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LOCAL AND PARALLEL FINITE ELEMENT ALGORITHMS FOR THE NAVIER-STOKES PROBLEM *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

Based on two-grid discretizations, in this paper, some new local and parallel finite element algorithms are proposed and analyzed for the stationary incompressible Navier-Stokes problem. These algorithms are motivated by the observation that for a solution to the Navier-Stokes problem, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedure. One major technical tool for the analysis is some local a priori error estimates that are also obtained in this paper for the finite element solutions on general shape-regular grids.

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1. Introduction

In this paper, we will propose some new parallel techniques for finite element computations of the stationary incompressible Navier-Stokes problem. These techniques are based on our understanding of the local and global properties of a finite element solution to the Navier-Stokes problem. Simply speaking, the global behavior of a solution is mostly governed by low frequency components while the local behavior is mostly governed by high frequency components. The main idea of our new algorithms is to use a coarse grid to approximate the low frequencies and then to use a fine grid to correct the resulted residual (which contains mostly high frequencies) by some local and parallel procedures. One technical tool for motivating this idea is the local error estimate for finite element approximations. Let (w_h, r_h) be a finite element approximation to the linearized Navier-Stokes problem on a quasi-uniform grid $T^h(\Omega)$, then the following kind of local estimate holds (see Lemma 3.2):

$$\|w_h\|_{1,D} + \|r_h\|_{0,D} \le c(\|w_h\|_{0,\Omega_0} + \|r_h\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}), \tag{1.1}$$

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where $D \subset\subset \Omega_0 \subset \Omega$, here $D \subset\subset \Omega_0$ means $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial\Omega_0 \setminus \partial\Omega) > 0$.

This paper may be considered as a sequel of papers [7,23,24,25] on designing local and parallel finite element algorithms. In [7,23,24,25], local and parallel algorithms for a class of elliptic problems and the Stokes problem are studied, based on the local behaviors of finite element approximations on sharp-regular grids.

The rest of the paper is organized as follows. In the coming section, some preliminary materials and assumptions of mixed finite element spaces are provided. In section 3, a number of local a priori error estimates are obtained for the finite element discretization of the Navier-Stokes problem. Based upon these local error estimates, several new local and parallel algorithms are devised and analyzed in section 4.

2. Preliminaries

In this section, we shall describe some basic notations and assumptions on the mixed finite element spaces and then study some properties of the mixed finite element approximation to the Navier-Stokes problem.

Let Ω be a bounded domain in $R^d (d = 2, 3)$ assumed to have a Lipschitz-continuous boundary $\partial\Omega$. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega), W^{s,p}(\Omega)^d$ and their associated norms and seminorms, see e.g. [1,4]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega), H^s(\Omega)^d = W^{s,2}(\Omega)^d$ and $H^1_0(\Omega) = \{v \in H^1(\Omega); v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|_{s,2,\Omega}$ and $|\cdot|_{s,\Omega} = |\cdot|_{s,2,\Omega}$. In some places of this paper, $\|\cdot\|_{2,\Omega}$ should be viewed as piecewise defined if it is necessary. The space $H^{-1}(\Omega)^n$, the dual of $H^1_0(\Omega)^d, d = 1, 2, 3$, will also be used. For $D \subset G \subset \Omega$, we use the notation $D \subset\subset G$ to mean that $\mathrm{dist}(\partial D \backslash \partial\Omega, \partial G \backslash \partial\Omega) > 0$.

Throughout this paper, we shall use the letter c (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences.

2.1. Mixed Finite Element Spaces

For generality, we will not concentrate on any specific mixed finite element space, rather we shall study a class of mixed finite element spaces that satisfy certain assumptions. We shall now describe such assumptions.

Assume that $T^h(\Omega) = \{\tau\}$ is a mesh of Ω with mesh-size function $h(x)$ whose value is the diameter h_τ of the element τ containing x. One basic assumption on the mesh is that it is not exceedingly over-refined locally, namely

*A*0**.** There exists $\gamma \geq 1$ such that

$$h^\gamma_\Omega \leq ch(x) \quad \forall x \in \Omega, \tag{2.1}$$

where $h_\Omega = \max_{x\in\Omega} h(x)$ is the largest mesh size of $T^h(\Omega)$.

This is apparently a very mild assumption and most practical meshes should satisfy this assumption. Sometimes, we will drop the subscript in h_Ω to h for the mesh size on a domain that is clear from the context.

Associated with a mesh $T^h(\Omega)$, let $X_h(\Omega) \subset H^1(\Omega)^d, M_h(\Omega) \subset L^2(\Omega)$ be two finite element subspaces on Ω and set

$$X^0_h(\Omega) = X_h(\Omega) \cap H^1_0(\Omega)^d, \quad M^0_h(\Omega) = M_h(\Omega) \cap L^2_0(\Omega),$$

where

$$L^2_0(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}.$$

Given $G \subset \Omega_0 \subset \Omega$, we define $(X_h(G), M_h(G))$ and $T^h(G)$ to be the restriction of $(X_h(\Omega), M_h(\Omega))$ and $T^h(\Omega)$ to G, and

$$X^h_0(G) = \{v \in X^0_h(\Omega); \mathrm{supp}\, v \subset\subset G\}, \quad M^h_0(G) = \{q \in M_h(\Omega); \mathrm{supp}\, q \subset\subset G\}.$$

We now state our basic assumptions on the mixed finite element spaces.

$A1$. **Approximation property.** For each $(u,p)\in(H^{t+1}(G)^d,H^t(G))(t\ge 1)$, then there exists an approximation $(\pi_h u,\rho_h p)\in(X_h(G),M_h(G))$ such that

$$\begin{cases} (\mathrm{div}(u-\pi_h u),q_h)=0 \quad \forall q_h\in M_h(G) \\ \|h^{-1}(u-\pi_h u)\|_{0,G}+\|u-\pi_h u\|_{1,G}\le ch_G^s\|u\|_{1+s,G},\ 0\le s\le t, \end{cases} \tag{2.2}$$

$$\|h^{-1}(p-\rho_h p)\|_{-1,G}+\|p-\rho_h p\|_{0,G})\le ch_G^s\|p\|_{s,G},\ 0\le s\le t. \tag{2.3}$$

$A2$. **Inverse estimate property.** For all $(v,q)\in(X_h(G),M_h(G))$, there holds

$$\|v\|_{1,G}\le c\|h^{-1}v\|_{0,G},\quad \|q\|_{0,G}\le c\|h^{-1}q\|_{-1,G}. \tag{2.4}$$

$A3$. **Superapproximation property.** Let $\omega\in C_0^\infty(\Omega)$ with supp $\omega\subset\subset G$. Then for any $(u,p)\in(X_h(G),M_h(G))$, there is $(v,q)\in(X_0^h(G),M_0^h(G))$ such that

$$\|h^{-1}(\omega u-v)\|_{1,G}\le c\|u\|_{1,G},\quad \|h^{-1}(\omega p-q)\|_{0,G}\le c\|p\|_{0,G}. \tag{2.5}$$

$A4$. **Stability property.** For $D\subset\subset\Omega_0$, there are two positive constants β and h_0 such that for all $h\in(0,h_0]$, there is a subdomain $G(=G_h):D\subset\subset G\subset\Omega_0$ for which

$$\beta\|q\|_{0,G}\le\sup_{v\in X_h^0(G)}\frac{(\mathrm{div}v,q)}{\|v\|_{1,G}}\quad\forall q\in M_h^0(G). \tag{2.6}$$

The approximation property $A1$ is referred to [5] and inverse estimate property $A2$ is referred to [23-25]. Many finite element spaces are known to have the superapproximation property, cf. [3, 13-15,18,23-25]. When $G=\Omega$, Assumption $A4$ is the standard stability condition for the Stokes finite elements. It will usually hold as long as G_h is chosen to be a union of elements in $T^h(\Omega)$.

2.2. The Navier-Stokes Problem

In this subsection, we shall study some basic properties of the Navier-Stokes problem and its mixed finite element approximation. First, we consider the following generalized Navier-Stokes problem:

$$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f \quad \text{in }\Omega, \tag{2.7}$$

$$\mathrm{div}u=g \quad \text{in }\Omega, \tag{2.8}$$

$$u=0 \quad \text{on }\partial\Omega, \tag{2.9}$$

where $u=(u_1,\cdots,u_d)$ is the velocity, p is the pressure, $\nu>0$ is the viscosity, the functions f and g are given.

In order to introduce a variational formulation, we set

$$((u,v))=(\nabla u,\nabla v),\quad a(u,v)=\nu((u,v))\quad\forall u,\ v\in H^1(\Omega)^d,$$

$$d(v,q)=(\mathrm{div}v,q),\quad\forall(v,q)\in(H^1(\Omega)^d,L_0^2(\Omega)),$$

$$\begin{aligned} b(u,v,w)&=((u\cdot\nabla)v,w)+\frac{1}{2}((\mathrm{div}u)v,w)\\ &=\frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v)\quad\forall u,v,w\in H^1(\Omega)^d. \end{aligned}$$

It is well known that $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ satisfy the following properties (see [5,7,12]):

$$\alpha\|v\|_{1,\Omega}^2\le a(v,v)=\nu|v|_{1,\Omega}^2,\ a(u,v)\le\nu|u|_{1,\Omega}|v|_{1,\Omega}\quad\forall u,v\in H^1(\Omega)^d, \tag{2.10}$$

$$d(v,q)\le d_1\|u\|_{1,\Omega}\|q\|_{0,\Omega}\quad\forall(v,q)\in(H^1(\Omega)^d,L^2(\Omega)), \tag{2.11}$$

$$b(u,v,w)=-b(u,w,v)\quad\forall u,v,w\in H^1(\Omega)^d, \tag{2.12}$$

$$|b(u,v,w)|\le c_1\|u\|_{1,\Omega}\|v\|_{1,\Omega}\|w\|_{1,\Omega}\quad\forall u,v,w\in H^1(\Omega)^d, \tag{2.13}$$

here α, d_1 are two positive constants. For given $f\in H^{-1}(\Omega)^d$ and $g\in L_0^2(\Omega)$, the variational formulation of (2.7)-(2.9) reads: find a pair $(u,p)\in(H_0^1(\Omega)^d,L_0^2(\Omega))$ such that for all $(v,q)\in(H_0^1(\Omega)^d,L_0^2(\Omega))$

$$a(u,v)+b(u,u,v)-d(v,p)+d(u,q)=(f,v)+(g,q). \tag{2.14}$$

Throughout this paper, we will assume that the linearized dual problem to (2.14) is $W^{2,2}$-regular, $0 < h \le h_0$, where h_0 is sufficiently small. Also, we assume that Assumptions $A0$, $A1 - A4$ and the sequel $A5$ hold.

Now, we consider the standard Galerkin finite element method for solving problem (2.14): Find $(u_\mu, p_\mu) \in (X_\mu^0(\Omega), M_\mu^0(\Omega))$ such that

$$a(u_\mu, v) + b(u_\mu, u_\mu, v) - d(v, p_\mu) + d(u_\mu, q) = (f, v) + (g, q), \tag{2.15}$$

for all $(v, q) \in (X_\mu^0(\Omega), M_\mu^0(\Omega))$, where $\mu = h, H$.

The following results on (u_μ, p_μ) are classical (see [5,8,16]).

Theorem 2.1. *Assume that Ω is C^{t+1}-smooth bounded domain in R^d for $t \ge 1$ or a bounded convex polygonal domain in R^d for $t = 1$, (u, p) is a nonsingular solution of problem (2.14). Then, there exists a small $h_0 > 0$ such that for $\mu \in (0, h_0]$, problem (2.15) has a unique solution (u_μ, p_μ). Moreover, if $(u, p) \in (H^{t+1}(\Omega)^d \cap H_0^1(\Omega)^d, H^t(\Omega) \cap L_0^2(\Omega))$, we have the error bound:*

$$\|u - u_\mu\|_{1,\Omega} + \|p - p_\mu\|_{0,\Omega} \le c\mu^s(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), 1 \le s \le t, \tag{2.16}$$

and

$$\|u_\mu\|_{1,\Omega} + \|p_\mu\|_{0,\Omega} \le c. \tag{2.17}$$

Furthermore, we need the $W^{2,2}$-regularity assumptions on the linearized dual problem of problem (2.14).

$A5$. **Regularity property.** If $\mu \in (0, h_0]$ and $(\varphi, \phi) \in (L^2(\Omega_0)^d, H^1(\Omega_0) \cap L_0^2(\Omega_0))$, the linearized dual problem of (2.14): Find $(\Phi, \Psi) \in (H^2(\Omega_0)^d \cap H_0^1(\Omega_0)^d, H^1(\Omega_0) \cap L_0^2(\Omega_0))$ such that for all $(v, q) \in (H_0^1(\Omega_0)^d, L_0^2(\Omega_0))$,

$$a(v, \Phi) + b(u_\mu, v, \Phi) + b(v, u_\mu, \Phi) + d(v, \Psi) - d(\Phi, q) = (\varphi, v) + (\phi, q), \tag{2.18}$$

admits a unique solution (Φ, Ψ) satisfying (c.f., e.g. [5,8,16])

$$\|\Phi\|_{2,\Omega_0} + \|\Psi\|_{1,\Omega_0} \le c(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}). \tag{2.19}$$

Moreover, a finite element scheme for solving problem (2.18) reads: Find $\Phi_\mu \in X_\mu^0(\Omega_0)$, $\Psi_\mu \in M_\mu^0(\Omega_0)$ such that for all $(v, q) \in (X_\mu^0(\Omega_0), M_\mu^0(\Omega_0))$,

$$a(v, \Phi_\mu) + b(u_\mu, v, \Phi_\mu) + b(v, u_\mu, \Phi_\mu) + d(v, \Psi_\mu) - d(\Phi_\mu, q) = (\varphi, v) + (\phi, q), \tag{2.20}$$

where $\mu = h, H$. It is shown that there is a unique solution (Φ_μ, Ψ_μ) satisfying the above equation and

$$\|\Phi - \Phi_\mu\|_{1,\Omega_0} + \|\Psi - \Psi_\mu\|_{0,\Omega_0} \le c\mu(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}), \tag{2.21}$$

$$\|\Phi_\mu\|_{1,\Omega_0} + \|\Psi_\mu\|_{0,\Omega_0} \le c(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}). \tag{2.22}$$

Throughout this paper, we will assume that $h < H \le h_0$ and Assumptions $A0 - A5$ hold.

Now, by applying the Aubin-Nitsche duality argument in [2,5] one can easily prove the following error estimates.

Theorem 2.2. *Under the assumptions of Theorem 2.1, for all $h \in (0, h_0]$, there holds*

$$\begin{aligned}\|u - u_h\|_{0,\Omega} + \|p - p_h\|_{-1,\Omega} &\le ch(\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega}) \\ &\le ch^{s+1}(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 0 \le s \le t. \end{aligned} \tag{2.23}$$

Proof. We will apply the Aubin-Nitsche duality argument to prove (2.23). For $(\varphi, \phi) \in (L^2(\Omega)^d, H^1(\Omega) \cap L_0^2(\Omega))$, there exists $(\Phi, \Psi) \in (H_0^1(\Omega)^d, L_0^2(\Omega))$ satisfying (2.18)-(2.19) with $\mu = h, \Omega_0 = \Omega$. Moreover, let $(\Phi_h, \Psi_h) \in (X_h^0(\Omega), M_h^0(\Omega))$ be the finite element approximation of (Φ, Ψ), then (2.20)-(2.22) hold with $\mu = h$ and $\Omega_0 = \Omega$.

Now, we obtain from (2.14) and (2.15) that

$$\begin{aligned}&a(u - u_h, \Phi_h) + b(u - u_h, u_h, \Phi_h) + b(u_h, u - u_h, \Phi_h) + b(u - u_h, u - u_h, \Phi_h) \\ &\quad + d(u - u_h, \Psi_h) - d(\Phi_h, p - p_h) = 0,\end{aligned} \tag{2.24}$$

which together with (2.18) yields

$$\begin{aligned}(\varphi, u-u_h)+(\phi, p-p_h) &= a(u-u_h, \Phi-\Phi_h)+b(u-u_h, u_h, \Phi-\Phi_h)\\ &\quad + b(u_h, u-u_h, \Phi-\Phi_h)+b(u-u_h, u-u_h, -\Phi_h)\\ &\quad + d(u-u_h, \Psi-\Psi_h)-d(\Phi-\Phi_h, p-p_h). \end{aligned} \tag{2.25}$$

Thanks to (2.10)-(2.13), (2.16)-(2.17), (2.21) and (2.22), we have

$$\begin{aligned}|a(u-u_h, \Phi-\Phi_h)| &\le \nu\|u-u_h\|_{1,\Omega}\|\Phi-\Phi_h\|_{1,\Omega} \le ch\|u-u_h\|_{1,\Omega}(\|\varphi\|_{0,\Omega}+\|\phi\|_{1,\Omega}),\\ |b(u-u_h, u_h, \Phi-\Phi_h)|+|b(u_h, u-u_h, \Phi-\Phi_h)| &\le ch\|u-u_h\|_{1,\Omega}(\|\varphi\|_{0,\Omega}+\|\phi\|_{1,\Omega}),\\ |b(u-u_h, u-u_h, \Phi_h)| &\le c\|u-u_h\|_{1,\Omega}^2\|\Phi_h\|_{1,\Omega} \le ch\|u-u_h\|_{1,\Omega}(\|\varphi\|_{0,\Omega}+\|\phi\|_{1,\Omega}),\\ |d(u-u_h, \Psi-\Psi_h)|+|d(\Phi-\Phi_h, p-p_h)|&\\ &\le d_1(\|\Phi-\Phi_h\|_{1,\Omega}\|p-p_h\|_{0,\Omega}+\|u-u_h\|_{1,\Omega}\|\Psi-\Psi_h\|_{0,\Omega})\\ &\le ch(\|u-u_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega})(\|\varphi\|_{0,\Omega}+\|\phi\|_{1,\Omega}),\end{aligned}$$

which together with (2.24) yields

$$\|u-u_h\|_{0,\Omega}+\|p-p_h\|_{-1,\Omega} \le ch_\Omega(\|u-u_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}).$$

Finally we obtain (2.23) from Theorem 2.1.

3. Local a Priori Error Estimates

In this section, we shall present some local a priori finite element error estimates for the Navier-Stokes problem on general shape regular grids. The results presented here generalize local a priori error estimates for the linear problem known in the literature (c.f. [3,7,13-15,23-25]), which will play a crucial role in our analysis. Although these general estimates are theoretically interesting in their own right, our main motivation is to use them to devise and analyze some new local and parallel algorithms to be presented in the following sections.

First, we need the following technical result, which can be derived from Lemma 3.1 in Xu and Zhou [24].

Lemma 3.1. *Let $\omega \in C_0^\infty(\Omega)$ such that $supp\omega \subset\subset \Omega_0$. Then*

$$\|\omega w\|_{1,\Omega}^2 \le ca(w, \omega^2 w)+c\|w\|_{0,\Omega_0}^2 \quad \forall w \in H_0^1(\Omega)^d, \tag{3.1}$$

We shall now present a local a priori estimate for finite element approximation for the Navier-Stokes problem, which will play a crucial role in our analysis. This type of estimates is an extension of the results in [3,7,23,24].

Lemma 3.2. *Suppose that $f \in H^{-1}(\Omega_0)^d$, $0<\mu \le h_0$. If $(w, r) \in (X_h^0(\Omega), M_h^0(\Omega))$ satisfies that for all $(v, q) \in (X_0^h(\Omega_0), M_0^h(\Omega_0))$,*

$$a(w, v)+b(u_\mu, w, v)+b(w, u_\mu, v)-d(v, r)+d(w, q)=(f, v), \tag{3.2}$$

then for $D \subset\subset \Omega_0 \subset \Omega$,

$$\|w\|_{1,D}+\|r\|_{0,D} \le c(\|w\|_{0,\Omega_0}+\|r\|_{-1,\Omega_0}+\|f\|_{-1,\Omega_0}), \tag{3.3}$$

where $\mu = h, H$.

Proof. Let s be an integral such that $s \ge \max\{2\gamma-1, \gamma+1\}$, D_j and $\Omega_j (j=1,2\cdots,s)$ satisfy

$$D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset D_i \subset\subset \cdots \subset\subset D_s \subset\subset \Omega_s,$$
$$\Omega_s \subset\subset \Omega_{s-1} \subset\subset \cdots \subset\subset \Omega_j \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Choose $G \subset \Omega$ satisfying $D \subset\subset G \subset\subset D_1$ and $\omega \in C_0^\infty(\Omega)$ such that $\omega \equiv 1$ on $\bar{G}$ and supp $\omega \subset\subset D_1$.

Note that

$$\|\omega r-\frac{1}{|D_1|}\int_{D_1}\omega r dx\|_{0,D_1} \le c \sup_{\phi \in X_h^0(D_1)}\frac{d(\phi, \omega r)}{\|\phi\|_{1,\Omega}} \le c \sup_{\phi \in X_h^0(\Omega)}\frac{d(\phi, \omega r)}{\|\phi\|_{1,\Omega}}$$

while for $\psi \in X_0^h(\Omega_0)$,

$$\begin{aligned} d(\phi, \omega r) &= (\omega \text{div } \phi, r) = d(\omega\phi, r) - (\phi\nabla\omega, r) \\ &= d(\omega\phi - \psi, r) + a(w, \psi) + b(u_\mu, w, \psi) + b(w, u_\mu, \psi) - (f, \psi) - (\phi\nabla\omega, r). \end{aligned}$$

Choose $\psi \in X_0^h(D_1)$ such that

$$\|\omega\phi - \psi\|_{1,D_1} \leq ch_{\Omega_0}\|\phi\|_{1,\Omega_0},$$

then

$$\begin{aligned} \|\psi\|_{1,D_1} &\leq c(h_{\Omega_0}\|\phi\|_{1,\Omega_0} + \|\omega\phi\|_{1,D_1}) \leq c\|\phi\|_{1,\Omega}, \\ |d(\omega\phi - \psi, r)| &\leq ch_{\Omega_0}\|r\|_{0,D_1}\|\phi\|_{1,\Omega}, \\ |a(w, \psi)| + |(f, \psi)| + |(g, r)| &\leq c(\|w\|_{1,D_1} + \|f\|_{-1,D_1})\|\phi\|_{1,\Omega}, \\ |b(u_\mu, w, \psi)| + |b(w, u_\mu, \psi)| &\leq c\|w\|_{1,D_1}\|\phi\|_{1,\Omega}, \\ |(\phi\nabla\omega, r)| &\leq c\|r\|_{-1,D_1}\|\phi\|_{1,\Omega}, \end{aligned}$$

and

$$|d(\phi, \omega r)| \leq c(h_{\Omega_0}\|\phi\|_{1,\Omega}\|r\|_{0,D_1} + (\|w\|_{1,D_1} + \|r\|_{-1,D_1} + \|f\|_{-1,\Omega_0})\|\phi\|_{1,\Omega}).$$

Namely,

$$\|\omega r - \frac{1}{|D_1|}\int_{D_1} \omega r dx\|_{0,D_1} \leq c(h_{\Omega_0}\|r\|_{0,D_1} + \|w\|_{1,D_1} + \|r\|_{-1,D_1} + \|f\|_{-1,\Omega_0}),$$

or

$$\|r\|_{0,D} \leq c(h_{\Omega_0}\|r\|_{0,D_1} + \|w\|_{1,D_1} + \|r\|_{-1,D_1} + \|f\|_{-1,\Omega_0}),$$

where

$$\|\frac{1}{|D_1|}\int_{D_1} \omega r dx\|_{0,D_1} \leq c\|r\|_{-1,D_1},$$

is used. Similarly, we have

$$\|r\|_{0,D_{i-1}} \leq c(h_{\Omega_0}\|r\|_{0,D_i} + \|w\|_{1,D_i} + \|r\|_{-1,D_i} + \|f\|_{-1,\Omega_0}), i = 1, 2 \cdots, s-1,$$

where $D_0 = D$. Thus

$$\begin{aligned} \|r\|_{0,D} &\leq c(h_{\Omega_0}^{s-1}\|r\|_{0,D_{s-1}} + \|w\|_{1,D_{s-1}} + \|r\|_{-1,D_{s-1}} + \|f\|_{-1,\Omega_0}) \\ &\leq c(\|r\|_{-1,D_{s-1}} + \|w\|_{1,D_{s-1}} + \|f\|_{-1,\Omega_0}) \end{aligned}$$

and

$$\|r\|_{0,D_1} \leq c(\|r\|_{-1,\Omega_s} + \|w\|_{1,\Omega_s} + \|f\|_{-1,\Omega_0}). \tag{3.5}$$

On the other hand, there exists $(v, q) \in (X_0^h(D_1), M_0^h(D_1))$ such that

$$\|\omega^2 w - v\|_{1,D_1} \leq ch_{\Omega_0}\|w\|_{1,D_1}, \quad \|\omega^2 r - q\|_{0,D_1} \leq ch_{\Omega_0}\|r\|_{0,D_1},$$

which imply

$$\begin{aligned} a(w, \omega^2 w - v) &\leq ch_{\Omega_0}\|w\|_{1,D_1}^2, \\ |d(v - \omega^2 w, r)| + |d(w, \omega^2 r - q)| &\leq ch_{\Omega_0}\|w\|_{1,D_1}\|r\|_{0,D_1}, \\ |(f, v)| \leq \|f\|_{-1,D_1}\|v\|_{1,D_1} &\leq c\|f\|_{-1,\Omega_0}(h_{\Omega_0}\|w\|_{1,D_1} + \|\omega w\|_{1,\Omega}), \end{aligned}$$

and for some $\varepsilon \in (0, 1)$,

$$\begin{aligned} |b(u_\mu, w, v)| &= |b(u_\mu, w, v - \omega^2 w) + b(u_\mu, w, \omega^2 w)| \\ &\leq ch_{\Omega_0}\|w\|_{1,D_1}^2 + \frac{1}{2}\left|\int_\Omega (u_\mu \cdot \nabla)w \cdot (\omega^2 w)dx - \int_\Omega (u_\mu \cdot \nabla)(\omega^2 w) \cdot w dx\right| \\ &= ch_{\Omega_0}\|w\|_{1,D_1} + \left|\int_\Omega (u_\mu \cdot \nabla)\omega \cdot \omega |w|^2 dx\right| \\ &\leq ch_{\Omega_0}\|w\|_{1,D_1}^2 + c\|u_\mu\|_{L^4}\|\omega w\|_{L^4}\|w\|_{0,\Omega_0} \\ &\leq ch_{\Omega_0}\|u_\mu\|_{1,\Omega_0}\|\omega w\|_{1,\Omega}\|w\|_{0,\Omega_0} \\ &\leq ch_{\Omega_0}\|w\|_{1,D_1}^2 + \varepsilon\|\omega w\|_{1,\Omega}^2 + c\varepsilon^{-1}\|w\|_{0,\Omega_0}^2, \\ |b(w, u_\mu, v)| &\leq ch_{\Omega_0}\|w\|_{1,D_1}^2 + \varepsilon\|\omega w\|_{1,\Omega}^2 + c\varepsilon^{-1}\|w\|_{0,\Omega_0}^2. \end{aligned}$$

Note that

$$d(\omega^2 w,r) = d(w,\omega^2 r) + (2\omega w\nabla\omega, r) = d(w,\omega^2 r - q) + (2\omega w\nabla\omega, r)$$
$$\le ch_{\Omega_0}\|w\|_{1,D_1}\|r\|_{0,D_1} + c\|\omega w\|_{1,D_1}\|r\|_{-1,D_1},$$

we have, from Lemma 3.1, that

$$\begin{aligned} c^{-1}\|\omega w\|_{1,\Omega}^2 &\le a(w,\omega^2 w) + \|w\|_{0,\Omega_0}^2 \\ &= a(w,\omega^2 w - v) + d(v,r) + (f,v) + \|w\|_{0,\Omega_0}^2 - b(u_\mu,w,v) - b(w,u_\mu,v) \\ &= a(w,\omega^2 w - v) + d(\omega^2 w,r) + d(v-\omega^2 w,r) + (f,v) + \|w\|_{0,\Omega_0}^2 \\ &\quad - b(u_\mu,w,v) - b(w,u_\mu,v) \\ &\le ch_{\Omega_0}(\|w\|_{1,D_1}^2 + \|w\|_{1,D_1}\|r\|_{0,D_1}) + c(1+\varepsilon^{-1})\|w\|_{0,\Omega_0}^2) + c\varepsilon\|\omega w\|_{1,\Omega}^2 \\ &\quad + c(\|\omega w\|_{1,D_1}\|r\|_{-1,D_1} + c\|f\|_{-1,\Omega_0}(h_{\Omega_0}\|w\|_{1,D_1} + \|\omega w\|_{1,\Omega}). \end{aligned}$$

Thus

$$\|\omega w\|_{1,\Omega} \le c(h_{\Omega_0}^{1/2}(\|w\|_{1,D_1} + \|r\|_{0,D_1}) + \|w\|_{0,\Omega_0} + \|r\|_{-1,D_1} + \|f\|_{-1,\Omega_0}),$$

which together with (3.5) leads to

$$\|w\|_{1,D} \le c(h_{\Omega_0}^{1/2}\|w\|_{1,\Omega_s} + \|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}).$$

Similarly,

$$\|w\|_{1,\Omega_j} \le c(h_{\Omega_0}^{1/2}\|w\|_{1,\Omega_{j-1}} + \|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}), \quad j = 1,2\cdots,s.$$

Hence

$$\|w\|_{1,D} \le c(h_{\Omega_0}^{(s+1)/2}\|w\|_{1,\Omega_0} + \|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}),$$

namely

$$\|w\|_{1,D} \le c(\|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}). \tag{3.6}$$

Combining (3.5) and a similar estimate

$$\|w\|_{1,\Omega_s} \le c(\|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}),$$

we obtain

$$\|r\|_{0,D} \le \|r\|_{0,D_1} \le c(\|w\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}),$$

which together (3.6) completes the proof.

4. New Local and Parallel Algorithms

In this section we shall present some new local and parallel finite element algorithms for the Navier-Stokes problem with $g = 0$. These algorithms are motivated by the local error estimates studied in the previous section. First, we shall discuss the local algorithms. The generalization of the local algorithms to parallel algorithms is straightforward.

For clarity, let Ω be a polygonal domain or polyhedral domain and $X_h(\Omega) \subset H^1(\Omega)^d, M_h(\Omega) \subset L^2(\Omega)$ be two finite element subspaces satisfying the assumptions $A1 - A4$ associated with a grid $T^h(\Omega)$ satisfying the assumption $A0$. Let $(u_h,p_h) \in (X_h^0(\Omega), M_h^0(\Omega))$ be the standard finite element solution of the Navier-Stokes problem:

$$a(u_h,v) + b(u_h,u_h,v) - d(v,p_h) + d(u_h,q) = (f,v) \ \ \forall (v,q) \in (X_h^0(\Omega), M_h^0(\Omega)). \tag{4.1}$$

Either locally or globally, with proper regularity assumption, we have the following error estimate:

$$\|u-u_h\|_{1,\Omega} + \|p-p_h\|_{0,\Omega} \le ch^s, 1 \le s \le t. \tag{4.2}$$

With this type of error estimates in mind, in the rest of this section, we will only compare the approximate solutions from our new algorithms with (u_h,p_h) instead of the exact solution (u,p).

4.1. Local Algorithms

The local algorithms we shall now present can be used to obtain approximate solution on a given subdomain mostly by local computation. The main idea is that the more global component of a finite element solution may be obtained by a relatively coarser grid and, the rest of the computation can then be localized.

Roughly speaking, our new algorithms will be based on sometimes one coarse grid of size H and one fine grid of size $h \ll H$, and sometimes on a grid that is fine in a subdomain and coarse on the rest of the domain.

The fine grid may be only defined locally. In our analysis, we shall use an auxiliary fine grid, say $T^h(\Omega)$, that is globally defined. One basic assumption for this auxiliary fine grid is that it should coincide with the local fine grid in the subdomain of interest.

Let $T^H(\Omega)$ be a shape-regular coarse grid of size $H \gg h$ which satisfies $A0$, so that the highly locally refined mesh $T^h(\Omega_0)$ can be obtained, where Ω_0 is a slightly larger subdomain containing a subdomain $D \subset \Omega$ (namely $D \subset\subset \Omega_0$). More precisely, we let $T^{H,h}(\Omega)$ denote a locally refined shape-regular mesh that may be viewed as being obtained by refining $T^H(\Omega)$ locally around the subdomain D in such a way that $T^{H,h}(\Omega_0) = T^h(\Omega_0)$.

We are interested in obtaining the approximation solution in the given subdomain D with an accuracy comparable to that from $T^h(\Omega)$. We shall propose two different gridding strategies for obtaining finite element approximations on the subdomain D. We denote the corresponding finite element spaces by $X^0_{H,h}(\Omega) \subset H^1_0(\Omega)^d, M^0_{H,h}(\Omega) \subset L^2_0(\Omega)$, which satisfies assumptions $A1 - A4$.

4.1.1. Implicity Approach

The first stragey is simply to solve a standard finite element solution in $(X^0_{H,h}(\Omega), M^0_{H,h}(\Omega))$.

Algorithm $A.0$. *Find* $(u^h_H, p^h_H) \in (X^0_{H,h}(\Omega), M^0_{H,h}(\Omega))$ *such that*

$$a(u^h_H, v) + b(u^h_H, u^h_H, v) - d(v, p^h_H) + d(u^h_H, q) = (f, v), \tag{4.3}$$

for all $(v, q) \in (X^0_{H,h}(\Omega), M^0_{H,h}(\Omega))$.

Strictly speaking, this algorithm is still a global algorithm as a global problem is solved. But it is designed to obtain a local approximation in the subdomain D and it makes use of a mesh that is much coarse away from D.

Theorem 4.1. *Assume that* $(u^h_H, p^h_H) \in (X^0_{H,h}(\Omega), M^0_{H,h}(\Omega))$ *is obtained by Algorithm* $A.0$. *Then*

$$\|u_h - u^h_H\|_{1,D} + \|p_h - p^h_H\|_{0,D} \le cH^{s+1}(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \tag{4.4}$$

Proof. By the definition of Algorithm $A.0$ and our assumption on the auxiliary grid $T^h(\Omega)$ that coincide with $T^{H,h}(\Omega)$ on Ω_0, we deduce from (4.1) and (4.3) that

$$\begin{aligned} a(u^h_H - u_h, v) &+ b(u^h_H - u_h, u_h, v) + b(u_h, u^h_H - u_h, v) \\ &- d(v, p^h_H - p_h) + d(u^h_H - u_h, q) = -b(u^h_H - u_h, u^h_H - u_h, v). \end{aligned}$$

for all $(v, q) \in (X^h_0(\Omega_0), M^h_0(\Omega_0))$. By Lemma 3.2 and (2.13), we get

$$\begin{aligned} &\|u_h - u^h_H\|_{1,D} + \|p_h - p^h_H\|_{0,D} \\ &\le c(\|u_h - u^h_H\|_{0,\Omega_0} + \|p_h - p^h_H\|_{-1,\Omega_0} + \|u_h - u^h_H\|^2_{1,\Omega_0}) \\ &\le c(\|u - u_h\|_{0,\Omega} + \|u - u^h_H\|_{0,\Omega} + \|u - u_h\|^2_{1,\Omega} + \|u - u^h_H\|^2_{1,\Omega}) \\ &\quad + c(\|p^h_H - p\|_{-1,\Omega} + \|p_h - p\|_{-1,\Omega}), \end{aligned}$$

which together with Theorem 2.1 and Theorem 2.2 finishes the proof.

4.1.2. Explicity Approach

Our second strategy is in a way an improvement of the first strategy . In this strategy, we first solve a global problem only on the given coarse grid $T^H(\Omega)$ and we then correct the residual locally on the fine mesh $T^h(\Omega_0)$. If we correct the residual globally on the fine mesh $T^h(\Omega)$, then we deduce the two-level finite element method for the Navier-Stokes equations. For some details of the two-level finite element, the reader can refer to Xu [21, 22] and Layton [9, 10, 11, 12]. Moreover, for some details of the two-level finite element for the time-dependent Navier-Stokes equations, the reader can refer to He [6].

Let $(X_H(\Omega), M_H(\Omega)) \subset (H_0^1(\Omega)^d, L_0^2(\Omega))$ be the finite element space pair on $T^H(\Omega)$ satisfying assumptions $A0 - A4$. A prototype of our new local algorithms is as follows.

Algorithm $B.0$. *1. Find a global coarse grid solution* $(u_H, p_H) \in (X_H^0(\Omega), M_H^0(\Omega))$*:*

$$a(u_H, v) + b(u_H, u_H, v) - d(v, p_H) + d(u_H, q) = (f, v) \quad \forall (v, q) \in (X_H^0(\Omega), M_H^0(\Omega)).$$

2. Find a local fine grid correction $(e_h, \eta_h) \in (X_h^0(\Omega_0), M_h^0(\Omega_0))$*:*

$$\begin{aligned} &a(e_h, v) + b(u_H, e_h, v) + b(e_h, u_H, v) - d(v, \eta_h) + d(e_h, q) \\ &= (f, v) - a(u_H, v) - b(u_H, u_H, v) + d(v, p_H) - d(u_H, q) \quad \forall (v, q) \in (X_h^0(\Omega_0), M_h^0(\Omega_0)). \end{aligned}$$

3.Update: $u^h = u_H + e_h, p^h = p_H + \eta_h$ *in* Ω_0.

Theorem 4.2. *Assume that* $(u^h, p^h) \in (X_h(\Omega_0), M_h(\Omega_0))$ *is obtained by Algorithm* $B.0$*. Then*

$$\begin{aligned} \|u_h - u^h\|_{1,D} + \|p_h - p^h\|_{0,D} &\le c(\|u_h - u_H\|_{0,\Omega} + \|p_h - p_H\|_{-1,\Omega}) \\ &\quad + cH(\|u_h - u_H\|_{1,\Omega} + \|p_h - p_H\|_{0,\Omega})) \\ &\le cH^{s+1}(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \end{aligned} \tag{4.5}$$

Consequently

$$\|u - u^h\|_{1,D} + \|p - p^h\|_{0,D} \le c(h^s + H^{s+1})(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \tag{4.6}$$

Proof. First of all, we derive from Algorithm $B.0$ and problem (4.1) that

$$\begin{aligned} &a(u_h - u^h, v) + b(u_h - u^h, u_H, v) + b(u_H, u^h - u_H, u_H, v) \\ &\quad - d(v, p_h - p^h) + d(u_h - u^h, q) + b(u_h - u_H, u_h - u_H, v) = 0, \end{aligned}$$

for all $(v, q) \in (X_0^h(\Omega_0), M_0^h(\Omega_0))$. Thus, Lemma 3.2 and (2.13) imply

$$\begin{aligned} \|u_h - u^h\|_{1,D} + \|p_h - p^h\|_{0,D} &\le c(\|u_h - u^h\|_{0,\Omega_0} + \|p_h - p^h\|_{-1,\Omega_0} + \|u_h - u_H\|_{1,\Omega_0}^2) \\ &\le c(\|u_h - u_H\|_{0,\Omega_0} + \|p_h - p_H\|_{-1,\Omega_0} + \|u_h - u_H\|_{1,\Omega}^2) \\ &\quad + c(\|e_h\|_{0,\Omega_0} + \|\eta_h\|_{-1,\Omega_0}). \end{aligned} \tag{4.7}$$

From Assumption $A5$, for $(\varphi, \phi) \in (L^2(\Omega_0)^d, H^1(\Omega_0) \cap L_0^2(\Omega_0))$, there exists $(w, r) \in (H_0^2(\Omega_0)^d \cap H_0^1(\Omega_0)^d, H^1(\Omega_0) \cap L_0^2(\Omega_0))$ such that

$$\left\{ \begin{array}{lcl} a(v, w) & + & b(u_H, v, w) + b(v, u_H, w) + d(v, t) - d(w, q) \\ & = & (\varphi, v) + (\phi, q) \quad \forall (v, q) \in (H_0^1(\Omega_0)^d, L_0^2(\Omega_0)), \end{array} \right.$$

and

$$\|w\|_{2,\Omega_0} + \|r\|_{1,\Omega_0} \le c(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}).$$

Let $(w_\mu, r_\mu) \in (X_\mu^0(\Omega_0), M_\mu^0(\Omega_0))$ be the finite element approximation of (w, r):

$$\begin{aligned} &a(v, w - w_\mu) + b(u_H, v, w - w_\mu) + b(v, u_H, w - w_\mu) \\ &\quad + d(v, r - r_\mu) - d(w - w_\mu, q) = 0 \quad \forall (v, q) \in (X_\mu^0(\Omega_0), M_\mu^0(\Omega_0)), \end{aligned}$$

where $\mu = h$ or H. Then

$$\|w - w_\mu\|_{1,\Omega_0} + \|r - r_\mu\|_{0,\Omega_0} \le c\mu(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}). \tag{4.8}$$

Note that

$$\begin{aligned} &a(e_h, v) + b(e_h, u_H, v) + b(u_H, e_h, v) - d(v, \eta_h) + d(e_h, q) \\ &= a(u_h - u_H, v) + b(u_h - u_H, u_H, v) \\ &+ b(u_H, u_h - u_H, v) + b(u_h - u_H, u_h - u_H, v) \\ &- d(v, p_h - p_H) + d(u_h - u_H, q) \quad \forall (v, q) \in (X_h^0(\Omega_0), M_h^0(\Omega_0)), \end{aligned}$$

and

$$a(u_h - u_H, w_H) + b(u_h - u_H, u_H, w_H) + b(u_H, u_h - u_H, w_H) \\ + b(u_h - u_H, u_h - u_H, w_H) - d(w_H, p_h - p_H) + d(u_h - u_H, r_H) = 0,$$

we have

$$\begin{aligned}(\varphi, e_h) + (\phi, \eta_h) &= a(e_h, w) + b(u_H, e_h, w) + b(e_h, u_H, w) + d(e_h, r) - d(w, \eta_h) \\ &= a(e_h, w_h) + b(u_H, e_h, w_h) + b(e_h, u_H, w_h) + d(e_h, r_h) - d(w_h, \eta_h) \\ &= a(u_h - u_H, w_h) + b(u_h - u_H, u_H, w_h) + b(u_H, u_h - u_H, w_h) \\ &\quad + b(u_h - u_H, u_h - u_H, w_h) - d(w_h, p_h - p_H) + d(u_h - u_H, r_h)\end{aligned}$$

and hence

$$\begin{aligned}(\varphi, e_h) + (\phi, \eta_h) = {} & a(u_h - u_H, w_h - w) + b(u_h - u_H, u_H, w_h - w) \\ & + b(u_H, u_h - u_H, w_h - w) + b(u_h - u_H, u_h - u_H, w_h - w) \\ & - d(w_h - w, p_h - p_H) + d(u_h - u_H, r_h - r) \\ & + a(u_h - u_H, w - w_H) + b(u_h - u_H, u_H, w - w_H) + b(u_H, u_h - u_H, w - w_H) \\ & + b(u_h - u_H, u_h - u_H, w - w_H) - d(w - w_H, p_h - p_H) + d(u_h - u_H, r - r_H).\end{aligned}$$

Therefore we obtain from (2.10)-(2.13) and (4.8) that

$$|(\varphi, e_h) + (\phi, \eta_h)| \le cH(\|u_h - u_H\|_{1,\Omega} + \|p_h - p_H\|_{0,\Omega})(\|\varphi\|_{0,\Omega_0} + \|\phi\|_{1,\Omega_0}), \tag{4.9}$$

or

$$\|e_h\|_{0,\Omega_0} + \|\eta_h\|_{-1,\Omega_0} \le cH(\|u_h - u_H\|_{1,\Omega} + \|p_h - p_H\|_{0,\Omega}),$$

which together with (4.2) and (4.7) finishes the proof.

4.2. New Parallel Algorithms Based on Local Algorithms

The parallel algorithms we shall present here naturally obtained from the local algorithms that we studied above. Given an initial coarse triangulation $T^H(\Omega)$, let us first divide Ω into a number of disjoint subdomains $D_1, \cdots, D_m$, then enlarge each D_j to obtain Ω_j that align with $T^H(\Omega)$. The basic idea of our parallel algorithm is very simple: we just apply the local algorithms in parallel in all Ω_j's.

Let us first discuss the parallel version of Algorithm $A0$. For each j, we use some adaptive process to obtain a shape-regular mesh $T_j(\Omega)$ and the corresponding finite element solution denoted by (u_j, p_j). We note that each $T_j(\Omega)$ has a substantially finer mesh inside Ω_j. We note that all $T_j(\Omega)$ are different triangulations for Ω and they can very arbitrary; but for simplicity of exposition, we assume each $T_j(\Omega)$ has the same size h in Ω_j (more precisely, $T_j(\Omega_j) = T^h(\Omega_j)$) and has the size H away from Ω_j. Let $X_{h_j}(\Omega) \subset H_0^1(\Omega)^d$, $M_{h_j}(\Omega) \subset L_0^2(\Omega)$ be the corresponding finite element spaces satisfying assumptions $A1 - A4$.

Algorithm $A.1$.

1. Find $(u_j, p_j) \in (X_{h_j}^0(\Omega), M_{h_j}^0(\Omega))(j = 1, 2, \cdots, m)$ in parallel:

$$a(u_j, v) + b(u_j, u_j, v) - d(v, p_j) + d(u_j, q) = (f, v) \quad \forall (v, q) \in (X_{h_j}^0(\Omega), M_{h_j}^0(\Omega)).$$

2. Set $(u^h, p^h) = (u_j, p_j)$ in $D_j (j = 1, 2, \cdots, m)$.

Define a piecewise norm

$$|||u_h - u^h|||_{1,\Omega} = (\sum_{j=1}^m \|u_h - u^h\|_{1,D_j}^2)^{1/2},$$

$$|||p_h - p^h|||_{0,\Omega} = (\sum_{j=1}^m \|p_h - p^h\|_{0,D_j}^2)^{1/2}.$$

By Theorem 4.1, we have

$$|||u_h - u^h|||_{1,\Omega} + |||p_h - p^h|||_{0,\Omega} \le cH^{s+1}(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t.$$

Consequently,

$$|||u - u^h|||_{1,\Omega} + |||p - p^h|||_{0,\Omega} \le c(h^s + H^{s+1})(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \tag{4.10}$$

We now discuss the parallel versions of Algorithm $B.0$, although there are many possibilities for the generalization. For clarity, it appears to be most convenient to discuss these using two globally defined grids: an initial coarse grid $T^H(\Omega)$ and a refined (from $T^H(\Omega)$) grid $T^h(\Omega)$ that satisfies $h \ll H$.

Algorithm $B.1$**.**

1. Find a global coarse grid solution $(u_H, p_H) \in (X_H^0(\Omega), M_H^0(\Omega))$*:*

$$a(u_H, v) + b(u_H, u_H, v) - d(v, p_H) + d(u_H, q) = (f, v) \quad \forall (v, q) \in (X_H^0(\Omega), M_H^0(\Omega)).$$

2. Find local fine grid corrections $(e_h^j, \eta_h^j) \in (X_h^0(\Omega_j), M_h^0(\Omega_j))(j = 1, \cdots, m)$ *in parallel:*

$$\begin{aligned} &a(e_h^j, v) + b(u_h, e_h^j, v) + b(e_h^j, u^h, v) - d(v, \eta_h^j) + d(e_h^j, q) \\ &= (f, v) - a(u_H, v) - b(u_H, u_H, v) + d(v, p_H) - d(u_H, q) \quad \forall (v, q) \in (X_h^0(\Omega_j), M_h^0(\Omega_j)). \end{aligned}$$

3. Set $(u^h, p^h) = (u_H, p_H) + (e_h^j, \eta_h^j)$*, in* $D_j (j = 1, 2, \cdots, m)$*.*

By Theorem 4.2, for this algorithm, we apparently have the following error result.

Theorem 4.3. *Assume that* (u^h, p^h) *is the solution obtained by Algorithm* $B.1$*. Then*

$$\begin{aligned} |||u_h - u^h|||_{1,\Omega} + |||p_h - p^h|||_{0,\Omega} &\le cH(\|u_h - u_H\|_{1,\Omega} + \|p_h - p_H\|_{0,\Omega}) \\ &\le cH^{s+1}(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t, \end{aligned} \tag{4.11}$$

and

$$|||u - u^h|||_{1,\Omega} + |||p - p^h\|_{0,\Omega} \le c(h^s + H^{s+1})(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \tag{4.12}$$

Proof. Note that

$$\begin{aligned} &\|u_h - u^h\|_{1,D_j} + \|p_h - p^h\|_{0,D_j} \\ &\le c(\|u_h - u_H\|_{0,\Omega_j} + \|p_h - p_H\|_{-1,\Omega_j} + H(\|u_h - u_H\|_{1,\Omega_j} + \|p_h - p_H\|_{0,\Omega_j})) \\ &\le cH(\|u_h - u_H\|_{1,\Omega} + \|p_h - p_H\|_{0,\Omega}), \end{aligned}$$

the desired result follows.

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