad (4.41)$$

Let $C_1 = \max\{c_1 c_4 c_6, c_1 c_6\}$, we get (4.39).

Lemma 4.4 *Let \tilde{u}_l, u_l^* be the solution defined in Algorithm I, then there exists a constant C_2 such that*

$$\begin{aligned} \|\tilde{u}_l - u_l^*\|_l &\leq \frac{C_2 h_l}{m_l^\gamma} \|f\|_{1,p} + \left(1 + \frac{C_2}{m_l^\gamma}\right) \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1} \\ &\quad + \frac{C_2 h_l^{-1}}{m_l^\gamma} \{ \|\tilde{u}_l - u_l\|_l + \|\tilde{u}_{l-1} - u_{l-1}\|_{l-1} \}. \end{aligned} \quad (4.42)$$

proof. Note that

$$\begin{aligned} \|\tilde{u}_l - u_l^*\|_l &= \|S_l^{m_l}(\tilde{u}_l - I_l u_{l-1}^*)\|_l \\ &\leq \|S_l^{m_l}(\tilde{u}_l - I_l \tilde{u}_{l-1})\|_l + \|S_l^{m_l} P_l(\tilde{u}_{l-1} - u_{l-1}^*)\|_l \\ &\quad + \|S_l^{m_l}(I_l - P_l)(\tilde{u}_{l-1} - u_{l-1}^*)\|_l \\ &\leq \frac{c_5 h_l^{-1}}{m_l^\gamma} \|\tilde{u}_l - I_l \tilde{u}_{l-1}\|_0 + \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1} \\ &\quad + \frac{c_5(c_6 + c_7)}{m_l^\gamma} \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1}. \end{aligned} \quad (4.43)$$

On the other hand,

$$\begin{aligned} \|\tilde{u}_l - I_l \tilde{u}_{l-1}\|_0 &\leq \|\tilde{u}_l - u_l\|_0 + \|u_l - I_l u_{l-1}\|_0 + \|I_l(u_{l-1} - \tilde{u}_{l-1})\|_0 \\ &\leq c_6 h_l^2 \|f\|_{1,p} + \|\tilde{u}_l - u_l\|_0 + c_6 \|u_{l-1} - \tilde{u}_{l-1}\|_0. \end{aligned} \quad (4.44)$$

So

$$\begin{aligned} \|\tilde{u}_l - u_l^*\|_l &\leq \frac{c_5 c_6 h_l}{m_l^\gamma} \|f\|_{1,p} + \left(1 + \frac{c_5(c_6 + c_7)}{m_l^\gamma}\right) \|\tilde{u}_{l-1} - u_{l-1}^*\|_{l-1} \\ &\quad + \frac{c_5 \max\{1, c_6\} h_l^{-1}}{\alpha_* m_l^\gamma} \{ \|\tilde{u}_l - u_l\|_l + \|\tilde{u}_{l-1} - u_{l-1}\|_{l-1} \}. \end{aligned} \quad (4.45)$$

Choosing $C_2 = \max\{c_5(c_6 + c_7), \frac{c_5 \max\{1, c_6\}}{\alpha_*}\}$ completes the proof of this lemma.

Based on the above two lemmas and using a similar argument of Theorem 3.3 in [26] leads to

Theorem 4.1 *If the meshsize h_0 of the coarsest partition is small enough such that (4.24) holds on every level l and*

$$C_1 h_0 < \frac{1}{4}, \quad (4.46)$$

moreover the number of the iteration steps on the last level satisfies

$$m_L^\gamma \geq \begin{cases} \max \left\{ \frac{C_2 \beta^\gamma}{\beta^\gamma - 1}, \exp(1)(C_2 + 10C_1 C_2) 2L \right\}, & \text{if } \beta^\gamma = 2, \\ \max \left\{ \frac{C_2 \beta^\gamma}{\beta^\gamma - 1}, \exp(1)(C_2 + 10C_1 C_2) \frac{4}{\beta^\gamma - 2} \right\}, & \text{if } \beta^\gamma > 2, \end{cases} \quad (4.47)$$

then under the same assumption as in Theorem 3.3, we have

$$\|u_L - u_L^*\|_L \leq C_3 (h_L^2 + \sum_{l=1}^L \frac{h_l}{m_l^\gamma}) \|f\|_{1,p}, \quad (4.48)$$

where $C_3 = \max \{ 2C_1, \exp(1)(C_2 + 10C_1 C_2) \}$.

Similar arguments to Lemma 1.3 and 1.4 in [3] leads to

Lemma 4.5 *Under the assumptions of Theorem 4.1, the accuracy of the cascadic multigrid **Algorithm I** is*

$$\|u_L - u_L^*\|_L \leq \begin{cases} C_3 \left(h_L + \frac{1}{1 - \frac{2}{\beta^\gamma}} \frac{1}{m_L^\gamma} \right) h_L \|f\|_{1,p}, & \text{for } \beta > 2^{\frac{1}{\gamma}}, p > 1, \\ C_3 \left(h_L + L \frac{1}{m_L^\gamma} \right) h_L \|f\|_{1,p}, & \text{for } \beta = 2^{\frac{1}{\gamma}}, p > 1. \end{cases} \quad (4.49)$$

Lemma 4.6 *The computational cost of the cascadic multigrid **Algorithm I** is proportional to*

$$\sum_{l=0}^L m_l n_l \leq \begin{cases} C \frac{1}{1 - \frac{2}{\beta^\gamma}} m_L n_L & \text{for } \beta < 4, \\ C L m_L n_L & \text{for } \beta = 4, \end{cases} \quad (4.50)$$

where n_l is the number of degrees of freedom on level l .

We call a cascadic multigrid algorithm is optimal in the energy norm on level l , if we obtain both the accuracy

$$\|u_l - u_l^*\|_l \approx \|u - u_l\|_l, \quad (4.51)$$

and the multigrid complexity

$$\text{amount of work} = O(n_l), \quad n_l = \dim V_l. \quad (4.52)$$

Summing the above two lemmas, we obtain

Theorem 4.2 Under the assumptions of Theorem 4.1, we have that
 (1). For $\gamma = 1$, if $2 \leq \beta < 4$, then the cascadic multigrid **Algorithm I** is optimal;
 (2). For $\gamma = \frac{1}{2}$, if $\beta = 4$ and m_L satisfies

$$m_L \geq L^2, \tag{4.53}$$

then the cascadic multigrid **Algorithm I** is quasi-optimal, i.e.,

$$\|u_L - u_L^*\|_{1,h_L} \leq C h_L \|f\|_{1,p}, \tag{4.54}$$

and the complexity of computation is

$$\sum_{l=0}^L m_l n_l \leq C n_L (1 + \log(n_L))^3. \tag{4.55}$$

5. Numerical Examples

In this section we will give two examples to confirm the theoretical results established above.

Example I. We consider the following problem:

$$\begin{cases} -\nabla \cdot (\mathbf{A}\nabla u) = f, & \text{in } \Omega = (0, 1) \times (0, 1), \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\mathbf{A} = \begin{pmatrix} e^{2x} + y^2 + 1 & e^{x+y} \\ e^{x+y} & x^2 + e^{2y} + 1 \end{pmatrix}$. let $u = \sin(2\pi x) \sin(2\pi y)(x^3 - y^4 + x^2 y^3)$ and f is determined by them.

For the three kinds of meshes depicted in Figure 6.-8., we use the P_1 -nonconforming quadrilateral finite volume element method to discretize (5.1) respectively and list the error between the finite volume element solution u_h and the exact solution u in Table 1.-3..

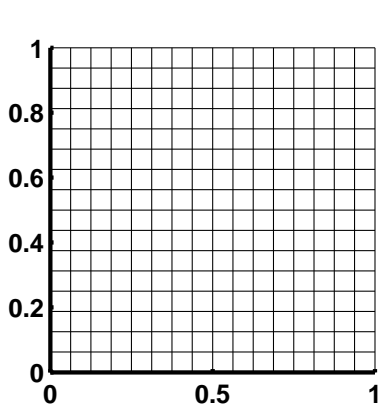


Figure 6. uniform mesh

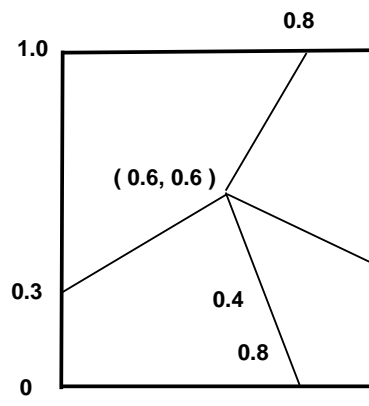


Figure 7. distorted mesh

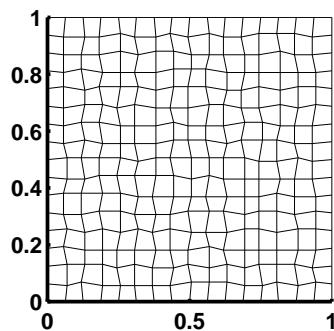


Figure 8. random-distorted mesh

For the partitions depicted in Figure 6. and Figure 7., which satisfy the Assumption 3.1, we can see from Table 1 and Table 2 that the convergence rate of the error under H^1 and L^2 -norm is optimal. This is in consistent with our theoretical results. But for the random-distorted mesh which does not satisfy the bi-section assumption, Table 3 shows that the convergence rate of the error is still optimal when the mesh is less distorted ($\leq 30\%$). Numerical behavior is better than that of theoretical analysis, which means in some sense that the L^2 -norm error estimates may can be improved.

Table 1 Error behavior on uniform mesh

#unknowns	$\ u - u_h\ _0$	#rate	$ u - u_h _{1,h}$	#rate
15×15	5.433179e-03		3.999064e-01	
31×31	1.355461e-03	2.0030	2.003736e-01	0.9970
63×63	3.386902e-04	2.0007	1.002405e-01	0.9992
127×127	8.465657e-05	2.0003	5.012701e-02	0.9998
255×255	2.114455e-05	2.0013	2.506435e-02	1.0000
511×511	5.338417e-06	1.9858	1.253228e-02	1.0000

Table 2 Error behavior on distorted bi-section mesh

#unknowns	$\ u - u_h\ _0$	#rate	$ u - u_h _{1,h}$	#rate
15×15	5.800791e-03		4.147051e-01	
31×31	1.506877e-03	1.9447	2.093579e-01	0.9861
63×63	3.885956e-04	1.9552	1.053572e-01	0.9907
127×127	9.778691e-05	1.9906	5.276996e-02	0.9975
255×255	2.456481e-05	1.9930	2.647886e-02	0.9949
511×511	6.133504e-06	2.0018	1.323951e-02	1.0000

Table 3 Error behavior on random-distorted mesh

#unknowns	$\ u - u_h\ _0$	#rate	$ u - u_h _{1,h}$	#rate
15×15	5.800791e-03		4.147051e-01	
31×31	1.506877e-03	1.9447	2.093579e-01	0.9861
63×63	3.885956e-04	1.9552	1.053572e-01	0.9907
127×127	9.778691e-05	1.9906	5.276996e-02	0.9975
255×255	2.456481e-05	1.9930	2.647886e-02	0.9949
511×511	6.133504e-06	2.0018	1.323951e-02	1.0000

Example II. In this example we use the **Algorithm I.** proposed in section 4 to solve the discrete problem in **Example I.**

Here we discuss the distorted mesh depicted in Figure 7., the subdivision of this mesh by the bi-section technique generates a set of nested partitions. Using the Gauss-seidel and CG iterations as the smoothing operator, we list the energy error between the cascadic multigrid solution u_L^* and the exact solution u on the last level L in Table 4-5 respectively. We can see

from the tables that for both of the smoothers, if the mesh is refined once, the energy error is decreasing by half independent of the coarse mesh. It means that the convergence rate of Algorithm I is one and independent of the refinement level for energy error.

Table 4 Gauss-seidel smoother
 $m_L = 2L^2, \beta = 4.0$

# unknowns	# L	$ u_L^* - u _{1,h_L}$
511 × 511	2	1.238004e-02
	3	1.235990e-02
	4	1.236131e-02
	5	1.235362e-02
	6	1.234954e-02
1023 × 1023	3	6.176383e-03
	4	6.180438e-03
	5	6.177397e-03
	6	6.175250e-03
	7	6.173981e-03
2047 × 2047	4	3.088991e-03
	5	3.088916e-03
	6	3.087799e-03
	7	3.087129e-03
	8	3.086699e-03

Table 5 CG smoother
 $m_L = 10, \beta = 3.0$

# unknowns	# L	$ u_L^* - u _{1,h_L}$
511 × 511	2	1.234826e-02
	3	1.236117e-02
	4	1.236476e-02
	5	1.236476e-02
	6	1.236476e-02
	7	1.236476e-02
1023 × 1023	3	6.176750e-03
	4	6.181118e-03
	5	6.181107e-03
	6	6.181096e-03
	7	6.181107e-03
2047 × 2047	4	3.089028e-03
	5	3.089067e-03
	6	3.089067e-03
	7	3.089067e-03
	8	3.089067e-03

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