Journal of Computational Mathematics, Vol.24, No.2, 2006, 209–224.

A PARALLEL NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR STOKES PROBLEMS *

Mei-qun Jiang
(*Department of Mathematics, Suzhou university, Suzhou 215006, China*)

Pei-liang Dai
(*Department of Mathematics, Changshu Institute of Technology, Changshu 215500, China*)

Abstract

A nonoverlapping domain decomposition iterative procedure is developed and analyzed for generalized Stokes problems and their finite element approximate problems in $\mathbf{R}^N$(N=2,3). The method is based on a mixed-type consistency condition with two parameters as a transmission condition together with a derivative-free transmission data updating technique on the artificial interfaces. The method can be applied to a general multi-subdomain decomposition and implemented on parallel machines with local simple communications naturally.

Mathematics subject classification: 65F10, 65N30.
Key words: Generalized Stokes problem, Nonoverlapping, Domain decomposition, Mixed finite element, Crouzeix-Raviart element.

1. Introduction

It is known that large scale simulation of viscous incompressible fluid flow requires the solution of the nonlinear time-dependent Navier-Stokes equations. The key and most time-consuming part of this process is the solution of the generalized Stokes problem at each nonlinear iteration. The numerical solution of the generalized Stokes problem plays a fundamental role in the simulation of viscous incompressible fluid flow. Therefore, efficient algorithms for the generalized Stokes problem are indispensable for the numerical solution of the Navier-Stokes equations. However, the systems arising from the generalized Stokes problem are indefinite. This then causes difficulty for solving systems by most preconditioner and iterative methods. On the other hand, it is also difficult to solve these systems directly since they are very large usually. Therefore, a nonoverlapping domain decomposition iterative procedure seems to be attractive for such ill-conditioned problems because it combines iterative methods on the artificial interfaces method on the small subdomains.

The motivation of this work is to develop and analyze a nonoverlapping domain decomposition iterative procedure for solving the generalized Stokes problem and its finite element approximate problems. The nonoverlapping domain decomposition iterative procedure is based on a mixed-type consistency condition with two parameters as a transmission condition together with a derivative-free transmission date updating technique on the artificial interfaces. The method can be applied in a general multi-subdomain decomposition and implemented on parallel machines with local multi-subdomain decomposition and implemented on parallel machines with local simple communications naturally. Firstly ,we consider the nonoverlapping domain decomposition method for the differential problems of generalized Stokes problem. In particular, its convergence is demonstrated by a "pseudo energy" technique. Then, we apply the method to the famous Crouzeix-Raviart linear nonconforming finite element problems of

* Received April 30, 2005; final revised August 31, 2005

generalized Stokes problem. A nonoverlapping domain decomposition iterative procedure is developed for solving the Crouzeix-Raviart linear nonconforming finite element problems. Its convergence is proved for a very general domain decomposition and finite element mesh even without the quasi-uniform and regular requirements. The algorithm is directly presented to the finite element problem without introducing any Lagrange multipliers.

Nonoverlapping domain decomposition methods have been studied extensively and become very attractive for their parallelism and flexibility (cf. [4-7,9-11,15-21]) . The basic idea to develop our method is originally from [5] as well as [6]. A mixed-type transmission condition with two parameters and its derivative-free updating technique are developed and analyzed for second order elliptic problems. In [6], the idea of [5] is extended and applied into mixed finite element problems for second order elliptic problems and mixed finite element methods of nearly elastic waves in frequency domain as well. The other closely related works are [4,11,16-21], In particular, the word of [4] is very similar to [15] but with only one parameter in the transmission condition. In fact, the method in [4] uses the same transmission data as the famous Lions method of [15] but different updating techniques. Moreover, the method of [4] can be regarded as a variant and improvement of Lions method for the continuous differential problems. In [4,11,16-21], the Lions method of [15] is applied into the generalized Stokes problem and its closely related problems, for instance, the Oseen equations, as well as their finite element approximations. All of the rest apply the Lions method to the various problems, for example, the mixed finite element problem (cf.[10]), by introducing Lagrange multipliers on the artificial interfaces.

2. Generalized Stokes Problem and Its Finite Element Approximations

Let Ω be a domain of $\mathbf{R}^N(N{=}2,3)$ and $\partial\Omega$ its boundary. For the sake of simplicity, this paper is to consider the following generalized Stokes problem over Ω.

$$\begin{cases} -\Delta u+\alpha u+\nabla p=f & \text{in } \Omega, \\ \nabla\cdot u=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u=(u_1(x),\cdots,u_N(x))$ is the velocity vector, $p=p(x)$ is pressure function, the $f\in\mathbf{L}^2(\Omega)\equiv[L^2(\Omega)]^N$ is the field of external forces, and α is a non-negative constant either 0 or $\alpha_0>0$. When $\alpha\equiv 0$ we have the Stokes problem, and the case $\alpha\equiv\alpha_0>0$ usually arises as part of the solution process for the Navier-Stokes equations or non-stationary Stokes equations by implicit difference discrete for time (cf.[12],[19]).

The most commonly used Galerkin-type weak formula for the generalize Stokes problem (2.1) is: Find $(u,p)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$\begin{cases} a(u,v)_\Omega+b(v,p)_\Omega=(f,v)_\Omega \quad \forall v\in\mathbf{H}_0^1(\Omega), \\ b(u,q)_\Omega=0 \quad \forall q\in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where $\mathbf{H}_0^1(\Omega)=[H_0^1(\Omega)]^N$, $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ whose integrable function with zero mean value, $(\cdot\ ,\ \cdot)_\Omega$ inner product over $L^2(\Omega)$ or $[L^2(\Omega)]^N$

$$a(u,v)_\Omega=(\nabla u,\nabla v)_\Omega+(\alpha u,v)_\Omega, \tag{2.3}$$

$$b(v,q)_\Omega=(q,\nabla\cdot v)_\Omega. \tag{2.4}$$

It is well-known that the generalized Stokes problem (2.1)-(2.2) has a unique solution $(u,p)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$. Moreover, there holds the regularity $(u,p)\in(\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega))\times(L_0^2(\Omega)\cap H^1(\Omega))$

for a suitably smooth domain Ω and the regularity $(u,p) \in (\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^{1.5}(\Omega))\times(L_0^2(\Omega)\cap H^{0.5}(\Omega))$ for a very general domain Ω (cf.[1],[14]) .

To describe mixed finite element approximations for (2.2), we begin with the triangulation of Ω. let $\mathcal{T}_h$ be a general finite element triangulation of Ω and, for simplicity, $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}_h} \overline{\tau}$. At this moment, $\mathcal{T}_h$ is not assumed it is either quasi-uniform or regular. For the sake of simplicity, we consider the famous nonconforming linear finite element, Crouzeix-Raviart element (cf. [3]) for the velocity field and piecewise constant finite element for the pressure field on the n-simplex (triangle, if n = 2, tetrahedron, if n = 3) triangulation $\mathcal{T}_h$. However, it is not difficult to see that the analysis and conclusion of this paper can be easily extended to other nonconforming finite elements, for instance, the nonconforming finite elements for n-quadrilateral partition or n-simplex partition (cf. [10]) and the nonconforming finite elements for n-rectangle partition (cf. [13]).

Assume that $V^h \in L^2(\Omega)$ is the Crouzeix-Raviart finite element approximate space of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$, that is,

$$V^h = \{v : v\,|_\tau \in P_1(\tau), \tau \in \mathcal{T}_h, v \ \ vanishes \ \ at \ \ k \in N_h \ \ and \ \ vanishes \ \ at \ \ k \in \Gamma_h\},$$

where N_h is the set of all face barycenters of $\mathcal{T}_h$'s element, i.e. n-simplex, in the interior of Ω and Γ_h as the set of all face barycenters of $\mathcal{T}_h$'s element on the boundary of $\partial\Omega$. Let $\mathbf{V}^h = [V^h]^N \subset \mathbf{L}^2(\Omega)$ as the finite dimensional approximate space of $\mathbf{H}_0^1(\Omega)$. Also, let $M^h \subset L_0^2(\Omega)$ be the piecewise constant finite element approximate space of $L_0^2(\Omega)$ associated with the triangle partition $\mathcal{T}_h$. The mixed finite element approximate problem for the problem (2.2) is: Find $(u,p) \in \mathbf{V}^h \times M^h$ such that

$$\begin{cases} a^h(u,v)_\Omega + b^h(v,p)_\Omega = (f,v)_\Omega \quad \forall v \in \mathbf{V}^h, \\ b^h(u,q)_\Omega = 0 \quad \forall q \in M^h, \end{cases} \tag{2.5}$$

where

$$a^h(u,v)_\Omega = \sum_{\tau\in\mathcal{T}_h} [(\nabla u, \nabla v)_\tau + (\alpha u, v)_\tau], \tag{2.6}$$

$$b^h(v,q)_\Omega = \sum_{\tau\in\mathcal{T}_h} (q, \nabla\cdot v)_\tau, \tag{2.7}$$

It is well-known that such $\mathbf{V}^h$ and M^h satisfy the Babuska-Brezzi condition:

$$\sup_{v\in V^h} \frac{|b^h(v,q)|}{||v||_h} \geq \gamma_0 ||q||_{L^2(\Omega)} \ \ \forall q \in M^h, \tag{2.8}$$

where

$$||v||_h = [(\nabla v, \nabla v)_\Omega + (v,v)_\Omega]^{1/2}, \ v \in \mathbf{V}^h, \tag{2.9}$$

and γ_0 is a positive number independent of the mesh size h. The Babuska-Brezzi condition (2.8) not only guarantees the existence and stability of the mixed finite element problem (2.5) but also makes the finite element approximate solution $(u^h\ ,p^h) \in \mathbf{V}^h \times M^h$ with the optimal error estimate if $\mathcal{T}_h$ is quasi-uniform, that is,

$$\begin{aligned} &||u^h - u||_{L^2(\Omega)} + h(||u^h - u||_h + ||p^h - p||_{L^2(\Omega)}) \\ &\leq Ch^2(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}) \end{aligned} \tag{2.10}$$

where $(u,p) \in (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times (L_0^2(\Omega) \cap H^1(\Omega))$ is the solution of the generalized Stokes problem (2.2) (cf. [1], [3], [12]).

3. Domain Decompositions and Consistency Conditions

To develop a nonoverlapping domain decomposition methods to solve the generalize Stokes problem (2.1) or (2.2) and its nonconforming finite element problem (2.5), we begin with decomposing Ω into an arbitrary $m(\geq 2)$ of disjoint subdomain (open sets) $\Omega_1, \Omega_2, \cdots, \Omega_m$ that is,

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \cdots \cup \overline{\Omega}_m = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m \cup \Sigma, \tag{3.1}$$

$$\Sigma = \bigcup_{1 \leq i \neq j \leq m} \Gamma_{ij}, \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j,\ \Gamma_i = \partial\Omega_i \cap \partial\Omega. \tag{3.2}$$

Moreover, while considering the finite element problem (2.5), we also need assume the above nonoverlapping domain decomposition is aligned with $\mathcal{T}_h$, that means, every Ω_i is a block (union) of some elements or even an individual element of $\mathcal{T}_h$.

Let us now decompose of the generalize Stokes problem (2.1) over $\{\Omega_i\}$ of the domain decomposition (3.1) -(3.2). In addition to requiring $(u_i\ ,p_i)$, $i = 1, 2, 3, \cdots, m$ to satisfy

$$\begin{cases} -\Delta u_i + \alpha u_i + \nabla p_i = f & \text{in} \quad \Omega_i, \\ \nabla \cdot u_i = 0 & \text{in} \quad \Omega_i, \\ u_i = 0 & \text{on} \quad \Gamma_{ij}, \end{cases} \tag{3.3}$$

It is necessary to impose the following consistency conditions (cf. [11], [16], [17], [18], [20], [21])

$$u_i = u_j, \quad \text{on} \quad \Gamma_{ij} \tag{3.4}$$

$$\frac{\partial u_i}{\partial \nu_i} - p_i\nu_i = -\frac{\partial u_j}{\partial \nu_j} + p_j\nu_j, \quad \text{on} \quad \Gamma_{ij} \tag{3.5}$$

where ν_i is the unit outward normal from Ω_i. It is much more convenient and efficient to replace the consistency conditions (3.4) - (3.5) by the following mixed-type consistency conditions (cf. [4], [5], [6]).

$$\beta_{ij}(\frac{\partial u_i}{\partial \nu_i} - p_i\nu_i) + \lambda_{ij}u_i = \beta_{ij}(\frac{\partial u_j}{\partial \nu_j} - p_j\nu_j) + \lambda_{ij}u_j, \quad \text{on} \quad \Gamma_{ij} \tag{3.6}$$

$$\beta_{ji}(\frac{\partial u_j}{\partial \nu_j} - p_j\nu_j) + \lambda_{ji}u_j = -\beta_{ji}(\frac{\partial u_i}{\partial \nu_i} - p_i\nu_i) + \lambda_{ij}u_i, \quad \text{on} \quad \Gamma_{ij} \tag{3.7}$$

where β_{ij}, called penalty coefficient, λ_{ij}, called transmission coefficient, satisfy

$$\beta_{ij} = \beta_{ji}, \quad 1 \leq i \neq j \leq m, \tag{3.8}$$

$$\lambda_{ij} = \lambda_{ji}, \quad 1 \leq i \neq j \leq m. \tag{3.9}$$

It will be clear later why we call β_{ij} penalty coefficient and λ_{ij} transmission coefficient.

By using the idea of [5] or [6] and the consistency condition (3.6)-(3.9), we can define a nonoverlapping domain decomposition iterative procedure for the the generalized Stokes problem (2.1) over the domain decomposition (3.1) - (3.2) as follows:

$$\begin{cases} -\Delta u_i + \alpha u_i + \nabla p_i = f & \text{in} \quad \Omega_i, \\ \nabla \cdot u_i = 0 & \text{in} \quad \Omega_i, \\ u_i = 0 & \text{on} \quad \Gamma_i, \end{cases} \tag{3.10}$$

$$\beta_{ij}(\frac{\partial u_i}{\partial \nu_i} - p_i\nu_i) + \lambda_{ij}u_i = g_{ij}^n, \quad \text{on} \quad \Gamma_{ij} \tag{3.11}$$

$$g_{ij}^{n+1} = 2\lambda_{ij} u_j^n - g_{ji}^n, \quad 1 \le i \ne j \le m, \quad \text{if} \quad \text{means}(\Gamma_{ij}) > 0, \tag{3.12}$$

where g_{ij}^0 are given. In fact, we understand precisely the iterative procedure (3.10) - (3.12) in the sense of the weak solution.

We now discuss how the iterative procedure (3.10) - (3.12) implements the consistency condition (3.6) - (3.7). It follows from (3.8) - (3.9) and (3.11) - (3.12) that we can derive that

$$\begin{aligned}&\beta_{ij}(\frac{\partial u_i^{n+1}}{\partial \nu_i} - p_i^{n+1}\nu_i) + \lambda_{ij} u_i^{n+1} = g_{ij}^{n+1} = 2\lambda_{ij} u_j^n - g_{ji}^n \\ &= 2\lambda_{ij} u_j^n - [\beta_{ji}(\frac{\partial u_j^n}{\partial \nu_j} - p_j^n \nu_j) + \lambda_{ji} u_j^n] \\ &= -\beta_{ij}(\frac{\partial u_j^n}{\partial \nu_j} - p_j^n \nu_j) + \lambda_{ji} u_j^n, \quad \text{on} \quad \Gamma_{ij.}\end{aligned} \tag{3.13}$$

Clearly, (3.13) implies that the iterative procedure (3.10) - (3.12) implements the consistency condition (3.6) - (3.7) through the iterative process. Hence, if the sequence $\{u_i^n\}$ converges to u_i, then u_i $(i = 1, 2, \cdots, m)$will satisfy the consistency condition (3.6) - (3.7).

Finally, we conclude this section by introducing the following special notations G_r $(r = 1, 2, \cdots)$, which will be used later.

$$\begin{cases} G_1 = \{\cup\Omega_k \,|\partial\Omega_k \cap \partial\Omega \; \textit{has positive measure}\} \\ G_{r+1} = \left\{\cup\Omega_k \,\middle|\partial\Omega_k \cap \overline{G_r} \; \textit{has positive measure}, \partial\Omega_k \cap \Gamma_l = \emptyset, \forall l \le r \right\} \end{cases} \tag{3.14}$$

4. A nonoverlapping Domain Decomposition Iterative Procedure for the Differential Problem

This section is to develop and analyze a nonoverlapping domain decomposition method for the generalized Stokes problem (2.2) based on the mixed-type consistency condition (3.6)-(3.7) and the domain decomposition (3.1)-(3.2). In particular, its convergence is demonstrated by a “pseudo energy” technique, which also is the motivation for proving the convergence in the case for the finite element problem next section.

We define the nonoverlapping domain decomposition iterative procedure based on the analysis in last section. In fact , we just need rewrite and (3.10)-(3.12) as in the sense of weak solution of generalized Stokes problems. The nonoverlapping domain decomposition iterative procedure for the generalized Stokes problem (2.2) is defined as follows.

Algorithm I

(i) Given $g_{ij}^0 \in \mathbf{L}^2(\Gamma_{ij}) = [L^2(\Gamma_{ij})]^2$, $1 \le i \ne j \le m$, $meas(\Gamma_{ij}) > 0$, arbitrarily;

(ii) Then recursively find $(u_i^n\, p_i^n) \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \times L^2(\Omega_i)$, $i = 1, 2, \cdots, m$, by solving the subproblems

$$a(u_i^n \;, v)_{\Omega_i} + b(v \;, p_i^n)_{\Omega_i} + \sum_{1\le i\ne j\le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} u_i^n v ds = (f \;, v)_{\Omega_i} \tag{4.1}$$

$$+ \sum_{1\le i\ne j\le m} \tfrac{1}{\beta_{ij}} \textstyle\int_{\Gamma_{ij}} g_{ij}^n v ds, \; \forall v \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$$

$$b(u_i^n, q)_{\Omega_i} = 0, \forall q \in L_0^2(\Omega_i). \tag{4.2}$$

(iii) Update the transmission condition data for $i = 1, 2, \cdots, m$

$$g_{ij}^{n+1} = 2\lambda_{ij} u_j^n - g_{ji}^n, \quad \text{in} \quad L^2(\Gamma_{ij}), \quad 1 \le i \ne j \le m, \quad \text{if} \quad \text{meas}(\Gamma_{ij}) > 0 \tag{4.3}$$

where β_{ij} and λ_{ij} satisfy (3.8)-(3.9), and $\mathbf{H}^1_{\Gamma_i}(\Omega_i)$ is the restriction of $\mathbf{H}^1_0(\Omega)$ over the subdomain Ω_i.

It is not hard to know that the subproblems (4.1)-(4.2) are always well-posed in the whole iterative process. Furthermore, it is easy to see that β_{ij} looks like a penalty put on the artificial interface Γ_{ij} in (4.1) and only λ_{ij} takes part in the update of transmission data but β_{ij} is not in the formula for updating the transmission data explicitly. That is why, like [5] and [6], β_{ij} is called "*penalty coefficient*" and λ_{ij} is called "*transmission coefficient*", respectively. It will be clear that β_{ij} and λ_{ij} play the key role in the convergence rate estimates.

We now discuss the convergence of *Algorithm I.* We first define the error function at the iterative step n:

$$\varepsilon_i^n = u_i^n - u \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \quad i = 1, 2, \cdots, m, \tag{4.4}$$

$$\rho_i^n = p_i^n - p \in L^2(\Omega_i), \quad i = 1, 2, \cdots, m, \tag{4.5}$$

where $(u, p) \in \mathbf{H}^1_0(\Omega) \times L^2(\Omega)$ is a solution of the generalized Stokes problem (2.2). Then we can have that

$$a(\varepsilon_i^n, v)_{\Omega_i} + b(v, \rho_i^n)_{\Omega_i} + \sum_{1\le i\ne j\le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} \varepsilon_i^n v ds = \tag{4.6}$$

$$\sum_{1\le i\ne j\le m} \tfrac{1}{\beta_{ij}} \int_{\Gamma_{ij}} g_{ij}^n v ds, \ \forall v \in \mathbf{H}^1_{\Gamma_i}(\Omega_i),$$

$$b(\varepsilon_i^n, q)_{\Omega_i} = 0, \quad \forall q \in L^2_0(\Omega_i), \tag{4.7}$$

$$g_{ij}^{n+1} = 2\lambda_{ij}\varepsilon_j^n - g_{ji}^n, \quad \text{in} \quad L^2(\Gamma_{ij}), \quad 1 \le i \ne j \le m, \quad \text{if} \quad \text{meas}(\Gamma_{ij}) > 0, \tag{4.8}$$

where we have used g_{ij}^n to replace $g_{ij}^n - [\beta_{ij}(\frac{\partial u}{\partial \nu_i} - p\nu_i) + \lambda_{ij}u]$. Furthermore, we have the following lemma, which play the key roles in the convergence analysis.

Lemma 4.1. *There holds the following identity*

$$|||g^{n+1}|||^2 = |||g^n|||^2 - 4\sum_{i=1}^m a(\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i}, \tag{4.9}$$

where $g^k = (g_{ij}^k)$, $1 \le i \ne j \le m$ *defined as in (4.6)-(4.8), and*

$$|||g^k|||^2 = \sum_{1\le i\ne j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}} |g_{ij}^k|^2 ds. \tag{4.10}$$

Proof. First, from (4.6)-(4.7), we can have that by taking $v = \varepsilon_i^n$ in (4.6)

$$a(\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i} = \sum_{j\ne i} \frac{1}{\beta_{ij}} \int_{\Gamma_{ij}} (g_{ij}^n - \lambda_{ij}\varepsilon_i^n)\varepsilon_i^n ds. \tag{4.11}$$

It then follows from (4.8), (4.10) and (4.11) that

$$\begin{aligned} |||g^{n+1}|||^2 &= \sum_{1\le i\ne j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}} |g_{ij}^{n+1}|^2 ds \\ &= \sum_{1\le i\ne j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}} |2\lambda_{ij}\varepsilon_j^n - g_{ji}^n|^2 ds \\ &= \sum_{1\le i\ne j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}} |g_{ji}^n|^2 ds - 4\sum_{i=1}^m \sum_{j\ne i} \frac{1}{\beta_{ij}} \int_{\Gamma_{ij}} (g_{ij}^n - \lambda_{ij}\varepsilon_i^n)\varepsilon_i^n ds \\ &= |||g^n|||^2 - 4\sum_{i=1}^m a(\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i}, \end{aligned} \tag{4.12}$$

where (3.8) and (3.9) have been used in (4.12). Then, the proof completes.

Theorem 4.2. *Let* $(u,p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ *be a weak solution of the generalized Stokes problem (2.4)-(2.5) with reasonable regularity*$(u,p) \in [H^{1.5}(\Omega)]^N \times H^{0.5}(\Omega)$*. Let* $(u_i^n, p_i) \in \mathbf{H}_{\Gamma_i}^1(\Omega_i) \times L^2(\Omega_i)(i = 1, 2, \cdots, m)$ *be the weak solution of the subproblem (4.1)-(4.2) of Algorithm I at iterative step n. Then we have that ,for any initial* $g_{ij}^0 \in \mathbf{L}^2(\Gamma_{ij})$

$$[\sum_{i=1}^{m}(||u_i^n - u||_{H^1(\Omega_i)}^2 + ||p_i^n - p||_{L^2(\Omega_i)}^2)]^{1/2} \to 0, \quad \text{as} \quad n \to \infty \tag{4.13}$$

Proof. Clearly, it suffices to show that, for any $i = 1, 2, \cdots, m$,

$$[\sum_{i=1}^{m}(||u_i^n - u||_{H^1(\Omega_i)}^2 + ||p_i^n - p||_{L^2(\Omega_i)}^2)] \to 0, \quad \text{as} \quad n \to \infty \tag{4.14}$$

From (4.9) and (4.10) of Lemma 4.1, we have that the sequence $\left\{g_{ij}^n\right\}$ is bounded in $\mathbf{L}^2(\Gamma_{ij})(1 \leq i \neq j \leq m$, $meas(\Gamma_{ij}) > 0)$. Hence, there exists a $g_{ij} \in \mathbf{L}^2(\Gamma_{ij})$ and a subsequence, still denoted as $\left\{g_{ij}^n\right\}$, such that, for any meas$(\Gamma_{ij}) > 0, \ 1 \leq i \neq j \leq m$

$$g_{ij}^n \to g_{ij}, \quad \text{weakly in} \quad \mathbf{L}^2(\Gamma_{ij}), \quad \text{as} \quad n \to \infty. \tag{4.15}$$

Also, from (4.9) of Lemma 4.1, we can have that, for any position integer M,

$$\sum_{n=0}^{M}\sum_{i=1}^{m} a(\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i} = \frac{1}{4}\left(\left|\left\|g^0\right\|\right|^2 - \left|\left\|g^{M+1}\right\|\right|^2\right) \geq 0,$$

which implies

$$\sum_{i=1}^{m} a(\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.16}$$

Thus, for any $i = 1, 2, \cdots, m$,

$$[(\nabla\varepsilon_i^n, \nabla\varepsilon_i^n)_{\Omega_i} + (\alpha\varepsilon_i^n, \varepsilon_i^n)_{\Omega_i}] \to 0, \quad \text{as} \quad n \to \infty. \tag{4.17}$$

Therefore, if $\alpha = \alpha_0 > 0$, then (4.17) directly implies that

$$\|\varepsilon_i^n\|_{\mathbf{H}^1(\Omega_i)} \to 0, \quad \text{as} \quad n \to \infty, \quad i = 1, 2, \cdots, m. \tag{4.18}$$

Thus, by Sobolev embedding theorem,

$$\|\varepsilon_i^n\|_{\mathbf{L}^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad 1 \leq i \neq j \leq m, \quad \text{meas}(\Gamma_{ij}) > 0. \tag{4.19}$$

Moreover, for any $1 \leq i \neq j \leq m, \ \ meas(\Gamma_{ij}) > 0$, and for any $\eta \in [C_0^\infty(\Gamma_{ij})]^N$, we define $(v,q) \in \mathbf{H}_{\Gamma_i}^1(\Omega_i) \times L_0^2(\Omega_i)$ satisfying

$$\begin{cases} -\Delta v + \nabla q = 0, & \text{in} \quad \Omega_i \\ \nabla \cdot v = 0, & \text{in} \quad \Omega_i \end{cases} \tag{4.20}$$

$$v = \begin{cases} \eta, & \text{on} \quad \Gamma_{ij} \\ 0, & \text{elsewhere on} \quad \partial\Omega_i. \end{cases} \tag{4.21}$$

Clearly, there exists such a $(v,q) \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \times L^2(\Omega_i)$ (cf. [1], [12]). Hence, Plugging this $v \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ into (4.6), we can derive from (4.18)-(4.21) that

$$\int_{\Gamma_{ij}} g^n_{ij} \eta ds \to 0, \quad \text{as} \quad n \to \infty, \forall \eta \in \mathbf{L}^2(\Gamma_{ij}). \tag{4.22}$$

That means the subsequence $\{g^n_{ij}\}$ is weakly convergent to 0. Therefore, by a standard argument of functional analysis, we immediately obtain that, for the whole sequence $\{g^n_{ij}\}$,

$$\|g^n_{ij}\|_{\mathbf{L}(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad 1 \le i \ne j \le m, \quad \text{meas}(\Gamma_{ij}) > 0. \tag{4.23}$$

Next we try to show that (4.18) and (4.23) still hold for $\alpha = 0$. First we consider the case of $\Omega_i \subset G_1$. It follows from (3.14) that

$$\varepsilon^n_i = 0, \quad \text{on} \quad \Gamma_i, \quad \text{meas}(\Gamma_i) > 0. \tag{4.24}$$

Hence, by the Poincaré inequality and (4.17),

$$\|\varepsilon^n_i\|^2_{\mathbf{H}^1(\Omega_i)} \le C(\nabla \varepsilon_i, \varepsilon_i)_{\Omega_i} \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_i \subset G_1. \tag{4.25}$$

Therefore, repeating the same arguments of (4.19)-(4.23), we have that, for any $\Omega_i \subset G_1$,

$$\|\varepsilon^n_i\|_{\mathbf{L}^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad (1 \le i \ne j \le m). \tag{4.26}$$

$$\|g^n_{ij}\|_{\mathbf{L}^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad (1 \le i \ne j \le m). \tag{4.27}$$

Then, we consider the case of $\Omega_i \subset G_2$. It follows from (4.8) and (4.26)-(4.27) that, for any $\Omega_i \subset G_2$,

$$\|g^n_{ij}\|_{\mathbf{L}^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_j \subset G_1. \tag{4.28}$$

For any $\Omega_i \subset G_2$, replacing (4.21) with

$$v = \begin{cases} \varepsilon^n_i, & \text{on} \quad \Gamma_{ij}, \quad \Omega_j \subset G_1, \\ 0, & \text{elsewhere} \quad \text{on} \quad \partial\Omega_i. \end{cases} \tag{4.29}$$

we then have $v \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ satisfying (4.20) and (4.29). Thus, plugging this $v \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ into (4.6) and using (4.17) and (4.28), we have that, for any $\Omega_i \subset G_2$

$$\|\varepsilon^n_i\|_{\mathbf{L}^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_j \subset G_1. \tag{4.30}$$

From (3.14), for any $\Omega_i \subset G_2$, there exists at least a $\Omega_j \subset G_1$ such that $meas(\Gamma_i) > 0$. Hence, it follows from (4.17), (4.30) and the Poincaré inequality that

$$\|\varepsilon^n_i\|^2_{\mathbf{H}^1(\Omega_i)} \le C[(\nabla \varepsilon_i, \nabla \varepsilon_i)_{\Omega_i} + \sum_{\Omega_i \subset G_1} \|\varepsilon^n_i\|^2_{\mathbf{L}^2(\Gamma_{ij})}] \to 0, \tag{4.31}$$

$$\text{as} \quad n \to \infty, \quad \forall \Omega_i \subset G_2.$$

Therefore, repeating the above arguments until the subdomains are exhausted, we have proved that (4.18) and (4.23) still hold for $\alpha = 0$.

Finally, we consider the convergence for the pressure. It the follows from the **LBB** condition and (4.6) that

$$\|\rho^n_i\|_{L^2(\Omega_i)} \le C \sup_{v \in \mathbf{H}^1_{\Gamma_i}} \frac{|b(v, \rho^n_i)|}{\|v\|_{\mathbf{H}^1(\Omega_i)}} \le C[\|\varepsilon^n_i\|_{\mathbf{H}^1(\Omega_i)} + \sum_{i \ne j} (\|\varepsilon^n_i\|_{\mathbf{L}^2(\Gamma_{ij})} + \|g^n_{ij}\|_{\mathbf{L}^2(\Gamma_{ij})})].$$

Moreover, by using (4.18), (4.24) and the embedding theorem, we get

$$\|\rho_i^n\|_{L^2(\Omega_i)} \to 0, \quad \text{as} \quad n \to \infty, \quad i = 1, 2, \cdots, m. \tag{4.32}$$

Thus, (4.14) can be obtained from (4.18), (4.23) and (4.32) easily. This completes the proof.

Corollary 4.3. *Under the assumptions of Theorem 4.2, we have that for any* $1 \le i \ne j \le m$, $meas(\Gamma_{ij}) > 0$

$$\left\| g_{ij}^n - [\beta_{ij}(\frac{\partial u}{\partial \nu_i} - p\nu_i) + \lambda_{ij} u] \right\|_{L^2(\Gamma_{ij})} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.33}$$

5. Finite Element Case

This section is to develop and analyze a nonoverlapping domain decomposition method for the nonconforming finite element problem (2.5). We define our algorithm as follows:

Algorithm II

(i) Given $g_{ij}^0 \in \mathbf{V}^h(\Gamma_{ij}) = [V^h(\Gamma_{ij})]^N$, $1 \le i \ne j \le m$, $meas(\Gamma_{ij}) > 0$, arbitrarily;

(ii) Then recursively find $(u_i^n, p_i^n) \in \mathbf{V}_i^h \times M_i^h$, $i = 1, 2, \cdots, m$, by solving the subproblems

$$a^h(u_i^n, v)_{\Omega_i} + b^h(v, p_i^n)_{\Omega_i} + \sum_{1 \le i \ne j \le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}}^* u_i^n v ds = (f, v)_{\Omega_i} \tag{5.1}$$

$$+ \sum_{1 \le i \ne j \le m} \frac{1}{\beta_{ij}} \int_{\Gamma_{ij}}^* g_{ij}^n v ds, \ \forall v \in \mathbf{V}_i^h,$$

$$b^h(u_i^n, q)_{\Omega_i} = 0, \quad \forall q \in M_i^h. \tag{5.2}$$

(iii) Update the transmission condition data for $i = 1, 2, \cdots, m$

$$g_{ij}^{n+1}(k) = 2\lambda_{ij} u_j^n(k) - g_{ji}^n(k), \quad k \in N_h \cap \Gamma_{ij}, \quad 1 \le j \ne i \le m, \tag{5.3}$$

where $V^h(\Gamma_{ij})$ is a piecewise constant finite dimension space over the partition of Γ_{ij} induced by the finite element triangulation $\mathcal{T}_h$; $\mathbf{V}_i^h = \mathbf{V}^h|_{\Omega_i}$, the restriction of $\mathbf{V}^h$ over Ω_i; $M_i^h = M^h|_{\Omega_i}$, the restriction of M^h over Ω_i; β_{ij} the penalty coefficient, and λ_{ij}, the transmission coefficient, satisfy (3.8) - (3.9); and

$$\int_{\Gamma_{ij}}^* uv ds = \sum_{k \in \Gamma_{ij} \cap N_h} u(k) v(k) meas(s_k), \tag{5.4}$$

where, and in this paper, $\{\varphi_p\}_{p \in N_h}$ is the node basis of the finite element space $\mathbf{V}^h$ and s_k is the element face with k as its barycenter. Clearly, (5.4) implies the numerical integration has been applied to compute the integration on the interfaces in (5.1).

Clearly, every subproblem (5.1) of *Algorithm II* is well-defined in the whole iterative process. We then consider the convergence of *Algorithm II.* Unlike *Algorithm I* for partial differential problem of the generalized Stokes problem, we have to show an equivalent splitting subproblem form with respect to the nonoverlapping domain decomposition for finite element problem (2.5) before proving the convergence of *Algorithm II.*

Theorem 5.1. *Let* $(u, p) \in \mathbf{V}^h \times M^h$ *be a solution of the finite element problem (2.5). Then, the problem (2.5) can be split into an equivalent subproblem form, that is, there exist* $g_{ij}^* \in \mathbf{V}^h(\Gamma_{ij})$, $1 \le i \ne j \le m$, $meas(\Gamma_{ij}) > 0$, *such that* $(u_i, p_i) \in \mathbf{V}_i^h \times M_i^h (i = 1, 2, \cdots, m)$, *satisfies*

$$a^h(u_i, v)_{\Omega_i} + b^h(v, p_i)_{\Omega_i} + \sum_{1 \le i \ne j \le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} u_i v ds = \tag{5.5}$$

$$\sum_{1\le i\ne j\le m} \tfrac{1}{\beta_{ij}} \int_{\Gamma_{ij}} g_{ij}^* v ds,\ \forall v \in \mathbf{V}_i^h,$$

$$b^h(u_i, q)_{\Omega_i} = 0, \quad \forall q \in M_i^h. \tag{5.6}$$

where $(u_i, p_i) = (u, p)|_{\Omega_i} = (u|_{\Omega_i}, p|_{\Omega_i})$, *the restriction on* Ω_i.

Proof. First, notice that M^h is a piecewise constant space, then it is easy to check that (2.5) can be rewritten as the splitting form (5.6),i.e.,

$$b^h(u_i, q)_{\Omega_i} = 0, \quad \forall q \in M_i^h, \quad i = 1, 2, \cdots, m. \tag{5.7}$$

Then we consider to split (2.5). Notice that $\{\varphi_p\}_{p\in N_h}$ is the nodal basis of the finite element space $\mathbf{V}^h$, it is then not difficult to check that $\{\varphi_p^l\}_{(p\in N_h, l=1,2,\cdots N)}$ consist of a basis of the finite dimension space $\mathbf{V}^h$, where φ_p^l is a N-vector function with φ_p as its *l-th* component and 0 as all of its other components. Hence, (2.5) can be rewritten as the following equivalent system.

$$a^h(u, \varphi_k)_{\Omega_i} + b^h(\varphi_k, p)_{\Omega_i} = (f, \varphi_k)_{\Omega_i}, \quad \forall k \in N_h, \quad i = 1, 2, \cdots, N. \tag{5.8}$$

Thus, it follows from the small support property of $\varphi_p(x)$, we have that, for any $k \in N_h \cap \Gamma_{ij}$,

$$a^h(u_i, \varphi_k)_{\Omega_i} + b^h(\varphi_k, p_i)_{\Omega_i} - (f, \varphi_k)_{\Omega_i} = -[a^h(u_j, \varphi_k)_{\Omega_j} + b^h(\varphi_k, p_j)_{\Omega_j} - (f, \varphi_k)_{\Omega_j}]. \tag{5.9}$$

Hence, for any $k \in N_h \cap \Gamma_{ij}$, we can define G_{ij}^k as follows

$$G_{ij}^k = \frac{1}{meas(s_k)}[a^h(u_j, \varphi_k)_{\Omega_i} + b^h(\varphi_k, p_j)_{\Omega_j} - (f, \varphi_k)_{\Omega_j}]. \tag{5.10}$$

Thus, we construct $g_{ij}^* \in \mathbf{V}^h(\Gamma_{ij})$, $meas(\Gamma_{ij}) > 0$, as

$$g_{ij}^* = \sum_{k\in\Gamma_{ij}\cap N_h} (\lambda_{ij} u(p)\nu_i - \beta_{ij} G_{ij}^k \varphi_k(k)). \tag{5.11}$$

Therefore, it follows from (5.8) - (5.11) and some direct calculation that, for any $k \in N_h \cap \overline{\Omega}_i$, $i = 1, 2, \cdots N$, (u_i, p_i) satisfies

$$a^h(u_i, \varphi_k)_{\Omega_i} + b^h(\varphi_k, p_i)_{\Omega_i} + \sum_{1\le i\ne j\le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} u_i \varphi_k ds = (f, \varphi_k)_{\Omega_i} \tag{5.12}$$

$$+ \sum_{1\le i\ne j\le m} \tfrac{1}{\beta_{ij}} \int_{\Gamma_{ij}} g_{ij}^* \varphi_k ds,\ i = 1, 2, \cdots, m.$$

Finally combining (5.7) and (5.12), we have (5.5)-(5.6). The proof is then completed.

We are now ready to consider the convergence of *Algorithm II* for a general domain decomposition and finite element triangulation even without quasi-uniform and regular requirements. We begin with introducing the error function at the iterative step n:

$$e_i^n = u_i^n - u \in \mathbf{V}_i^h, \quad i = 1, 2, \cdots, m. \tag{5.13}$$

$$r_i^n = p_i^n - p \in M_i^h, \quad i = 1, 2, \cdots, m. \tag{5.14}$$

where $(u, p) \in \mathbf{V}^h \times M^h$ is a solution of the finite element problem (2.5). Then we have the error equation as follows:

$$a^h(e_i^n, v)_{\Omega_i} + b^h(v, r_i^n)_{\Omega_i} + \sum_{1\le i\ne j\le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} e_i^n v ds = \tag{5.15}$$

$$\sum_{1\le i\neq j\le m} \tfrac{1}{\beta_{ij}} \int_{\Gamma_{ij}} g_{ij}^n v ds,\ \forall v \in \mathbf{V}_i^h,$$

$$b^h(e_i^n, q)_{\Omega_i} = 0, \quad \forall q \in M_i^h, \tag{5.16}$$

$$g_{ij}^{n+1}(k) = 2\lambda_{ij} e_j^n(k) - g_{ji}^n(k), \quad k \in N_h \cap \Gamma_{ij}, \quad 1 \le i \neq j \le m, \tag{5.17}$$

where we have used g_{ij}^n to replace $g_{ij}^n - g_{ij}^*$ in which g_{ij}^* is defined in Theorem 5.1.

We have the following lemmas, which will be used in the convergence analysis of *Algorithm II* later. First we can obtain the following lemma by taking $v = e_i^n$ in (5.15).

Lemma 5.2. *There holds the following identity*

$$a^h(e_i^n, e_i^n)_{\Omega_i} = \sum_{i\neq j} \frac{1}{\beta_{ij}} \int_{\Gamma_{ij}}^{*} (g_{ij}^n - \lambda_{ij} e_i^n) e_i^n ds. \tag{5.18}$$

Lemma 5.3. *There then holds the following identity*

$$|||g^{n+1}|||_*^2 = |||g^n|||_*^2 - 4\sum_{i=1}^m a^h(e_i^n, e_i^n)_{\Omega_i}. \tag{5.19}$$

where $g^k = (g_{ij}^k)_{1\le i\neq j\le m}$ *defined as in (5.15) - (5.17), and*

$$|||g^k|||_*^2 = \sum_{1\le i\neq j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}}^{*} |g_{ij}^k|^2 ds. \tag{5.20}$$

Proof. It follows from (5.15) - (5.17) and (5.18) of Lemma 5.2 that

$$\begin{aligned} |||g^{n+1}|||_*^2 &= \sum_{1\le i\neq j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}}^{*} |g_{ij}^{n+1}|^2 ds \\ &= \sum_{1\le i\neq j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}}^{*} |2\lambda_{ij} e_j^n - g_{ji}^n|^2 ds \\ &= \sum_{1\le i\neq j\le m} \frac{1}{\beta_{ij}\lambda_{ij}} \int_{\Gamma_{ij}}^{*} |g_{ji}^n|^2 ds - 4\sum_{i=1}^m \sum_{j\neq i} \frac{1}{\beta_{ij}} \int_{\Gamma_{ij}}^{*} (g_{ij}^n - \lambda_{ij} e_i^n) e_i^n ds \\ &= |||g^n|||_*^2 - 4\sum_{i=1}^m a^h(e_i^n, e_i^n)_{\Omega_i}, \end{aligned} \tag{5.21}$$

where (3.8) and (3.9) have been used in (5.21). Hence the proof has been completed.

Theorem 5.4. *Let* $(u, p) \in \mathbf{V}^h \times M^h$ *be a solution of the finite element problem (2.5). Let* $(u_i^n, p_i) \in \mathbf{V}_i^h \times M_i^h$ *(i = 1, 2, ..., m) be the solutions of the subproblem (5.1)-(5.2) of Algorithm II at iterative step n. Then we have that for any initial* $g_{ij}^0 \in \mathbf{L}^2(\Gamma_{ij})$,

$$[\sum_{i=1}^m (||u_i^n - u||_{h,\Omega_i}^2 + ||p_i^n - p||_{L^2(\Omega_i)}^2)]^{1/2} \to 0, \quad \text{as} \quad n \to \infty \tag{5.22}$$

where

$$||v||_{h,\Omega_i} = (\sum_{\tau\in T_h, \tau\subset\Omega_i} (||\nabla v||_{L^2(\tau)}^2 + ||v||_{L^2(\Omega_i)}^2))^{1/2} \tag{5.23}$$

Proof. It suffices to show that, for any $i = 1, 2, \cdots, m$

$$\left[\|u_i^n - u\|_{h,\Omega_i}^2 + \|p_i^n - p\|_{L_0^2(\Omega_i)}^2\right] \to 0, \quad \text{as} \quad n \to \infty. \tag{5.24}$$

From (5.19) and (5.20) of Lemma 5.3, we have that the sequence $\left\{g_{ij}^n\right\}$ is bounded in the finite dimension space $\mathbf{V}^h(\Gamma_{ij})(1 \leq i \neq j \leq m, \text{meas}(\Gamma_{ij}) > 0)$. Hence, there exists a $g_{ij} \in \mathbf{L}^2(\Gamma_{ij})$ and a subsequence, still denoted as $\left\{g_{ij}^n\right\}$, such that, for any $1 \leq i \neq j \leq m,$ meas$(\Gamma_{ij}) > 0$,

$$\int_{\Gamma_{ij}}^{*} \left|g_{ij}^n - g_{ij}^*\right| ds \to 0, \quad \text{as} \quad n \to \infty. \tag{5.25}$$

Also, from Lemma 5.3, we have that, for any position integer M,

$$\sum_{n=0}^{M}\sum_{i=1}^{m} a^h(e_i^n, e_i^n)_{\Omega_i} = \frac{1}{4}\left(\left|\left|\left|g^0\right|\right|\right|^2 - \left|\left|\left|g^{M+1}\right|\right|\right|^2\right) \geq 0,$$

which implies

$$\sum_{i=1}^{m} a^h(e_i^n, e_i^n)_{\Omega_i} \to 0, \quad \text{as} \quad n \to \infty. \tag{5.26}$$

Thus, for any $i = 1, 2, \cdots, m$

$$[(\nabla e_i^n, \nabla e_i^n)_{\Omega_i} + (\alpha e_i^n, e_i^n)_{\Omega_i}] \to 0, \quad \text{as} \quad n \to \infty. \tag{5.27}$$

Therefore, if $\alpha = \alpha_0 > 0$, then (5.27) directly implies that

$$\|e_i^n\|_{h,\Omega_i}^2 \to 0, \quad \text{as} \quad n \to \infty, \quad i = 1, 2 \cdots m. \tag{5.28}$$

Hence,

$$\int_{\Gamma_{ij}}^{*} |e_i^n|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad 1 \leq i \neq j \leq m, \quad \text{meas}(\Gamma_{ij}) > 0. \tag{5.29}$$

Moreover, for any $1 \leq i \neq j \leq m,$ meas$(\Gamma_{ij}) > 0$**,** we define $(v, q) \in \mathbf{V}_i^h \times M_i^h$ satisfying

$$-\Delta v + \nabla q = 0, \quad \text{in}\Omega_i. \tag{5.30}$$

$$\nabla \cdot v = 0, \quad \text{in}\Omega_i. \tag{5.31}$$

$$v = \begin{cases} g_{ij}^n(k), & \text{at } k \in \Gamma_{ij} \cap N_h, \\ 0, & \text{other nodal points on } \partial\Omega_i. \end{cases} \tag{5.32}$$

It is easy to see that, there exists such a $(v, q) \in \mathbf{V}_i^h \times M_i^h$(cf. [1], [12]). Hence, plugging this $v \in \mathbf{V}_i^h$ into (5.15), we can derive from (5.28)-(5.32) that

$$\int_{G_{ij}}^{*} \left|g_{ij}^n\right|^2 ds \to 0, \quad \text{as} \quad n \to \infty,$$

which means the subsequence $\left\{g_{ij}^n\right\}$ is convergent to 0. Thus, by a standard argument of functional analysis, we obtain that, for the whole sequence $\left\{g_{ij}^n\right\}$,

$$\int_{G_{ij}}^{*} \left|g_{ij}^n\right|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad 1 \leq i \neq j \leq m, \quad \text{meas}(\Gamma_{ij}) > 0. \tag{5.33}$$

We now try to show that (5.28) and (5.33) still hold for $\alpha = 0$. First, we consider the case of $\Omega_i \subset G_1$. It follows from (3.14) that

$$e_i^n = 0, \quad \text{on} \quad \Gamma_i, \quad \text{meas}(\Gamma_i) > 0. \tag{5.34}$$

Hence,

$$\|e_i^n\|_{h,\Omega_i}^2 \le C(\nabla e_i^n, \nabla e_i^n)_{\Omega_i} \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_i \subset G_1. \tag{5.35}$$

Therefore, repeating the arguments of (5.29)-(5.33), we have that, for any $\Omega_i \subset G_1$,

$$\int_{\Gamma_{ij}}^{*} |e_i^n|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad (1 \le i \ne j \le m). \tag{5.36}$$

$$\int_{\Gamma_{ij}}^{*} \left|g_{ij}^n\right|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad (1 \le i \ne j \le m). \tag{5.37}$$

Then, we consider the case of $\Omega_i \subset G_2$. It follows from (5.17) and (5.36) - (5.37) that, for any $\Omega_i \subset G_2$,

$$\int_{\Gamma_{ij}}^{*} \left|g_{ij}^n\right|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_j \subset G_1. \tag{5.38}$$

for any $\Omega_i \subset G_2$, replacing (5.32) with

$$v = \begin{cases} e_i^n, & \text{on } \Gamma_{ij}, \quad \Omega_j \subset G_1, \\ 0, & \text{elsewhere on } \partial\Omega_i. \end{cases} \tag{5.39}$$

We then have $v \in \mathbf{V}_i^h$ satisfying (5.30) - (5.31) and (5.39) . Thus, plugging $v \in \mathbf{V}_i^h$ of (5.39) into (5.15) and using (5.27) and (5.38), we have that, for any $\Omega_i \subset G_2$,

$$\int_{\Gamma_{ij}}^{*} |\varepsilon_i^n|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_j \subset G_1. \tag{5.40}$$

Notice that (3.14) , for any $\Omega_i \subset G_2$, there is at least a $\Omega_j \subset G_1$ such that $meas(\Gamma_i) > 0$. Hence, it follows from (5.27) and (5.40) that

$$\|e_i^n\|_{h,\Omega_i}^2 \le C\left[(\nabla e_i^n, \nabla e_i^n)_{\Omega_i} + \sum_{\Omega_j \subset G_1} \int_{\Gamma_{ij}}^{*} |e_i^n|^2 \, ds\right] \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \Omega_j \subset G_2. \tag{5.41}$$

Repeating the above arguments until the subdomains are exhausted, we have proved that (5.28) and (5.33) still hold for $\alpha = 0$.

Finally, we consider the convergence for the pressure. It follows from the Babuska-Brezzi condition (2.8) and (5.15) that

$$\|r_i\|_{L^2(\Omega_i)} = \sup_{v \in \mathrm{V}_i^h} \frac{\left|b^h(v, r_i)\right|}{\|v\|_{h,\Omega_i}} \le C\left[\|e_i^n\|_{h,\Omega_i} + \sum_{j \ne i} \left(\int_{\Gamma_{ij}}^{*} |e_i^n|^2 \, ds + \int_{\Gamma_{ij}}^{*} \left|g_{ij}^n\right|^2 ds\right)^{1/2}\right]$$

Moreover, by using (5.28) , (5.33) and the embedding theorem, we have

$$\|r_i^n\|_{L^2(\Omega_i)}^2 \to 0, \quad \text{as} \quad n \to \infty, \quad i = 1, 2 \cdots m. \tag{5.42}$$

Thus, (5.24) can be obtained from (5.28), (5.33) and (5.42) easily. This completes the proof.

On the other hand, we can rewrite *Algorithm II* as an iterative procedure to find the fixed point of an affined map. Assume that

$$\mathbf{W} = \prod_{i=1}^{n} (\mathbf{V}_i^h \times M_i^h), \quad \Lambda = \prod_{1 \le i \ne j \le m} V^h(\Gamma_{ij}). \tag{5.43}$$

Let $\mathbf{A}_f$**:** $\mathbf{W}\times\Lambda \to W\times\Lambda$ be an affine mapping defined by (5.1) - (5.3) of *Algorithm II.* That means,

$$[(u^{n+1}, p^{n+1}), g^{n+1}] = \mathbf{A}_f[(u^n, p^n), g^n], \tag{5.44}$$

where $[(u^{n+1}, p^{n+1}), g^{n+1}]$ satisfies (5.1) - (5.3) . We then have the following lemma from the definition of $\mathbf{A}_f$ directly.

Lemma 5.5. *Let $(u, p) \in \mathbf{V}^h \times M^h$ be a solution of the problem (2.5) and denote by $(u, p) \equiv (u\,|_{\Omega_i}\,, p\,|_{\Omega_i}) \in W$, then there exists $g \in \Lambda$ such that $[(u, p), g]$ is a fixed point of $\mathbf{A}_f$. Conversely, let $[(u, p), g] \in W\times\Lambda$ is a fixed point of $\mathbf{A}_f$ and $(u, p) \in \mathbf{L}^2(\Omega)\times L^2(\Omega)$ satisfying $(u\,|_{\Omega_i}\,, p\,|_{\Omega_i}) = (u_i, p_i)$. Then, $(u, p) \in \mathbf{V}^h \times M^h$ is a solution of the problem (2.5).*

Furthermore, if let $\mathbf{A}= \mathbf{A}_f\,|_{f=0}\,,\ F = \mathbf{A}_f(0,0)$. Then, $\mathbf{A}$ is a linear mapping, indeed, which is the iterator (iterative matrix) of *Algorithm II,* and satisfies

$$\mathbf{A}_f[(u, p), g] = \mathbf{A}[(u, p), g] + F, \tag{5.45}$$

which means that, we have

$$[(e^{n+1}, r^{n+1}), g^{n+1}] = \mathbf{A}[(e^n, r^n), g^n]. \tag{5.46}$$

Theorem 5.6. *There holds the following estimate*

$$\sigma(\mathbf{A}) < 1 \tag{5.47}$$

where $\sigma(\mathbf{A})$ *is the spectral radius* $\mathbf{A}$.

Proof. Let μ be an eigenvalue of $\mathbf{A}$ and $[(e, r), g] \neq [(0, 0), 0]$ be its corresponding eigenvector. Then we have that $\mathbf{A}[(e, r), g] = \mu[(e, r), g]$, and,

$$a^h(e_i, v)_{\Omega_i} + b^h(v, r_i)_{\Omega_i} + \sum_{1\le i\neq j\le m} \frac{\lambda_{ij}}{\beta_{ij}} \int_{\Gamma_{ij}} e_i v ds = \tag{5.48}$$

$$\sum_{1\le i\neq j\le m} \tfrac{1}{\beta_{ij}} \int_{\Gamma_{ij}} g_{ij} v ds,\ \forall v \in \mathbf{V}_i^h,$$

$$b^h(e_i, q)_{\Omega_i} = 0, \quad \forall q \in M_i^h, \tag{5.49}$$

$$\mu g_{ij}(k) = 2\lambda_{ij} e_j(k) - g_{ji}(k), \quad k \in N_h \cap \Gamma_{ij}, \quad 1 \le i \neq j \le m, \tag{5.50}$$

which, together with Lemma 5.2 and Lemma 5.3, imply that

$$\mu\, |||g|||_*^2 = |||g|||_*^2 - 4\sum_{i=1}^m a^h(e_i, e_i)_{\Omega_i}, \tag{5.51}$$

which means $|\mu| \le 1$ and $|\mu| = 1$ if and only if

$$a^h(e_i, e_i)_{\Omega_i} = (\nabla e_i, \nabla e_i)_{\Omega_i} + (\alpha e_i, e_i)_{\Omega_i} = 0, \quad \forall\ \ i = 1, 2\cdots m. \tag{5.52}$$

We next show $|\mu| < 1$, that is, $|\mu| \neq 1$. Clearly, if $|\mu| = 1$, then each e_i is a constant over Ω_i. Then we have that since e_i vanishes at nodal points on $\partial\Omega_i \subset G_1$.

$$e_i = 0, \ \text{ in }\ \Omega_i, \ \ \forall \Omega_i \subset G_1. \tag{5.53}$$

With similar arguments as in the proof of Theorem 5.4, taking $v \in \mathbf{V}_i^h$ satisfying (5.30)-(5.31) and

$$v = \begin{cases} g_{ij}(k), & \text{at } k \in \Gamma_{ij} \cap N_h, \\ 0, & \text{other nodal points on } \partial\Omega_i. \end{cases} \tag{5.54}$$

Plugging it into (5.48)-(5.50), we derive that

$$g_{ij} = 0, \quad \text{on} \quad \Gamma_{ij}, \quad \text{meas}(\Gamma_{ij}) > 0, \quad \forall \Omega_i \subset G_1, \tag{5.55}$$

$$g_{ij} = 0, \quad \text{on} \quad \Gamma_{ij}, \quad \text{meas}(\Gamma_{ij}) > 0, \quad \forall \Omega_i \subset G_1, \quad \Omega_j \subset G_2, \tag{5.56}$$

$$e_i = 0, \quad \text{on} \quad \Gamma_{ij}, \quad \forall \Omega_i \subset G_1, \quad \Omega_j \subset G_2. \tag{5.57}$$

Hence, it follows from (3.14) and (5.52)-(5.57) that

$$e_i = 0, \quad \text{in} \quad \Omega_i, \quad \forall \Omega_i \subset G_2. \tag{5.58}$$

Repeating the above arguments , we have that

$$e_i = 0, \quad \text{in} \quad \Omega_i, \quad \forall \quad i = 1, 2, \cdots, m. \tag{5.59}$$

$$g_{ij} = 0, \quad \text{on} \quad \Gamma_{ij}, \quad \text{meas}(\Gamma_{ij}) > 0, \quad 1 \le i \ne j \le m. \tag{5.60}$$

Moreover, from (5.48)-(5.50) and (5.59)-(5.60),

$$r_i = 0, \quad \text{in} \quad \Omega_i, \quad \forall \quad i = 1, 2, \cdots, m. \tag{5.61}$$

Therefore, $[(e, r), g] = [(0, 0), 0]$, which is a contradiction. Hence, $|\mu| < 1$. This completes the proof.

Finally, we emphasize an interesting fact. In Theorem 5.4 and Theorem 5.6 and their proofs, we have neither assumed nor used any restrictions for both the domain decomposition (3.1)-(3.2) and the finite element triangulations $\mathcal{T}_h$. Therefore, the convergence results, Theorem 5.4 and Theorem 5.6, hold for the general domain decomposition and finite element triangulation even without both quasi-uniform and regular requirements.

6. Concluding Remark

In this paper we have developed a parallel nonoverlapping domain decomposition iterative procedure for the generalized Stokes problems and their Crouzeix-Raviart finite element approximate problems. The basic idea to define the method is to use an mixed-type consistency condition together with a derivative-free transmission data updating technique on the artificial interfaces between two subdomains, which is originally from [5]. The method can be implemented on a massive parallel machine naturally as each subdomain is assigned to its own process.

References

[1] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Varlag, berlin, 1991.
[2] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[3] M. Crouzeix and P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations I, R.A.I.R.O. Modél. Math. Anal. Numér., R-3(1973), 33-75.
[4] Q. Deng, An analysis for a nonoverlapping domain decomposition iterative procedure, *SIAM J. Sci. Comp.*, **18** (1997), 444-456.
[5] Q. Deng, An optimal nonoverlapping domain decomposition iterative procedure, *SIAM J. Numer. Anal.*, **41** (2003), 964-982.
[6] Q. Deng, A nonoverlapping domain decomposition method for nonconforming finite element problems, *Comm. Pure. Appl. Anal.*, **3** (2003), 295-306.
[7] Q. Deng and X. Feng, Two-level overlapping Schwarz methods for plate elements on unstructured meshes using non-matching grids, in CRM Proceedings and Lecture Notes, Volume 21, 160-170.

[8] Q. Deng and X. Feng, Multigrid methods for the generalized Stokes equations based on mixed finite element methods, *J. Comp. Math.*, **20** (2002), 129-152.
[9] B. Després, Méthods de décomposition de domains pourles problémes de propagation d'ondes en régime harmonique, Thése, Université Paris IX Dauphine, UER Mathématiques de la Décision,1991.
[10] J. Douglas Jr., P. Paes Leme, J.E. Roberts and J. Wang, A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods, *Numer. Math.*, **65** (1993), 95-108.
[11] L. Fatone, P. Gervasio and A. Quarteroni, Numerical solution of vascular flows by heterogeneous domain decomposition methods, in domain decomposition Methods in sciences and Engineering, T. Chan, T. Kako, H. Kawarada, and O.Pironeau, eds., DDM.org, 2001, 297-303.
[12] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-stokes Equations, Springer-Varlag ,berlin, 1986.
[13] H. Han, Nonconforming element in the mixed finite element method, *J. Comput. Math.*, **3** (1984), 223-233.
[14] R.B. Kellogg and J.B. Osborn, A regularity result for the stokes problem on a convex polygon, *J. Funct. Anal.* **21** (1976), 397-431.
[15] P.L. Lions, On the Schwarz alternating method III: a variant for nonoverlapping subdomains, in Third International Symposium on Domain Decomposition Methods For Partial Differential Equations, T.F.Chan, R. Glowinski,J.Periaux and O. B. Widlund eds., SIAM, Philadelphia, 1990, 202-223.
[16] G. Lube, L. Müller and F.C. Otto, A nonoverlapping domain decomposition method for stabilized finite element approximations of the Oseen equations, *J. Comput. Appl. Math.*, **132** (2001), 211-236.
[17] F.C. Otto and G. Lube, Non-overlapping domain decomposition applied to incompressible flow problems, *Contemporary Mathematics*, **218** (1998), 507-514.
[18] L.F. Pavarino and O.B. Widlund, Balancing Neumann-Neumann methods for incompressible Stokes equations, *Comm. Pure Appl. Math.*, **55** (2002), 302-335.
[19] A. Quarteroni and A. Valli, Domain Decomposition Methods for Partial Differential Equations, Oxford Science Publications, 1999.
[20] T.C. Rebollo and E.C. Vera, A non-overlapping domain decomposition method for Stokes equations via a penalty term on the interface, *C. R. Acad. Sci. Paris, Ser I,* **334** (2002), 1-6.
[21] P. Le Tallec and A. Patra, Non-overlapping domain decomposition methods for adaptive hp-approximations of Stokes problem with discontinuous pressure fields, *Comp. Meths. Appl. Mech. Eng.*, **145** (1997), 361-379.