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A NEW SQP-FILTER METHOD FOR SOLVING NONLINEAR PROGRAMMING PROBLEMS *1)

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Abstract

In [4], Fletcher and Leyffer present a new method that solves nonlinear programming problems without a penalty function by SQP-Filter algorithm. It has attracted much attention due to its good numerical results. In this paper we propose a new SQP-Filter method which can overcome Maratos effect more effectively. We give stricter acceptant criteria when the iterative points are far from the optimal points and looser ones vice-versa. About this new method, the proof of global convergence is also presented under standard assumptions. Numerical results show that our method is efficient.

Mathematics subject classification: 65K05, 49M37, 90C30, 90C26.
Key words: Nonlinear programming, Sequential quadratic programming, Filter, Restoration phase, Maratos affects, Global convergence, Multi-objective optimization, Quadratic programming subproblem.

1. Introduction

This paper is concerned with a new SQP-Filter method which not only has good convergence properties but also shows encouraging numerical results. We consider the general nonlinear programming problem(NLP)

$$\begin{array}{rl} min & \qquad f(x) \\ s.t. & c_i(x) = 0, i = 1, \ldots, m_e, \\ & c_i(x) \geq 0, i = m_e + 1, \ldots, m, \end{array} \tag{1.1}$$

where f and $c_i (i = 1, 2, \cdots, m)$ are twice continuously differentiable functions. Let $c(x) = (c_1(x), c_2(x), \cdots, c_m(x))^T$, $c_{\mathcal{E}} = (c_i)_{i\in\mathcal{E}}$ and $c_{\mathcal{I}} = (c_i)_{i\in\mathcal{I}}$, where $\mathcal{E} = \{1, \ldots, m_e\}$ and $\mathcal{I} = \{m_e + 1, \ldots, m\}$ denote the indices of equality and inequality constraints, respectively. With these notations the Lagrangian function associated with (1.1) is given by

$$\mathbb{L}(x, \lambda) = f(x) - \lambda^T c(x), \tag{1.2}$$

where $\lambda = (\lambda_{\mathcal{E}}^T,\ \lambda_{\mathcal{I}}^T) \in \Re^m$, $\lambda_{\mathcal{I}} \geq 0$. We denote the gradients of f by $g = g(x) = \nabla f(x)$, the Jacobian of the constraints by $A(x) = \nabla c(x)^T$, $A_{\mathcal{E}} = A_{\mathcal{E}}(x) = \nabla c_{\mathcal{E}}^T$ and $A_{\mathcal{I}} = A_{\mathcal{I}}(x) = \nabla c_{\mathcal{I}}^T$ corresponding to the equality and inequality constraints respectively. Superscript $^{(k)}$ refers to iterations indices and $f^{(k)}$ is taken to mean $f(x^{(k)})$ etc. Quantities relating to a local solutions x^* of (1.1) are superscripted with a $*$.

In the pioneering work[4], Fletcher and Leyffer present the filter technique for the globalization of SQP methods, which avoids the use of multipliers or penalty parameters. Instead, the

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acceptability of steps is determined by comparing the constraint violation and objective function value. The new iterative point is acceptable to the filter if either feasibility or objective function value is sufficiently improved in comparison to all iterative points bookmarked in the current filter. The promising numerical results in[4] led to a growing interest in filter methods in recent years. By strengthening the acceptant criteria of the filter, the global convergence of three variants of trust-region SQP-Filter methods is shown in Fletcher and Leyffer [4].

In this paper we propose a new SQP-Filter method that ensures global convergence. The original SQP-Filter methods in [1, 5, 6] can be affected by the well known Marotos effect. In fact, these methods use constraint violation and objective value in the filter. By the Marotos effect a full SQP-step can lead to an increase of both these filter components even arbitrarily close to a regular minimizer(see [15], P.539). This makes the full SQP-step unacceptable for the filter and can prohibit fast local convergence. To avoid this, we use different filters and different acceptant criteria by the measurement of constraint violation. And two obvious advantages are that our method converges quickly when the points are far from the optimal points and our method can prevent Marotos effect efficiently when the iterative points come closer to the optimal points.

The paper is organized as follows. In section 2 we describe briefly the trust-region SQP-Filter method of Fletcher, Leyffer and Toint[4]. In section 3 we present the main idea of our new SQP-Filter method. In section 4 we give out the SQP-Filter algorithm and its convergence analysis. Finally at section 5 we give some numerical results which show that our method is very efficient.

2. Fletcher's Trust-region SQP-Filter Method

In this section we recall the trust-region SQP-Filter method proposed by Fletcher and Leyffer[4] for (1.1).

Given the current iterative point $x^{(k)}$ and a trust-region radius $\rho^{(k)}$, the computation of the step is based on the approximate solution of the QP-subproblems $QP_{FL}(x^{(k)}, \rho^{(k)})$

$$\begin{array}{rl} min & \hat{\Psi}^{(k)}(d) = \frac{1}{2} d^T W^{(k)} d + d^T g^{(k)} \\ s.t. & c_i^{(k)} + d^T a_i^{(k)} = 0, i = 1, \ldots, m_e, \\ & c_i^{(k)} + d^T a_i^{(k)} \geq 0, i = m_e + 1, \ldots, m, \\ & \|d\|_\infty \leq \rho^{(k)}, \end{array} \tag{2.1}$$

where $W^{(k)} = \nabla^2 \mathbb{L}(x^{(k)}, \lambda)$ denotes the exact Hesse matrix of Lagrangian function at point $x^{(k)}$. For convenience of description, Fletcher and Leyffer in [4] give the concepts of constraint violation, dominance and filter as follows

Definition 2.1. *For the nonlinear programming problem (*1.1*), we define its constraint violation function $h(c(x))$ by*

$$h(c(x)) = \sum_{i \in \mathcal{E}} |c_i(x)| + \sum_{i \in \mathcal{I}} max\{-c_i(x), 0\}. \tag{2.2}$$

For simplicity, we use $h^{(k)}$ and $f^{(k)}$ to denote values of $h(c(x))$ and $f(x)$ evaluated at point $x^{(k)}$, respectively.

Definition 2.2. *A pair $(h^{(k)}, f^{(k)})$ is said to dominate another pair $(h^{(l)}, f^{(l)})$ if and only if both $h^{(k)} \leq h^{(l)}$ and $f^{(k)} \leq f^{(l)}$.*

With this concept Fletcher and Leyffer give the definition of a filter, which will be used in a trust-region type algorithm as criteris for accepting or rejecting a trial step.

Definition 2.3. *A filter is a list of pairs* $((h^{(l)}, f^{(l)}))$ *such that no pair dominates any other. A point* $(h^{(k)}, f^{(k)})$ *is said to be acceptable for inclusion in the filter if it is not dominated by any points in the filter.*

From now on, we use $\mathcal{F}^{(k)}$ to denote the set of iteration indices $j(j < k)$ such that $(h^{(j)}, f^{(j)})$ is an entry in the current filter. Since a local solution $\bar{x}$ of (1.1) minimizes f locally under the constraint $h(c(x)) = 0$, a promising trial step d should either reduce the constraint violation h or the objective function value f. To ensure sufficient decrease of at least one of the two criteria, in [3] Fletcher et al. strengthen the acceptant criteria, they call x acceptable to the filter $\mathcal{F}^{(k)}$ if for all $(h^{(j)}, f^{(j)}) \in \mathcal{F}^{(k)}$

$$\textit{either} \quad h(c(x)) \leq \beta h^{(j)} \qquad \textit{or} \qquad f(x) + \gamma h(c(x)) \leq f^{(j)} \tag{2.3}$$

is satisfied with constants $0 < \gamma < \beta < 1$. To update the information in the filter appropriate point x, which is acceptable to the filter, is added into the filter $\mathcal{F}^{(k)}$. This operation means that $(h(c(x)), f(x))$ is added into the set of pairs $(h^{(j)}, f^{(j)})$ in $\mathcal{F}^{(k)}$ and subsequently any old pair that is dominated by the new pair $(h(c(x)), f(x))$ is removed from the filter.

The acceptance test (2.3) has the useful inclusion property that after adding a pair $(h(c(x)), f(x))$ into the filter all unacceptable points of the old filter remain unacceptable for the new filter(although dominated pairs are removed). This allows to show that the infeasibility tends to zero for any infinite subsequence of iterates added into the filter.

The key idea of Fletcher et al. in [4] is a minimalistic approach to SQP. The main reason for doing this lies in that the SQP method without any line search, penalty function or trust region technique has shown encouraging numerical results. For instance, Zoppke-Donaldson in [2] presents an SQP algorithm which does not require any penalty parameter or backtracking as in the watchdog technique. Now we turn back to consider the QP-subproblem (2.1). After getting the solution $d^{(k)}$ to the QP-subproblem (2.1), we decide to accept or reject the trial step d by a certain acceptant criteria such as (2.3). The new trial point $x^{(k+1)} = x^{(k)} + d^{(k)}$ is accepted by the filter if the corresponding pair $(h^{(k+1)}, f^{(k+1)})$ is not dominated by any point in the filter. Otherwise the step is rejected and the trust region radius $\rho^{(k)}$ is reduced. Thus the usual criteria of descent in the merit function are replaced by the requirement that the new point is acceptable to the filter. It is easily to see that the filter criteria are more looser than the usual criteria of descent in the merit function which could lead a minimalistic approach to SQP. Obviously, (2.1) may be inconsistent and if this case occurs the algorithm enters the restoration phase.

The aim in the restoration phase is to find a point $x^{(k+1)}$ which both is acceptable to the filter $\mathcal{F}^{(k)}$ and makes the QP-subproblem is consistent. Various restoration algorithms can be used and Fletcher et al. in [4] still suggest that the SQP-Filter method be used in this phase. To learn the SQP-Filter method suggested by Fletcher and Leyffer in details, one can see [4].

3. The Main Idea of Our SQP-Filter Method

We notice that the acceptant criteria suggested by Fletcher and Leyffer are generally more looser than line search conditions on penalty function which probably can decrease the number of the computation of QP steps. But this also may invoke such a problem that the convergence is very slow when the iterative points are far from the optimal point. The above disadvantage is partly caused by the use of the same acceptant criteria which ignore the iterative points are near to or far from the optimal point. For that reason we present a new SQP-Filter method.

The main idea of our SQP-Filter method is as follows. When the iterates are far from the optimal points, the filter acceptant criteria are much stricter than those presented by Fletcher and Leyffer which can impede the iterates come close to the optimal point more quickly; on the other hand when iterates come nearer to the optimal points the criteria of our new method are much looser than those proposed by Fletcher and Leyffer which improves the local convergence property. So we give different acceptant criteria according to the measurement of constraint violation $h(c(x))$. We give three types of filters, that is to say, $\mathcal{F}_{coarse}, \mathcal{F}_{moderate}$ and $\mathcal{F}_{fine}$ corresponding to the situations that $h(c(x))$ is large, moderate and small respectively. In different types of filters we decide to accept or reject a point by the criteria of the corresponding filter. We call x acceptable to the filter $\mathcal{F}^{(k)}$(here $\mathcal{F}^{(k)}$ is one type of filters described above) if for all $(h^{(j)}, f^{(j)}) \in \mathcal{F}^{(k)}$, at least one of the following two inequalities

$$f(x) \leq f^{(j)} + \tan(\pi/4 - \theta^{(j)})(h(c(x)) - \beta h^{(j)}) - \gamma h(c(x)), \tag{3.1}$$

$$f(x) \leq f^{(j)} + \tan(\pi/4 + \theta^{(j)})(h(c(x)) - \beta h^{(j)}) - \gamma h(c(x)), \tag{3.2}$$

holds, where $\theta^{(j)}$ adaptively adjusts by the measurement of constraint violation. Now we describe the method that adjusts the parameter $\theta^{(j)}$ in details.

(1) *When $h(c(x))$ is small, firstly we use* (2.3) *to decide whether to accept the point x or not. If x could be accepted by the fine filter($\mathcal{F}^{(k)}_{fine}$), then we let $x^{(k+1)} = x$, possibly increase the trust region radius $\rho^{(k)}$ and go to the next iteration. Otherwise we do not reject the point immediately as Fletcher and Leyffer has done in* [*4*]. *We continue to compute a trial step $\hat{d}$ at point x and judge whether the point $x + \hat{d}$ could be accepted by the fine filter. If $x + \hat{d}$ could be accepted then we let $x^{(k+1)} = x + \hat{d}$, and go to the next iteration. Otherwise we decrease the trust region radius and solve the QP-subproblem (*2.1*) at point $x^{(k)}$ again. When the constraint violation is small, the advantage of our method lies in it can accept the QP step at the greatest possibility. This could partly overcome the slow local convergence near the optimal point.*

(2) When the measurement of $h(c(x))$ is moderate, *we choose* $\theta^{(j)} = \pi/4$. In this case our filter acceptant criteria are *the same as those proposed by Fletcher and Leyffer.*

(3) When the measurement of $h(c(x))$ is large, *we choose $\theta^{(j)}$ in the interval* $(\pi/4, \pi/2)$. *In this case our filter acceptant criteria are much stricter than those proposed by Fletcher and Leyffer in* [*4*]. *This could prevent the iterative points from converging slowly to the optimal point when they are far from the optimal points.*

The filter acceptant criteria above are related to the measurement of constraint violation, so now we are in the position to give some standards about how to measure the constraint violation.

(1) *If the constraint violation $h(c(x))$ is numerically small or it is relatively small to the initial constraint violation $h^{(0)}$, that is to say $h(c(x)) \leq max\{M \cdot \epsilon, 0.01h^{(0)}\}$, we consider the constraint violation is small.* Here ϵ is the tolerance of error and $M > 0$ is a big real number. For convenience we use $hbound1$ to denote $max\{M \cdot \epsilon, 0.01h^{(0)}\}$.

(2) *If at point x the constraint violation $h(c(x))$ is no less than $max\{0.5h^{(0)}, 0.1ubd\}$, then we consider that $h(c(x))$ is large.* Here "u" is defined as one upper bound of constraint violation and is the same as the one defined in [4] by Fletcher and Leyffer. For convenience we use $hbound2$ to denote $max\{0.5h^{(0)}, 0, 1u\}$.

(3) *If the constraint violation $h(c(x))$ is in the interval (ubound1, ubound2), then we consider the measurement of the constraint violation is moderate.*

The different filters of our new SQP-Fiter method is illustrated in Figure 3.1.

Now we give definitions of dominance and a filter. Firstly we give the definition of dominance.

Definition 3.1. *A pair $(h^{(k)}, f^{(k)})$ is said to dominate another pair $(h^{(l)}, f^{(l)})$ if and only if (1) If $h^{(k)}$ is small or moderate, then both $h^{(k)} \leq \beta h^{(l)}$ and $f^{(k)} \leq f^{(l)} - \gamma h^{(k)}$ hold. (2) If $h^{(k)}$ is large, then both $f^{(k)} \leq f^{(l)} + \tan(\pi/4 - \theta^{(l)})(h^{(k)} - \beta h^{(l)}) - \gamma h^{(k)})$ and $f^{(k)} \leq f^{(l)} + \tan(\pi/4 + \theta^{(l)})(h^{(k)} - \beta h^{(l)}) - \gamma h^{(k)})$ hold.*

Now we give the concept of the filter about our new method.

Definition 3.2. *A filter is a list of pairs $((h^{(l)}, f^{(l)}))$ such that no pair dominates any other(Because there are three types of filters, that is to say $\mathcal{F}_{coarse}$,$\mathcal{F}_{moderate}$ and $\mathcal{F}_{fine}$, the relationship of dominance is considered in its own filter). A pair $(h^{(k)}, f^{(k)})$ is said to be acceptable for inclusion in the filter if it is not dominated by any point in the filter.*

For convenience, from now on we use the point to replace the pair in a certain filter. For instance, we use $x^{(l)}$ instead of the corresponding pair $((h^{(l)}, f^{(l)}))$.

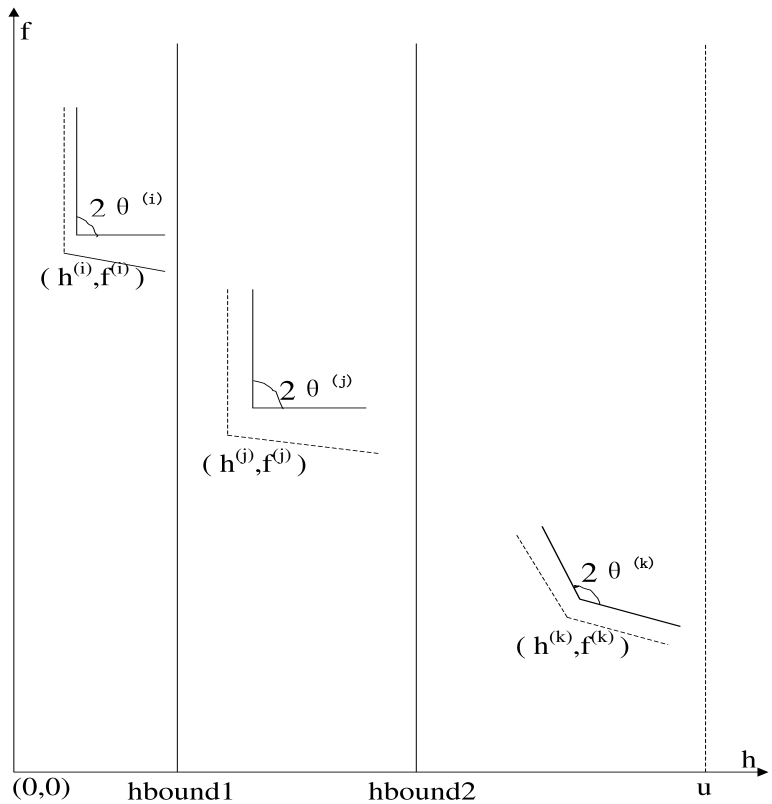

Figure 3.1:An NLP Filter with slanting Envelope

At the end of this section, we give an example to illustrate the difference between our new SQP-Filter method and the SQP-Filter method proposed by Fletcher et al. in [4].
Example 3.1(see [15], P.539)

$$\begin{array}{rl} min & f(u,v) = 3v^2 - 2u \\ s.t. & c(u,v) = u - v^2 = 0 \end{array} \tag{3.3}$$

It is easily to know that there is only one optimal solution $x^* = (0,0)^T$ of the problem (3.3). We consider the point $\bar{x}(\varepsilon) = (u(\varepsilon), v(\varepsilon))^T = (\varepsilon^2, \varepsilon)^T$ which is close to the optimal solution x^* if ε is sufficiently small. We choose $\nabla\mathbb{L}(x^*, \lambda^*)$ as Hessian matrix, so the QP-subproblem corresponding to the point $\bar{x}(\varepsilon)$ has the following form

$$\begin{array}{ll} min & d^T \begin{pmatrix} -2 \\ 6\varepsilon \end{pmatrix} + \frac{1}{2} d^T \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} d, \\ s.t. & d^T \begin{pmatrix} 1 \\ -2\varepsilon \end{pmatrix} = 0, \\ & \|d\|_\infty \le \rho^{(0)}, \end{array} \tag{3.4}$$

where $d \in \Re^2$ and $\rho^{(0)}$ is the trust region radius. We can know that the solution of QP-subproblem (3.4) is

$$\bar{d}(\varepsilon) = \begin{pmatrix} -2\varepsilon^2 \\ -\varepsilon \end{pmatrix} \tag{3.5}$$

We know that $\bar{d}(\varepsilon)$ is a superlinearly convergent step. At point $\bar{x}(\varepsilon) + \bar{d}(\varepsilon)$, both the objective function value and the constraint violation become larger than those corresponding to the point $\bar{x}(\varepsilon)$. So this superlinearly convergent step $\bar{d}(\varepsilon)$ will be rejected by the filter criteria presented in [4]. But this superlinearly convergent step could be accepted by our method because $h(c(\bar{x}(\varepsilon) + \bar{d}(\varepsilon)))$ is small.

4. The Algorithm and Its Convergence Results

4.1 The Algorithm of the New SQP-Filter Method

Suppose that the initial point $x^{(0)}$ is given, how can we solve the NLP problem (1.1) with the SQP-Filter method? First of all, the SQP-Filter method still use SQP to solve the SQP-Filter method. Hence the first step is to give out the following QP-subproblem

$$\begin{array}{ll} min & \Psi^{(k)}(d) = \frac{1}{2} d^T B^{(k)} d + d^T g^{(k)} \\ s.t. & c_i^{(k)} + d^T a_i^{(k)} = 0, i = 1, \ldots, m_e, \\ & c_i^{(k)} + d^T a_i^{(k)} \ge 0, i = m_e + 1, \ldots, m, \\ & \|d\|_\infty \le \rho^{(k)}. \end{array} \tag{4.1}$$

From now on we use $QP(x^{(k)}, \rho^{(k)})$ to denote the QP-subproblem above. There is a slight difference between (2.1) and (4.1), that is to say, the Hessian matrix in (4.1) is not an exact Hessian matrix but an approximate matrix updated by the BFGS formulae presented by Powell in [11, 12] or presented by Stoer in [14]. The new Hessian matrix $B^{(k+1)}$ is just relevant to $B^{(k)}, y^{(k)}$ and the solution $d^{(k)}$, where the vector $y^{(k)}$ is defined as follows

$$y^{(k)} = \nabla_x \mathbb{L}(x^{(k+1)}, \lambda^{(k)})) - \nabla_x \mathbb{L}(x^{(k)}, \lambda^{(k)})). \tag{4.2}$$

In general we let the unit matrix as the initial Hesse and $B^{(k+1)} = \prod(B^{(k)}, \hat{y}^{(k)}, d^{(k)})$, where $\hat{y}^{(k)}$ and $\prod(B, y, d)$ are defined as follows

$$\prod(B, y, d) = B + \frac{yy^T}{y^T d} - \frac{Bdd^T B}{d^T B d}, \tag{4.3}$$

$$\begin{cases} \hat{y}^{(k)} = y^{(k)}, if \quad d^{(k)T} \hat{y}^{(k)} > 0, \\ \hat{y}^{(k)} = y^{(k)} + 0.1 d^{(k)}, if \quad d^{(k)T} \hat{y}^{(k)} = 0. \end{cases} \tag{4.4}$$

And we let $B^{(k+1)}$ equal to $B^{(k)}$ if $(\hat{y}^{(k)})^T d^{(k)}$ is less than 0.

When we try to solve the QP-subproblem (4.1), the following two cases need to be considered. (1) The QP-subproblem (4.1) is consistent. In this case we can get the solution $d^{(k)}$ of (4.1).

If this trial step could be accepted by our new filter acceptant criteria then we let $x^{(k+1)} = x^{(k)} + d^{(k)}$, increase the trust region radius and solve the QP-subproblem (4.1) at the new point again. Otherwise we would decrease the trust region radius and solve the QP-subproblem (4.1) again.
(2) The QP-subproblem (4.1) is inconsistent. The algorithm will enter the restoration phase to find a point $x^{(k+1)}$ at which both the QP-subproblem (4.1) is consistent and $x^{(k+1)}$ is acceptable to $\mathcal{F}^{(k)}$. And we solve the QP-subproblem (4.1) at the new point again.

For the restoration algorithm plays an important part in the SQP-Filter method, now we describe our restoration algorithm in details. When the QP-subproblem (4.1) is inconsistent, the QP-solver will exit with a feasible solution $d^{(k)}$ of the following *Phase II* subproblem

$$\begin{aligned} min \quad & \sum_{i\in J_{\mathcal{E}}^{(k)}} |d^T a_i^{(k)}| - \sum_{i\in J_{\mathcal{I}}^{(k)}} d^T a_i^{(k)} \\ s.t. \quad & c_i^{(k)} + d^T a_i^{(k)} = 0, i \in {J_{\mathcal{E}}^{(k)}}^{\perp}, \\ & c_i^{(k)} + d^T a_i^{(k)} \geq 0, i \in {J_{\mathcal{I}}^{(k)}}^{\perp}, \\ & \|d\|_\infty \leq \rho^{(k)}, \end{aligned} \tag{4.5}$$

where ${J_{\mathcal{E}}^{(k)}}^{\perp} = \{i||c_i^{(k)} + d^{(k)T} a_i^{(k)}| \leq \varepsilon, i \in \mathcal{E}\}$, $J_{\mathcal{E}}^{(k)} = \mathcal{E} \setminus {J_{\mathcal{E}}^{(k)}}^{\perp}$, ${J_{\mathcal{I}}^{(k)}}^{\perp} = \{i|c_i^{(k)} + d^{(k)T} a_i^{(k)} > 0, i \in \mathcal{I}\}$ and $J_{\mathcal{I}}^{(k)} = \mathcal{I} \setminus {J_{\mathcal{I}}^{(k)}}^{\perp}$.

Because the aim of the restoration phase is to find a point $x^{(k+1)}$ such that the QP-subproblem (4.1) is consistent at that point, we now turn to consider the following problem

$$\begin{aligned} min \quad & \sum_{i\in J_{\mathcal{E}}^{(k)}} (\pm) c_i(x) - \sum_{i\in J_{\mathcal{I}}^{(k)}} c_i(x) \\ s.t. \quad & c_i(x) = 0, i \in {J_{\mathcal{E}}^{(k)}}^{\perp}, \\ & c_i(x) \geq 0, i \in {J_{\mathcal{I}}^{(k)}}^{\perp}, \\ & \|d\|_\infty \leq \rho^{(k)}. \end{aligned} \tag{4.6}$$

"+" or "−"in the first term of objective function is decided by $c_i^{(k)} + d^{(k)T} a_i^{(k)}$ greater or no more than 0 respectively. We also use the SQP-Filter method to solve (4.6), that is to say, we solve (4.6) by solving a sequence of the following QP-subproblems($l = 0, 1, \cdots$)

$$\begin{aligned} min \quad & \sum_{i\in J_{\mathcal{E}}^{(k)}} (\pm) d^T a_i^{(k,l)} - \sum_{i\in J_{\mathcal{I}}^{(k)}} d^T a_i^{(k,l)} + \tfrac{1}{2} d^T B_{res}^{(k,l)} d \\ s.t. \quad & c_i^{(k,l)} + d^T a_i^{(k,l)} = 0, i \in {J_{\mathcal{E}}^{(k)}}^{\perp} \\ & c_i^{(k,l)} + d^T a_i^{(k,l)} \geq 0, i \in {J_{\mathcal{I}}^{(k)}}^{\perp} \\ & \|d\|_\infty \leq \rho^{(k,l)}, \end{aligned} \tag{4.7}$$

where the matrix $B_{res}^{(k,l)}$ is the Hessian matrix of the restoration phase and can be updated by the BFGS formulae.

We judge the consistency of the following LP-subproblem at point $x^{(k,l+1)} = x^{(k,l)} + d^{(k,l)}$ when we have successfully gotten the solution $d^{(k,l)}$ of (4.7)

$$\begin{aligned} & d^T a_i^{(k,l+1)} + c_i^{(k,l+1)} = 0, i \in \mathcal{E}, \\ & d^T a_i^{(k,l+1)} + c_i^{(k,l+1)} \geq 0, i \in \mathcal{I}, \\ & \|d\|_\infty \leq \rho^{(k,l)}. \end{aligned} \tag{4.8}$$

If the subproblem (4.8) is consistent and $x^{(k,l+1)}$ is acceptable to the normal(*Phase I*) filter, the algorithm return to the normal SQP. Otherwise we judge whether the point $x^{(k,l+1)}$ could be accepted by the restoration phase (*Phase II*) filter or not. If $x^{(k,l+1)}$ could be accepted by the *Phase II* filter, then the *Phase II* would be updated and the restoration algorithm continues to solve (4.7) at the point $x^{(k,l+1)}$. Otherwise we decrease the trust region radius and continue to solve (4.7) at the point $x^{(k,l)}$. The process above would be repeated until the subproblem (4.8) is consistent.

For convenience we give some notations

$$\Delta\Psi^{(k)} = \Psi^{(k)}(0) - \Psi^{(k)}(d) = -d^T g^{(k)} - \tfrac{1}{2} d^T B^{(k)} d,$$
$$\Delta f^{(k)} = f(x^{(k)}) - f(x^{(k)} + d).$$

The complete algorithm of our new SQP-Filter method and the restoration algorithm can be stated as follows.

Algorithm 4.1. *(A new SQP-Filter method)*

Step 0 *Given* $x^{(0)} \in \Re^n$, $\rho^{(0)} > 0$, *ubd, tt, tolerance of error* ϵ, $\sigma \in (0,1)$, $B^{(0)} \in \Re^{n\times n}$ *symmetric, numberhsmall* $= 0$, *compute* $f(x^{(0)})$ *and* $c_i(x^{(0)})$, $i = 1, 2, \cdots, m$. *Let* $u = max\{ubd, tt \cdot h^{(0)}\}$ *and* $\mathcal{F}^{(0)}_{coarse} = \{(u, -\infty)\}$, $\mathcal{F}^{(0)}_{fine} = \mathcal{F}^{(0)}_{moderate} = \emptyset$, $k := 0$;

Step 1 *compute* $g^{(k)} = \nabla f(x^{(k)}), a_i^{(k)} = \nabla c_i(x^{(k)})$, $i = 1, 2, \cdots, m$, *generate the Hesse* $B^{(k)}$ *by the BFGS formulae if k is greater than 0;*
generate the QP-subproblem (4.1)*;*

Step 2 *Try to solve QP-subproblem* (4.1)*;*
If QP-subproblem (4.1) *is inconsistent then*

if numberhsmall does not equal to 1, then add $(h^{(k)}, f^{(k)})$ *into the filter and enter the restoration phase (see Algorithm* 4.2*), return to normal SQP when* $x^{(k+1)}$ *is found where the subproblem* (4.8) *is feasible, initialize* $\rho = \rho^0$ *and go to Step 1;*

else $\rho^{(k)} = \rho^{(k)}/2$, $x^{(k)} =$*tempx*, *numberhsmall* $= 0$, *go to Step 2;*

Else go to the Step 3;

Step 3 *Use d to denote the solution of* (4.1)*;*
if $d = 0$, *then the KT point of* (1.1) *has been found, stop;*
otherwise compute $f(x^{(k)}+d)$, $h(c(x^{(k)}+d))$, *judge whether to accept* $x^{(k)}+d$ *in its own filter (according to the measurement of* $h(c(x^{(k)} + d))$ *to decide which filter would be used);*
if $x^{(k)} + d$ *could be accepted,go to Step 4; otherwise go to Step 5;*

Step 4 *If* $\Delta f < \sigma\Delta\Psi$ *and* $\Delta\Psi > 0$ *then goto step 5;*
else let $d^{(k)} = d$, $x^{(k+1)} = x^{(k)} + d^{(k)}$*;*
add $x^{(k+1)}$ *in corresponding filter while* $\Delta\Psi \le 0$ *and update the filter;*
initialize trust region radius as $\rho^{(0)}$, $k := k + 1$ *and go to Step 1;*

Step 5 *If* $h(c(x^{(k)} + d)) \le hbound1$ *then go to step 6;*
else let $\rho^{(k)} = \rho^{(k)}/2$, *numberhsmall* $= 0$, *go to Step 2;*

Step 6 *If numberhsmall equals to 1 then let* $x^{(k)} =$*tempx*, $h^{(k)} =$*temph*, $f^{(k)} =$*tempf*, $\rho^{(k)} = \rho^{(k)}/2$, *numberhsmall* $= 0$, *go to Step 2;*
else let numberhsmall $=$ *numberhsmall* $+ 1$, *temph* $= h^{(k)}$, *tempf* $= f^{(k)}$, *tempx* $= x^{(k)}$, *go to Step 2;*

The algorithm above can also be given in the following diagram(the loop in the dashed line is inner loop).

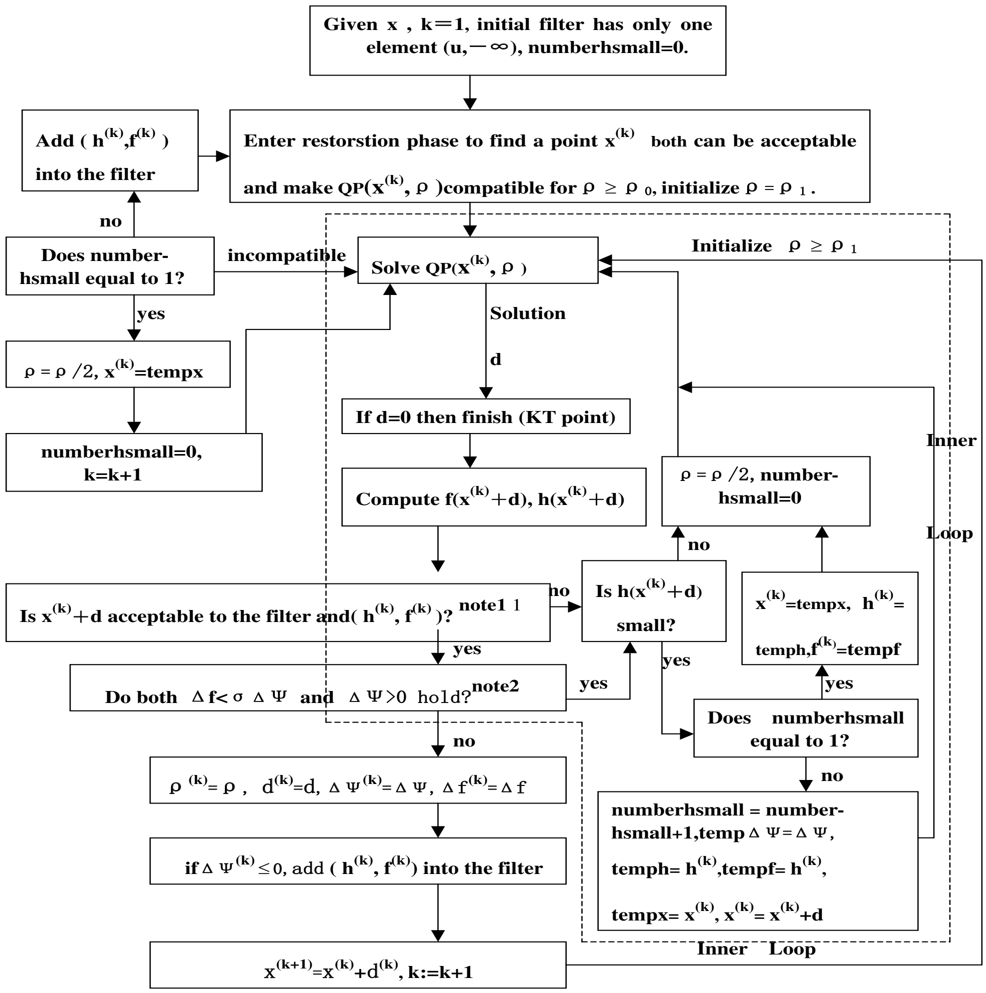

Figure 4.1

Algorithm 4.2. *(The algorithm of restoration phase)*

Step 0 Let $x^{(k,0)} = x^{(k)}$, $\rho^{(k,0)} = \rho^{(k)}$, $c_i^{(k,0)} = c_i^{(k)}$, $a_i^{(k,0)} = \nabla c_i^{(k)}$;
get sets ${J_{\mathcal{E}}^{(k)}}^{\perp}$, $J_{\mathcal{E}}^{(k)}$, ${J_{\mathcal{I}}^{(k)}}^{\perp}$ and $J_{\mathcal{I}}^{(k)}$;
give out the approximate Hesse $B_{res}^{(k,0)}$ and let $\mathcal{F}_{res}^{(k,0)} = \{x^{(k)}\}$, $l := 0$;

Step 1 *Solve the QP-subproblem* (4.7) *of restoration phase;*
use d *to denote the solution and temporarily let* $x^{(k,l+1)} = x^{(k,l)} + d$;
for $i = 1, \cdots, m$, *compute* $c_i^{(k,l+1)} = c_i(x^{(k,l+1)})$ *and* $a_i^{(k,l+1)} = \nabla c_i(x^{(k,l+1)})$;

Step 2 *Judge feasibility of the subproblem* (4.8)*;*
if (4.8) *is feasible then remove all elements in* Phase II *filter and return to the normal SQP phase;*
otherwise go to Step 3;

Step 3 *If* $x^{(k,l+1)}$ *could be accepted by the* Phase II *filter, then add it to the filter and update the filter, increase the trust region radius, update the approximate Hesse* $B^{(k,l+1)}$ *by the BFGS formulae and let* $l := l+1$, *go to Step 1;*
Otherwise compute the second order correction step $\hat{d}^{(k,l)}$ *and let* $\hat{x}^{(k,l+1)} = x^{(k,l)} + d + \hat{d}^{(k,l)}$, *go to Step 4;*

Step 4 *If* $\hat{x}^{(k,l+1)}$ *could be accepted by the* Phase II *filter, then add it to the filter and update the filter, let* $x^{(k,l+1)} = \hat{x}^{(k,l+1)}$, *increase the trust region radius, update the approximate Hesse* $B^{(k,l+1)}$ *by the BFGS formulae and let* $l := l+1$, *go to Step 1;*
Otherwise let $x^{(k,l+1)} = x^{(k,l)}$ *and decrease the trust region radius,* $l := l+1$, *go to step 1;*

4.2 Convergence Results of the New Algorithm

In the section we will give some convergence results about our new SQP-Filter algorithm. First of all we give some assumptions.

Assumption 4.3. *(Standard Assumptions)*

1. *All points* x *that are sampled by our algorithm lie in a nonempty closed and bounded set* X*;*
2. *The objective function and constraint functions* $c_i(x)(i = 1, \cdots, m)$ *are twice continuously differentiable on an open set containing* X*;*
3. *There exists an* $\bar{M} > 0$ *such that the Hessian matrices* $B^{(k)}$ *satisfy* $\|B^{(k)}\|_2 \le \bar{M}$ *for all* k.

To prove the convergence results, we give some lemmas at first.

Lemma 4.4. *(inclusion property) If a pair* (h, f) *is acceptable to the filter, then the area rejected by the new filter includes the one rejected by the old filter.*

Proof. We consider three conditions, that is, the measurement of h is small, moderate and large. We use $\bar{\mathcal{F}}_{new}$ and $\bar{\mathcal{F}}_{old}$ to denote the area rejected by the new and the old filter respectively. We will give proof to these three situations in the following. (1) The measurement of h is large. In this case the constraint violation h is in the interval $(hbound2, u)$ and the filter acceptant criteria of our new method are that (3.1) holds or (3.2) holds. Suppose that (a, b) belongs to $\bar{\mathcal{F}}_{old}$, then there exists a pair $(h^{(l)}, f^{(l)}) \in \mathcal{F}_{old}$ such that both

$$b > f^{(l)} + \tan(\pi/4 - \theta)(a - \beta h^{(l)}) - \gamma a \tag{4.9}$$

and

$$b > f^{(l)} + \tan(\pi/4 + \theta)(a - \beta h^{(l)}) - \gamma a \tag{4.10}$$

hold, where $\theta \in (\pi/4, \pi/2)$ is a constant. Now if $(h^{(l)}, f^{(l)}) \in \mathcal{F}_{new}$, then obviously $(a, b) \in \bar{\mathcal{F}}_{new}$ holds and the lemma is true. So we just need to consider the case that the pair $(h^{(l)}, f^{(l)})$ does not belong to $\mathcal{F}_{new}$. Because we only add one pair (h, f) into $\mathcal{F}_{old}$ which leads to $(h^{(l)}, f^{(l)})$ be removed from $\mathcal{F}_{old}$. We can conclude that the pair $(h^{(l)}, f^{(l)})$ is dominated by (h, f), i.e., both

$$f^{(l)} > f + \tan(\pi/4 - \theta)(h^{(l)} - \beta h) - \gamma h^{(l)} \tag{4.11}$$

and

$$f^{(l)} > f + \tan(\pi/4 + \theta)(h^{(l)} - \beta h) - \gamma h^{(l)} \tag{4.12}$$

hold. From (4.9) and (4.11) we know that

$$\begin{aligned} b &> \{f + \tan(\pi/4 - \theta)(h^{(l)} - \beta h) - \gamma h^{(l)}\} + \tan(\pi/4 - \theta)(a - \beta h^{(l)}) - \gamma a \\ &= \{f + \tan(\pi/4 - \theta)(a - \beta h) - \gamma a\} + \{\tan(\pi/4 - \theta)(h^{(l)} - \beta h^{(l)}) - \gamma h^{(l)}\}. \end{aligned}$$

Because $\theta \in (\pi/4, \pi/2)$ we know that $\tan(\pi/4 - \theta) < 0$ which leads to

$$tan(\pi/4 - \theta)(h^{(l)} - \beta h^{(l)}) - \gamma h^{(l)} < 0.$$

So we know that

$$b > f + \tan(\pi/4 - \theta)(a - \beta h) - \gamma a.$$

Similarly we can get

$$b > f + \tan(\pi/4 - \theta)(a + \beta h) - \gamma a.$$

The above two inequalities imply that $(a, b) \in \bar{\mathcal{F}}_{new}$.

(2) The measurement of h is moderate. In this case the constraint violation h is in the interval$(hbound1, hbound2)$ and the filter acceptant criteria of our new method are the same as those proposed by Fletcher et al.[4]. Suppose that (a, b) belongs to $\bar{\mathcal{F}}_{old}$, then there exists a pair $(h^{(l)}, f^{(l)}) \in \mathcal{F}_{old}$ such that both $a > \beta h^{(l)}$ and $b > f^{(l)} - \gamma h^{(l)}$ hold. If $(h^{(l)}, f^{(l)}) \in \mathcal{F}_{new}$, then obviously $(a, b) \in \bar{\mathcal{F}}_{new}$ holds and the lemma is true. So we just need to consider the case that the pair $(h^{(l)}, f^{(l)})$ does not belong to $\mathcal{F}_{new}$. Because we only add one pair (h, f) into $\mathcal{F}_{old}$ which leads to $(h^{(l)}, f^{(l)})$ be removed from $\mathcal{F}_{old}$. We can conclude that the pair $(h^{(l)}, f^{(l)})$ is dominated by (h, f), i.e.,

$$h \le \beta h^{(l)} \quad \text{and} \quad f \le f^{(l)} - \gamma h^{(l)}.$$

It is easily seen from the above inequalities that the following equalities hold

$$a > \beta h^{(l)} > h > \beta h,$$
$$b > f^{(l)} - \gamma h^{(l)} > f > f - \gamma h.$$

These imply that the pair (a, b) belongs to $\bar{\mathcal{F}}_{new}$.

(3) h is small. In this case the proof is similar to the case (2).

Before giving the main convergence theorem we will give some lemmas and theorems.

Theorem 4.5. *If $f(x)$ is bounded below on $\mathcal{F}_{coarse}$ generated by Algorithm 4.1, $\mathcal{F}_{coarse}$ is finite.*

Proof. We suppose that $\mathcal{F}_{coarse} = \{(h^{(l_1)}, f^{(l_1)}), \cdots, (h^{(l_k)}, f^{(l_k)}), (h^{(l_{k+1})}, f^{(l_{k+1})}), \cdots\}$. We know that either (3.1) or (3.2) holds for the elements in $\mathcal{F}_{coarse}$. They are equivalent to one of the following three conditions

(1) both $hbound2 \le h^{(l_{k+1})} \le \beta h^{(l_k)}$ and $f^{(l_k)} - \gamma h^{(l_{k+1})} \le f^{(l_{k+1})} \le f^{(l_k)} - \gamma h^{(l_{k+1})} + \tan(\pi/4 + \theta^{(l_k)})(h^{(l_{k+1})} - \beta h^{(l_k)})$,

(2) both $h^{(l_{k+1})} \le \beta h^{(l_k)}$ and $f^{(l_{k+1})} \le f^{(l_k)} - \gamma h^{(l_{k+1})}$,

(3) both $u \ge h^{(l_{k+1})} \ge \beta h^{(l_k)}$ and $f^{(l_{k+1})} \le f^{(l_k)} - \gamma h^{(l_{k+1})} + \tan(\pi/4 - \theta^{(l_k)})(h^{(l_{k+1})} - \beta h^{(l_k)})$.

According to (1), we know that $h^{(l_{k+1})} \le \beta h^{(l_k)}$ hold; About (3), according to $h^{(l_{k+1})} \ge \beta h^{(l_k)}$ and $\tan(\pi/4 - \theta^{(l_k)}) < 0$, we can conclude that

$$f^{(l_{k+1})} \le f^{(l_k)} - \gamma h^{(l_{k+1})} + \tan(\pi/4 - \theta^{(l_k)})(h^{(l_{k+1})} - \beta h^{(l_k)}) \le f^{(l_k)} - \gamma h^{(l_{k+1})}.$$

So for all $(h^{(l_k)}, f^{(l_k)}) \in \mathcal{F}_{coarse}$, either $h^{(l_{k+1})} \le \beta h^{(l_k)}$ or $f^{(l_{k+1})} \le f^{(l_k)} - \gamma h^{(l_{k+1})}$ holds.

Now if $\mathcal{F}_{coarse}$ is infinite, then exact one of the following two conclusions holds

(I) The sequence $\mathcal{F}_{coarse}$ has one accumulation point,

(II) The sequence $\mathcal{F}_{coarse}$ has no accumulation points.
If (I) holds and we suppose that one accumulation point is $\eta(\geq hbound2 > 0)$, we can find an infinite subsequence $\{x^{(l_k)}\}_{l_k\in\mathcal{Q}} \subseteq \mathcal{F}_{coarse}$ on which

$$|h^{(l_k)} - \eta| < (\frac{1-\beta}{1+\beta})\eta. \tag{4.13}$$

Let l_k^+ denote the successor to l_k in $\mathcal{Q}$. It follows that $h^{(l_k^+)} > 2\eta\beta/(1+\beta)$ and $h^{(l_k)} < 2\eta/(1+\beta)$ and hence $h^{(l_k^+)} > \beta h^{(l_k)}$. Because the pair $(h^{(l_k^+)}, f^{(l_k^+)})$ is added into the filter, it is a consequence of the inclusion property that it is acceptable to $(h^{(l_k)}, f^{(l_k)})$, even if the latter pair has been deleted from the filter on an intermediate iteration. According to $h^{(l_k^+)} > \beta h^{(l_k)}$ we know that $f^{(l_k)} - f^{(l_k^+)} \geq \gamma h^{(l_k^+)} > 0$. Thus $f^{(l_k)}$ is monotonically decreasing for all $l_k \in \mathcal{Q}$. Summing over all indices $l_k \in \mathcal{Q}$ contradicts the fact that $f^{(l_k)}$ is bounded below. So (I) does not hold.
If (II) holds then it follows that we can find an infinite subsequence $\{x^{(l_k)}\}_{l_k\in\mathcal{Q}} \subseteq \mathcal{F}_{coarse}$ on which $h^{(l_k)} > \eta$ is monotonically increasing, where $\eta > 0$ is a constant. So it follows that $f^{(l_k)} - f^{(l_k^+)} \geq \gamma h^{(l_k^+)} \geq \gamma\eta$, with the same conclusion. So (II) does not hold.
Consequently, the assumption that $\mathcal{F}_{coarse}$ is infinite does not hold.

Similar to Theorem 4.5, we can get the following results.

Theorem 4.6. *If $f(x)$ is bounded below on $\mathcal{F}_{moderate}$ and $\mathcal{F}_{fine}$ generated by Algorithm 4.1, then*
(1) $\mathcal{F}_{moderate}$ is finite ;
(2) If $\mathcal{F}_{fine}$ is infinite, we can get that $h^{(k)} \to 0$, for $(h^{(k)}, f^{(k)}) \in \mathcal{F}_{fine}$.

Proof. The proof of case (1) is similar to that of Theorem 4.5. Now we just give a proof to case (2). If it were not true, $\{h^{(l_k)}\}_{l_k\in\mathcal{F}_{fine}}$ would have an accumulation point $\eta > 0$. Define $\mathcal{Q}$ and l_k^+ as in Theorem 4.5. Because the pair $(h^{(l_k^+)}, f^{(l_k^+)})$ is added into the filter, it is a consequence of the inclusion property that it is acceptable to $(h^{(l_k)}, f^{(l_k)})$, even if the latter pair has been deleted from the filter on an intermediate iteration. We know that $h^{(l_k^+)} > \beta h^{(l_k)}$ is not satisfied, and hence that $f^{(l_k)} - f^{(l_k^+)} \geq \gamma h^{(l_k^+)} > 0$. Thus $f^{(l_k)}$ is monotonically decreasing for all $l_k \in \mathcal{Q}$. Summing over all indices $l_k \in \mathcal{Q}$ contradicts the fact that $f^{(l_k)}$ is bounded below. There is also the possibility that $\{h^{(l_k)}\}$ has no accumulation points. In this situation we can find an infinite subsequence $\{x^{(l_k)}\}_{l_k\in\mathcal{Q}} \subseteq \mathcal{F}_{coarse}$ on which $h^{(l_k)} > \eta$ is monotonically increasing, where $\eta > 0$ is a constant. So we know that $f^{(l_k)} - f^{(l_k^+)} \geq \gamma h^{(l_k^+)} \geq \gamma\eta$, with the same conclusion. So we know $f^{(l_k)} \to 0$ for $l_k \in \mathcal{F}_{fine}$.

A common feature in a trust region algorithm for unconstrained minimization is the use of a sufficient reduction criterion

$$\Delta f \geq \sigma\Delta\Psi \tag{4.14}$$

where $\Delta\Psi$ is positive, and $\sigma \in (0, 1)$ is a preset parameter. However, in an NLP algorithm, $\Delta\Psi$ may be negative or even 0, in which case this test is no longer appropriate. A feature of our algorithm that it uses (4.14) only when $\Delta\Psi$ is greater than 0.

For a trial step $d^{(k)}$ which could be accepted by the filter, we call it an *f-type step* if both (4.14) and $\Delta\Psi > 0$ hold and this iteration is called an *f-type iteration*; we call it an *h-type step* if either $\Delta\Psi \leq 0$ holds or QP-subproblem (4.1) is inconsistent and this iteration is called an *h-type iteration*. As the trust region radius is reduced in the inner loop, the value of $\Delta\Psi$ is reduced. Thus the status test $\Delta\Psi > 0$ may go from true to false, but not vice-versa. Consequently, the inner loop always samples the possibility for an *f-type iteration* before that of an *h-type iteration*.

We notice that not all points that could be accepted by the filter are added into the filter but only those that are generated by *h-type iterations*. One direct result is that the following lemma holds.

Lemma 4.7. *For any $(h^{(j)}, f^{(j)})$ in the current filter $\mathcal{F}^{(k)}$, it follows that $h^{(j)} > 0$.*

Proof. We suppose that the conclusion is false, then there exists a pair $(h^{(l)}, f^{(l)}) \in \mathcal{F}^{(k)}$ such that $h^{(l)}$ equals to 0. As $d = 0$ is a feasible solution of QP-subproblem (4.1), QP-subproblem (4.1) is always consistent. Thus if $x^{(l)}$ is not a KT point then $\Delta\Psi^{(l)}$ is greater then 0. Consequently, the successive iteration is an *f-type iteration* by the fact that $h^{(l)}$ equals to 0 and $x^{(l)}$ could not been added into the filter. This leads to a contradiction. Thus, the lemma is true.

For convenience, we also give the following notation as Fletcher et al. has done in [6]

$$\tau^{(k)} = \min_{j \in \mathcal{F}^{(k)}} h^{(j)} > 0.$$

We notice that only *h-type iterations* can reset $\tau^{(k)}$. Now we list two lemmas presented in [6].

Lemma 4.8. *Consider minimizing a quadratic function $\phi(\alpha)(\Re \to \Re)$ on the interval $\alpha \in [0, 1]$, when $\phi'(0) < 0$. A necessary and sufficient condition for the minimizer to be at $\alpha = 1$ is $\phi'' + \phi'(0) \le 0$. In this case it follows that $\phi(0) - \phi(1) \ge -\frac{1}{2}\phi'(0)$.*

Lemma 4.9. *If Assumption* 4.3 *holds, and let $d \ne 0$ be a feasible point of $QP(x^{(k)}, \rho)$. It then follows that*

$$\begin{aligned} &\nabla f \ge \nabla\Psi - n\rho^2\bar{M}, \\ &|c_i(x^{(k)} + d)| \le \tfrac{1}{2}n\rho^2\bar{M}, i = 1, \ldots, m_e, \\ &c_i(x^{(k)} + d) \le \tfrac{1}{2}n\rho^2\bar{M}, i = m_e + 1, \ldots, m. \end{aligned} \tag{4.15}$$

About Algorithm 4.1, we can get the following conclusion.

Lemma 4.10. *If the inner loop does not terminate finitely, it follows that the trust region radius $\rho \to 0$.*

Proof. (1) If d equals to 0, then the inner loop terminates(a KT point is found).
(2) If the point $x^{(k)}+d$ could be accepted by $(h^{(k)}, f^{(k)})$ and $\mathcal{F}^{(k)}$, then the inner loop terminates.
(3) If the point $x^{(k)}+d$ could not be accepted by $(h^{(k)}, f^{(k)})$ and $\mathcal{F}^{(k)}$, then in the two successive iterations, there is at least one iteration which makes ρ be halved. So if we consider every two steps as one iteration, then there is at least one step which makes ρ be halved.

From above we know that the trust region radius $\rho \to 0$.

Lemma 4.11. *If Assumption* 4.3 *holds, then there exists a ρ_0 such that for any $\rho \le \rho_0$, if d solves $QP(x^{(k)}, \rho)$, then $x^{(k)} + d$ is acceptable to the filter.*

Proof. According to the definition of constraint violation and Lemma 4.9, we know the following inequality

$$h(c(x^{(k)} + d)) \le m \cdot \tfrac{1}{2}n\rho^2\bar{M} = \tfrac{mn\rho^2\bar{M}}{2}$$

holds. From above we can see if $\frac{mn\rho^2\bar{M}}{2} \le \beta\tau^{(k)}$ holds, then $x^{(k)} + d$ is acceptable to the filter. We denote $\rho_0 \triangleq \sqrt{\frac{2\beta\tau^{(k)}}{mn\bar{M}}}$. The condition $\frac{mn\rho^2\bar{M}}{2} \le \beta\tau^{(k)}$ is equivalent to $\rho \le \rho_0$.

So we can conclude that for any $\rho \le \rho_0$, if d solves $QP(x^{(k)}, \rho)$, then $x^{(k)} + d$ is acceptable to the filter.

Lemma 4.12. *If Assumption* 4.3 *holds, then the inner loop terminates finitely.*

Proof. (1) $x^{(k)}$ is a KT point, then $d = 0$ is the solution of QP-subproblem (4.1). So the inner loop terminates.
(2) $x^{(k)}$ is not a KT point. In this case if the inner loop does not terminate finitely, then by Lemma 4.10 we know it follows that $\rho \to 0$. The remaining part of this proof is the same with Lemma 6 in [6].

Our global convergence theorem concerns Kuhn-Tucker(KT) necessary condition under a Mangasarian-Fromowitz constraint qualification (MFCQ)(see [9]). Now we give the definition of Mangasarian-Fromowitz constraint qualification.

Definition 4.13. *(Mangasarian-Fromowitz constraint qualification) A feasible point of problem* (1.1) *satisfies MFCQ if and only if both (i)the vectors a_i^0, $i \in \mathcal{E}$ are linearly independent, and (ii)there exists a direction s that satisfies $s^T a_i^0 = 0$, $i \in \mathcal{E}$ and $s^T a_i^0 > 0$, $i \in \mathcal{A}^0$, where $\mathcal{A}^0 \subset \mathcal{I}$ denotes the set of active inequality constraints at x^0.*

Theorem 4.14. *If Assumption* 4.3 *holds and let $x^0 \in X$ be a feasible point of problem* (1.1) *at which MFCQ holds, but which is not a KT point. Then there exists a neighborhood $\mathcal{N}^0$ of x^0 such that $QP(x,\rho)$ is feasible for any $x \in \mathcal{N}^0$, $\rho \in [\delta h(c(x)), b_0]$, where b_0 and $\delta > 0$ are constants.*

Proof. For x^0 is a feasible point but not a point of problem (1.1) at which MFCQ holds, we can get the following conclusion by the definition of MFCQ and one order optimal condition
(1) the vectors a_i^0, $i \in \mathcal{E}$ are linearly independent,
(2) the set $\{s^0 | {s^0}^T g^0 < 0, {s^0}^T a_i^0 = 0, i = 1, 2, \cdots, m_e, {s^0}^T a_i^0 > 0, i \in \mathcal{A}_0\} \neq \emptyset$, i.e., there exists a vector $s^0 \in \Re^n$ such that ${s^0}^T g^0 < 0, {s^0}^T a_i^0 = 0, i = 1, 2, \cdots, m_e$ and ${s^0}^T a_i^0 > 0, i \in \mathcal{A}_0$ hold.

We notice that for any $\alpha \in [0,1]$, $\alpha\rho s^0$ is a feasible point of $QP(x^0,\rho)$. Since s^0 does not equal to 0, for convenience we can suppose that $\|s^0\|_2 = 1$. According to (1) and (2) we can conclude that $card(\mathcal{E}) < n$. Now we give a illustration for the reason. If $card(\mathcal{E}) = n$, then s^0 could be linearly expressed by $a_i^0, i \in \mathcal{E}$, i.e., that there exist constants $l_i (i \in \mathcal{E})$ such that $s^0 = \sum_{i\in\mathcal{E}} l_i a_i^0$. We know that ${s^0}^T s^0 = \sum_{i\in\mathcal{E}} l_i {s^0}^T a_i^0 = \sum_{i\in\mathcal{E}} l_i \cdot 0 = 0$. So s^0 equals to 0 and this contradicts the fact that $\|s^0\|_2$ equals to 1.

$(A^T A)^{-1} A^T$ is a general inverse of matrix A and we denote $A^+ \triangleq (A^T A)^{-1} A^T$. We use $A_\mathcal{E}$ to denote the matrix whose columns are formed by the vectors $a_i(x) = \nabla c_i(x) (i \in \mathcal{E})$. Since (1) holds and $\nabla c_i(x) (i \in \mathcal{E})$ are continuously differentiable, we know that there exists a neighborhood $\mathcal{N}^1$ of x^0 such that for any $x \in \mathcal{N}^1$ the matrix $A_\mathcal{E}^+$ is bounded.

We notice that $-{A_\mathcal{E}^+}^T c_\mathcal{E}$ satisfies equality constraints of problem $QP(x^0,\rho)$. Let $P = -A_\mathcal{E}^{+T} c_\mathcal{E}$, $p = \|P\|_2$ and if $\mathcal{E} = \emptyset$ then let $P = 0$, $p = 0$. Because $\alpha\rho s^0 (\forall \alpha \in [0,1])$ are feasible solutions of problem $QP(x^0,\rho)$ and constraint functions are twice continuously differentiable, we can get a feasible solution of problem $QP(x,\rho)$. Let $s = (I - A_\mathcal{E} A_\mathcal{E}^{+T}) s^0 / \|(I - A_\mathcal{E} A_\mathcal{E}^{+T}) s^0\|_2$.It is obvious that s belongs to $Null(A_\mathcal{E}^T)$. If $\mathcal{E} = \emptyset$, then we let $s = s^0$.

Because the objective and constraint functions are twice continuously differentiable and (2) holds, we know there exist a neighborhood $\mathcal{N}^2$ of x^0 and a positive constant such that
$$\text{both} \quad s^T g \leq -\varepsilon \quad \text{and} \quad s^T \nabla c_i(x) \geq -\varepsilon$$
for all $x \in \mathcal{N}^2$. If we let $\mathcal{N}^0 = \mathcal{N}^1 \cap \mathcal{N}^2$, then for any $x \in \mathcal{N}^0$, all formulae above hold. By definition of P it follows that $p = O(h(c)) \Rightarrow h(c) \geq b_1 p$, where b_1 is a positive constant. So if δb_1 is no less than 1, which just needs δ sufficiently large, then $\rho \geq \delta h(c(x)) \geq (\delta b_1) p \geq p$ holds for all $x \in \mathcal{N}^0$.

Let $d_\theta = P + \theta(\rho - p)s$, where $\rho > p$ is a constant and $\theta \in [0,1]$ which is used to adjust the norm of d. By the analysis above we know that d_θ satisfies $c_\mathcal{E} + A_\mathcal{E}^T d = 0$. As $A_\mathcal{E}^+$ satisfies $A_\mathcal{E}^+ A_\mathcal{E} A_\mathcal{E}^+ = A_\mathcal{E}^+$, $P^T s = -c_\mathcal{E}^T (A_\mathcal{E}^+ - A_\mathcal{E}^+ A_\mathcal{E} A_\mathcal{E}^+) s^0 / \|(I - A_\mathcal{E} A_\mathcal{E}^{+T}) s^0\|_2 = 0$ holds. For all $\theta \in [0,1]$,
$$\|d_\theta\|_2 = \sqrt{p^2 + \theta^2(\rho - p)^2} \leq \sqrt{p^2 + (\rho - p)^2} = \|d_1\|_2 = \sqrt{(\rho^2 + 2p(p - \rho))} \leq \rho.$$ It follows that d_θ satisfies the trust region constraint.

In the following we will prove that d_θ satisfies the inequality constraints. First of all we consider the inactive constraints, i.e., $i \in \mathcal{I} \setminus \mathcal{A}^0$. For any $x \in \mathcal{N}^0$, because the constraint functions are twice continuously differentiable, we know that there exist positive constants $const1$ and $const2$ such that

$$\text{both} \quad c_i(x) \geq const1 \quad \text{and} \quad \|\nabla c_i(x)\|_2 \leq const2.$$

From the inequalities above we know

$$c_i(x) + d_\theta^T a_i(x) \geq const1 - \rho \cdot const2.$$

So if $\rho < const1/const2$ holds, then we can conclude that $const1 - \rho \cdot const2 > 0$, i.e., d_θ satisfies the inactive constraints.

Now we consider the active constraints, i.e., $i \in \mathcal{A}^0$. For all $x \in \mathcal{N}_0$ and d_θ, where $\theta \in (0, 1]$, the following inequality

$$c_i(x) + d_\theta^T a_i(x) = c_i + P^T a_i + \theta(\rho - p)s^T a_i \geq c_i + P^T a_i - \theta(\rho - p)\varepsilon \geq 0$$

holds if

$$\theta\rho\varepsilon \leq \theta p\varepsilon + c_i + P^T \nabla c_i \Leftrightarrow \rho \leq \frac{p + (c_i + P^T \nabla c_i)}{\theta\varepsilon}.$$

By the definition of P, we know $p = O(h(c))$ and $\frac{p+(c_i+P^T\nabla c_i)}{\theta\varepsilon} = O(h(c))$. So if we choose δ sufficiently large it will follow that $c_i(x) + d_\theta^T \nabla c_i(x) \geq 0$.

By the discussion above we know that if we let $b_0 = const1/const2$, then there a neighborhood of x^0 such that for all $x \in \mathcal{N}^0$ and $\rho \in [\delta h(c(x)), b_0]$, d_θ is a feasible solution of problem $QP(x, \rho)$, where θ belongs to $(0, 1]$ and δ is a positive constant.

Lemma 4.15. *Let $\varepsilon_0 > 0$. If $0 < a < \varepsilon_0/2, b < -\varepsilon_0$, then $|a + b| \geq \varepsilon_0/2$.*

Proof. Because $0 < a < \varepsilon_0/2, b < -\varepsilon_0$, it follows that $a + b \leq -\varepsilon_0/2$. Thus the lemma is true.

Theorem 4.16. *Under the assumptions of Theorem 4.14, it follows that both $\Delta\Psi \geq \frac{1}{16}\rho\varepsilon$ and $\Delta f \geq \sigma\Delta\Psi$ hold. Furthermore, the actual reduction Δf satisfies the following inequality, i.e., $\Delta f \geq \gamma h(c(x + d))$.*

Proof. By Theorem 4.14, we know that both $d_{1/2}$ and d_1 are feasible solutions of problem $QP(x, \rho)$, where $x \in \mathcal{N}_0$ and $\rho \in [\delta h(c(x)), b_0]$. As a consequence

$$\begin{aligned}
2\Delta\Psi &\geq |(\Psi(0) - \Psi(d_{1/2})) - (\Psi(0) - \Psi(d_1))| \\
&= |\Psi(d_1) - \Psi(d_{1/2})| \\
&= |\tfrac{1}{2}(P + (\rho - p)s)B(P + (\rho - p)s) - \tfrac{1}{2}(P + \tfrac{1}{2}(\rho - p)s)B(P + \tfrac{1}{2}(\rho - p)s) + \tfrac{1}{2}(\rho - p)g^T s| \\
&= |\tfrac{1}{2}(\rho - p)\{s^T BP + \tfrac{3}{4}(\rho - p)s^T Bs + g^T s\}| \\
&= |\tfrac{3}{8}\rho^2 s^T Bs + \tfrac{1}{2}\rho g^T s| + O(p) \\
&= \tfrac{1}{2}\rho|\tfrac{3}{4}\rho s^T Bs + g^T s| + O(p).
\end{aligned}$$

Because $g^T s \leq -\varepsilon$, it follows that $\frac{3}{4}\rho s^T Bs \leq \frac{3}{4}\bar{M}\rho \leq \frac{1}{2}\varepsilon$ when $\rho \leq \frac{2\varepsilon}{3\bar{M}}$. If we replace b_0 in Theorem 4.14 with $min\{b_0, \frac{2\varepsilon}{3\bar{M}}\}$, then it follows that both $g^T s \leq -\varepsilon$ and $\frac{3}{4}\rho s^T Bs \leq \varepsilon/2$ hold, for all $x \in \mathcal{N}_0$, $\rho \in [\delta h(c(x)), b_0]$. According to Lemma 4.15, we know that

$$\begin{aligned}
2\Delta\Psi &\geq \tfrac{1}{2}\rho|\tfrac{3}{4}\rho s^T Bs + g^T s| + O(p) \\
&\geq \tfrac{1}{4}\rho\varepsilon + O(p).
\end{aligned}$$

It follows from the above inequality that $\Delta\Psi \geq \frac{1}{8}\rho\varepsilon + O(p)$. Because $p = O(h(c))$, there exists a sufficiently large positive ζ which is irrelevant to ρ such that

$$\Delta\Psi \geq \tfrac{1}{8}\rho\varepsilon + O(h(c)) \geq \tfrac{1}{8}\rho\varepsilon - \zeta h(c) \geq \tfrac{1}{16}\rho\varepsilon,$$

if $\rho \geq \frac{16\zeta}{\varepsilon}h(c)$. It is obvious that if we let δ presented in Theorem 4.14 equal to $max\{\delta, 16\zeta/\varepsilon\}$, then all conditions above could be satisfied.

By Lemma 4.9 we know that

$$\Delta f/\Delta\Psi \geq 1 - n\rho^2\bar{M}/\Delta\Psi \geq 1 - 16n\rho\bar{M}/\varepsilon \geq \sigma,$$

if $\rho \leq (1 - \rho)\varepsilon/(16n\bar{M})$. Let $b_0 = min\{b_0, (1 - \rho)\varepsilon/(16n\bar{M})\}$, it follows that

$$\Delta f \geq \sigma\Delta\Psi.$$

Now we can conclude that $\Delta f \geq \gamma h(c(x+d))$ holds.
$\Delta f - \gamma h(c(x+d)) \geq \frac{1}{16}\sigma\rho\varepsilon - \gamma h(c(x+d))$
$\geq \frac{1}{16}\sigma\rho\varepsilon - \frac{1}{2}nm\gamma\rho^2\bar{M}$ (By Lemma 4.9).
So it follows that $\Delta f \geq \gamma h(c(x+d))$ if $\rho \leq \sigma\varepsilon/(8mn\gamma\bar{M})$ which could be satisfied by letting $b_0 = min\{b_0, \sigma\varepsilon/(8mn\gamma\bar{M})\}$.

We have a main theorem similar to the one presented in [6] by Fletcher et al..

Theorem 4.17. *If Assumption* 4.3 *holds, the outcome of applying Algorithm* 4.1 *is one of the followings*
(1)The restoration phase fails to find a point x which is both acceptable to the filter and for which $QP(x,\rho)$ is compatible for some $\rho \geq \rho^0$.
(2)A KT point of problem (1.1) *is found($d = 0$ solves $QP(x^{(k)},\rho)$ for some k).*
(3)Any accumulation point is feasible, and is either a KT point or fails to satisfy MFCQ.

Proof. We just need to consider the case in which neither (1) nor (2) occurs. By Lemma 4.12 we know that inner loop terminates finitely and by Assumption 4.3 we know that the sequence $\{x^{(k)}, k = 1, 2, \cdots\}$ generated by Algorithm 4.1 lies in a bounded set X. According to Theorem 4.5 and Theorem 4.6 we can conclude that $h(c(\hat{x})) \to 0$ for any accumulation point $\hat{x}$, i.e., any accumulation point is a feasible point.

Now we consider the proposition(to be contradicted)that there exists some accumulation point which is a feasible point of (1.1) and at which MFCQ holds, but it is not a KT point point of (1.1). First of all we will prove that there does not exist an infinite subsequence which is formed by *h-type iterations*. were it not for the above case, there exists an infinite indices set $\mathcal{K}$ such that all points $x^{(k)} (k \in \mathcal{K})$ are *h-type iterations* and $h^{(k)} \to 0$. By the definition of $\tau^{(k)}$ we know $\tau^{(k)} \to 0$. Because $\tau^{(k)} \to 0$ and it only can be reset by an *h-type iteration*, there exists a subsequence $\{x^{(k)}, k \in \mathcal{K}_1 \subseteq \mathcal{K}\} \subset \{x^{(k)}, k \in \mathcal{K}\}$ such that for this subsequence, $\tau^{(k+1)} = h^{(k)} < \tau^{(k)}$ holds.

As $\{x^{(k)}, k \in \mathcal{K}_1\} \subseteq X$ is bounded we know there exists a subsequence $\{x^{(k)}, k \in \mathcal{K}_2 \subseteq \mathcal{K}_1\}$ such that $x^{(k)} \to x^\infty$ and $h^{(k)} \to 0$ hold. Because of the feasibility of (1.1) at x^∞, if we suppose that MFCQ hold at x^∞ then x^∞ is not a KT point of (1.1). By discussion above we know the following two conditions hold
(i) The vectors $\nabla c_i(x^\infty), i \in \mathcal{E}$ are linearly independent,
(ii)There exists a vector $s \in \Re^n$ such that $s^T\nabla c_i(x^\infty) = 0, i \in \mathcal{E}$ and $s^T\nabla c_i(x^\infty) > 0, i \in \mathcal{A}^\infty \subseteq \mathcal{I}$ hold. Furthermore, if k is sufficiently large, then $x^{(k)} \in \mathcal{N}^\infty$, where the definition of $\mathcal{N}^\infty$ is the same as that definition in Theorem 4.14 or Theorem 4.16.

By Theorem 4.14 or Theorem 4.16 we know that problem $QP(x^{(k)},\rho)$ is consistent and the solution d of which satisfy the following inequalities

$$\Delta\Psi \geq \tfrac{1}{16}\rho\varepsilon,$$
$$\Delta f \geq \sigma\Delta\Psi,$$
$$\Delta f \geq \gamma h(c(x+d)).$$

So d satisfy all conditions of an *f-type step* and by Lemma 4.11 we know that $x^{(k)}+d$ is acceptable to the filter if $\rho \leq \sqrt{\frac{2\beta\tau^{(k)}}{mnM}}$ holds. If $\rho \in [\delta h^{(k)}, min\{\sqrt{\frac{2\beta\tau^{(k)}}{mnM}}, b_0\}]$, where b_0 is independent with ρ, then d satisfy all conditions of an *f-type step* and $x^{(k)} + d$ is acceptable to the filter.

Now we will prove that ρ will lie in the interval $[\delta h^{(k)}, min\{\sqrt{\frac{2\beta\tau^{(k)}}{mn\bar{M}}}, b_0\}]$ because of the inner loop. Since $\tau^{(k)} \to 0$, if $k \in \mathcal{K}_2$ is sufficiently large then the range $\delta h^{(k)} \leq \rho \leq min\{\sqrt{\frac{2\beta\tau^{(k)}}{mnM}}, b_0\}$ becomes $\delta h^{(k)} \leq \rho \leq \sqrt{\frac{2\beta\tau^{(k)}}{mnM}}$. In this case, since $h^{(k)} \to 0, h^{(k)} < \tau^{(k)}$, there exists a sufficiently large k_2 such that if k satisfies $k_2 < k \in \mathcal{K}_2$ then $\sqrt{\frac{2\beta\tau^{(k)}}{mnM}} \geq \delta h^{(k)}$. In fact this only need $h^{(k)} \leq \beta/(mn\bar{M}\delta^2)$. At the beginning of the inner loop, $\rho = \bar{\rho} \geq \rho_0$ is given and $\rho > \sqrt{\frac{2\beta\tau^{(k)}}{mnM}}$.

But the successive iterations of inner loop, ρ *at least will be halved at every two steps,* which will lead one of the two following conditions occurs

(a) ρ enters the interval$[\delta h^{(k)}, \sqrt{\frac{2\beta\tau^{(k)}}{mn\bar{M}}}]$, (b) $\rho > \sqrt{\frac{2\beta\tau^{(k)}}{mn\bar{M}}}$ still holds.

The case (a) will result an *f-type iteration* and the case (b) also could not result an *f-type iteration* because $\Delta\Psi$ will decrease whilst ρ decreases and $\rho \geq \delta h^{(k)}$. If $k \in \mathcal{K}_2$ is sufficiently large, an *f-type iteration* will be generated which leads to a contradiction. So if there exists some accumulation point which is a feasible point of (1.1) and at which MFCQ holds, but it is not a KT point point of (1.1), then we can conclude that the main iterative sequence only contains a finite of *h-type iterations*. From that we know that there exists a k_3 such that while $k \geq k_3$, all iterations are *f-type iterations*.

According to Assumption 4.3, $f(x)$ is twice continuously differentiable on set X which is bounded. So $\sum\limits_{k\in\bar{\mathcal{K}}} \Delta f^{(k)}$ converges on any infinite set $\bar{\mathcal{K}}$. Since $(h^{(k+1)}, f^{(k+1)})$ is not dominated by $(h^{(k)}, f^{(k)})$ we know that $h^{(k)} \to 0$, i.e., any accumulation point is a feasible point, according to Theorem 4.5 and Theorem 4.6. Now we consider such an accumulation point x^∞ which is not a KT point but at which MFCQ holds. For any infinite subsequence $\{x^{(k)}, k \in \bar{\mathcal{K}}, k \geq k_3\}$, all successive iterations are *f-type iteration* because of $k \geq k_3$ which leads to no new points are added into the filter. So the value of τ is not reset, i.e., for any $k \geq k_3$, $\tau^{(k)} = \tau^{(k_3)}$ holds. For convenience, we can suppose that all points $\{x^{(k)}, k \in \bar{\mathcal{K}}\}$ lie in the neighborhood of $\mathcal{N}^\infty$. Thus if $\delta h^{(k)} \leq \rho \leq min\{\sqrt{\frac{2\beta\tau^{(k_3)}}{mn\bar{M}}}, b_0\} \triangleq \hat{\rho}$ then some *f-type step* is accepted, where $\hat{\rho}$ is positive and independent with ρ. According to $h^{(k)} \to 0$, $\hat{\rho} \geq 2\delta h^{(k)}$ holds for sufficiently large k. So in the successive series iterations of the inner loop, also only the case (a) or (b) occurs. We can always guarantee that $\rho^{(k)} \geq min\{\hat{\rho}/2, \rho_0\}$. By Theorem 4.16, we can get the following result

$$\Delta f \geq \sigma\Delta\Psi \geq \tfrac{1}{16}\sigma\rho\varepsilon \geq \tfrac{1}{16}\sigma\varepsilon min\{\hat{\rho}/2, \rho_0\},$$

which is contradicted with the result that $\sum\limits_{k\in\bar{\mathcal{K}}} \Delta f^{(k)}$ converges. The contradiction implies that the theorem is true.

5. Numerical Results

This section presents some numerical results obtained by an implementation of our new method. There are many standard test sets for nonlinear programming problems, such as CUTE of I.BongartZ et al. in [7], test set of J.J.Moré in [10], etc. In our numerical test, the test problems are taken from W.Hock and K.Schittkowski[8].

In our numerical test, the exact first order derivatives of objective functions and constraint functions are used. All routines are written in Visual Fortran. The QP solver used is QL0001 which is distributed by A.L.Tits and J.L.Zhou of University of Maryland. For the initial trust region radius we choose $\rho = 10$ and the tolerance is set to $\epsilon = 1.0E-8$. Initially we set $\beta = 0.99$, $\gamma = 0/01$, $UBD = 100$, $tt = 1.25$, and the upper bound of constraint violation $u = max\{UBD, tt\cdot h^{(0)}\}$. Hessian matrices are updated by the BFGS formulae and the initial Hessian matrices are set to the identity matrices.

The test problems are classified by objective functions and constraint functions as follows. (1) simple boundary constraints, equality constraints, equality mixed with inequality constraints, linear constraints and nonlinear constraints; (2) linear and nonlinear objective functions. The scale of test problem is small, where $m \leq 100$, $n \leq 100$, and objective functions and constraint functions are all smooth.

5.1 Comparison to NLPQL in [13] and Our New SQP-Filter Method on Programming with Simple Boundary Constraints

We compare our method to NLPQL presented by K.Schittkowski in [13] on programming just with boundary constraints. The results about the two method is given in Table 5.1.

Table 5.1 (Testing results about NLP problems just with boundary constraints)

TP	N	M	NEF	FEX	F	DFX	DGX
3 NFT	2	0	301	0.00000000E+00	0.66935221E-03	0.67E-03	0.00E+00
HS**	2	0	90	0.00000000E+00	0.16103740E-19	0.16E-19	0.00E+00
4 NFT**	2	0	7	0.26666667E+01	0.26666667E+01	0.00E+00	0.00E+00
HS	2	0	18	0.26666667E+01	0.26666667E+01	0.00E+00	0.00E+00
5 NFT**	2	0	55	-0.19132230E+01	-0.19132222E+01	0.79E-06	0.00E+00
HS	2	0	56	-0.19132230E+01	-0.19132230E+01	0.11E-10	0.00E+00
25 NFT**	3	0	233	0.00000000E+00	0.72953587E-05	0.73E-05	0.00E+00
HS	3	0	251	0.00000000E+00	0.57804601E-02	0.58E-02	0.00E+00
38 NFT	4	0	501	0.00000000E+00	0.59461998E-05	0.59E-05	0.00E+00
HS**	4	0	459	0.00000000E+00	0.29634809E-07	0.30E-07	0.00E+00
45 NFT**	5	0	121	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
HS	5	0	168	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
110 NFT**	10	0	271	-0.45778470E+02	-0.45778469E+02	0.40E-06	0.00E+00
HS	10	0	290	-0.45778470E+02	-0.45778470E+02	0.28E-09	0.00E+00
201 NFT**	2	0	11	0.00000000E+00	0.00000000E+00	0.00E+00	0.00E+00
HS	2	0	29	0.00000000E+00	0.45089348E-17	0.45E-17	0.00E+00
110 NFT**	10	0	271	-0.45778470E+02	-0.45778469E+02	0.40E-06	0.00E+00
HS	10	0	290	-0.45778470E+02	-0.45778470E+02	0.28E-09	0.00E+00
201 NFT**	2	0	11	0.00000000E+00	0.00000000E+00	0.00E+00	0.00E+00
HS	2	0	29	0.00000000E+00	0.45089348E-17	0.45E-17	0.00E+00
204 NFT	2	0	75	0.18360100E+00	0.18363583E+00	0.35E-04	0.00E+00
HS**	2	0	55	0.18360100E+00	0.18360120E+00	0.11E-05	0.00E+00
207 NFT**	2	0	178	0.00000000E+00	0.40667874E-04	0.41E-04	0.00E+00
HS	2	0	118	0.00000000E+00	0.27034955E-07	0.27E-07	0.00E+00
210 NFT	2	0	289	0.00000000E+00	0.44579570E-05	0.45E-05	0.00E+00
HS**	2	0	47	0.00000000E+00	0.39924439E-05	0.40E-05	0.00E+00
213 NFT	2	0	295	0.00000000E+00	0.29861841E-05	0.30E-05	0.00E+00
HS**	2	0	145	0.00000000E+00	0.15179672E-06	0.15E-06	0.00E+00
256 NFT**	4	0	501	0.00000000E+00	0.39098840E-06	0.39E-06	0.00E+00
HS	4	0	697	0.00000000E+00	0.91827889E-07	0.92E-07	0.00E+00
258 NFT	4	0	417	0.00000000E+00	0.21183635E-05	0.21E-05	0.00E+00
HS**	4	0	395	0.00000000E+00	0.20582604E-07	0.21E-07	0.00E+00
260 NFT	4	0	101	0.00000000E+00	0.36199999E-01	0.36E-01	0.00E+00
HS**	4	0	395	0.00000000E+00	0.20582607E-07	0.21E-07	0.00E+00
261 NFT**	4	0	481	0.00000000E+00	0.17657579E-07	0.18E-07	0.00E+00
HS	4	0	547	0.00000000E+00	0.58742233E-07	0.59E-07	0.00E+00
267 NFT	5	0	571	0.00000000E+00	0.17420919E-01	0.17E-01	0.00E+00
HS**	5	0	675	0.00000000E+00	0.15955255E-04	0.16E-04	0.00E+00
272 NFT**	6	0	623	0.00000000E+00	0.15243581E-01	0.15E-01	0.00E+00
HS	6	0	804	0.00000000E+00	0.56556841E-02	0.57E-02	0.00E+00
274 NFT	2	0	284	0.00000000E+00	0.47576904E-11	0.48E-11	0.00E+00
HS**	2	0	46	0.00000000E+00	0.35247538E-22	0.35E-22	0.00E+00
275 NFT	4	0	497	0.00000000E+00	0.32239150E-04	0.32E-04	0.00E+00
HS**	4	0	69	0.00000000E+00	0.13975486E-06	0.14E-06	0.00E+00
276 NFT	6	0	689	0.00000000E+00	0.13891247E-02	0.14E-02	0.00E+00
HS**	6	0	201	0.00000000E+00	0.13183587E-10	0.13E-10	0.00E+00
287 NFT**	20	0	2101	0.00000000E+00	0.14181070E-06	0.14E-06	0.00E+00
HS	20	0	2187	0.00000000E+00	0.14817396E-06	0.15E-06	0.00E+00
289 NFT	30	0	1901	0.00000000E+00	0.57394416E-08	0.57E-08	0.00E+00
HS**	30	0	848	0.00000000E+00	0.10087201E-09	0.10E-09	0.00E+00
290 NFT**	2	0	55	0.00000000E+00	0.10139202E-09	0.10E-09	0.00E+00
HS	2	0	155	0.00000000E+00	0.10788891E-06	0.11E-06	0.00E+00
291 NFT**	10	0	1031	0.00000000E+00	0.13378455E-05	0.13E-05	0.00E+00
HS	10	0	1709	0.00000000E+00	0.11976216E-04	0.12E-04	0.00E+00
294 NFT**	6	0	701	0.00000000E+00	0.46525950E-03	0.47E-03	0.00E+00
HS	6	0	1654	0.00000000E+00	0.39714708E-07	0.40E-07	0.00E+00
295 NFT**	10	0	1101	0.00000000E+00	0.90823797E-03	0.91E-03	0.00E+00
HS	10	0	4272	0.00000000E+00	0.22913957E-07	0.23E-07	0.00E+00
296 NFT**	16	0	1701	0.00000000E+00	0.15129161E-02	0.15E-02	0.00E+00
HS	16	0	2015	0.00000000E+00	0.15280757E-02	0.15E-02	0.00E+00
297 NFT**	30	0	3101	0.00000000E+00	0.29039802E-02	0.29E-02	0.00E+00
HS	30	0	3993	0.00000000E+00	0.29172835E-02	0.29E-02	0.00E+00

298 NFT**	50	0	5101	0.00000000E+00	0.48856172E-02	0.49E-02	0.00E+00
HS	50	0	6633	0.00000000E+00	0.48981547E-02	0.49E-02	0.00E+00
299 NFT**	100	0	8081	0.00000000E+00	0.98401575E-02	0.98E-02	0.00E+00
HS	100	0	13233	0.00000000E+00	0.98482202E-02	0.98E-02	0.00E+00
307 NFT	2	0	277	0.00000000E+00	0.61679140E-01	0.62E-01	0.00E+00
HS**	2	0	132	0.00000000E+00	0.12373060E-04	0.12E-04	0.00E+00
309 NFT	2	0	271	-0.39871708E+01	-0.39871048E+01	0.66E-04	0.00E+00
HS**	2	0	94	-0.39871708E+01	-0.39871708E+01	-0.19E-08	0.00E+00
312 NFT**	2	0	291	0.00000000E+00	0.22251541E-01	0.22E-01	0.00E+00
HS	2	0	298	0.00000000E+00	0.59232518E-03	0.59E-03	0.00E+00
314 NFT	2	0	293	0.16904000E+00	0.16906186E+00	0.22E-04	0.00E+00
HS**	2	0	56	0.16904000E+00	0.16904268E+00	0.16E-04	0.00E+00
328 NFT**	2	0	50	0.17441520E+01	0.17441520E+01	0.56E-08	0.00E+00
HS	2	0	99	0.17441520E+01	0.17441520E+01	0.17E-07	0.00E+00
357 NFT**	4	0	6	0.35845660E+00	0.36733387E+00	0.89E-02	0.00E+00
HS	4	0	170	0.35845660E+00	0.35845711E+00	0.14E-05	0.00E+00
368 NFT**	8	0	254	-0.75000000E+00	-0.75000000E+00	0.74E-11	0.00E+00
HS	8	0	297	-0.75000000E+00	-0.75000000E+00	0.92E-09	0.00E+00
379 NFT**	11	0	981	0.40137700E-01	0.49532121E-01	0.94E-02	0.00E+00
HS	11	0	2401	0.40137700E-01	0.40137773E-01	0.18E-05	0.00E+00

Remarks.

The notations about some expressions in the Table 5.1 are given as follows
TP—test problem number, N—number of variables, M—number of constraints,
NF—number of objective function evaluations,
NDF—number of gradient evaluations of objective function,
NEF— number of equivalent function evaluations, i.e., NF plus number of function calls needed for gradient computation,
FEX—exact objective function values, F—computed objective function values,
DFX—relative error in objective function,
DGX—sum of constraint violations,
HS—results which are gotten by the method in [13],
NFT—results which are gotten by our new SQP-Filter method,
**—superscript ** denotes that corresponding method has a better behavior.

For the 39 test problems above, total number of objective function evaluations of our new method is 33799 while total number of objective function evaluations of NLPQL presented in [13] is 45801. The results illustrate that our method use less computation of objective function values. Our new method behaves better than NLPQL on 23 test problems of all 39 test problems and NLPQL behaves better than our new method on 16 test problems of all 39 test problems. The performance of our new method is satisfactory.

5.2 Comparison to Fletcher's SQP-Filter and Our New SQP-Filter Method on General Nonlinear Programming

Next, our new SQP-Filter method is compared with Fletcher's SQP-Filter method on 161 general general nonlinear programming problems. Our new method fails on 31 test problems and Fletcher's method fails on 34 test problems of all 161 test problems. Table 5.2 and table 5.3 give the results of our new method and Fletcher's method respectively.

Table 5.2(Testing results about NLP problems by using our new SQP-Filter method)

TP	N	ME	M	NF	NDF	NG	NDG	FEX	F	DFX	DGX
6	2	1	1	20	10	20	10	0.00000000E+00	0.29677103E-16	0.30E-16	0.00E+00
7	2	1	1	16	11	16	11	-0.17320508E+01	-0.17320508E+01	0.18E-14	0.00E+00
8	2	2	2	9	7	11	8	-0.10000000E+01	-0.10000000E+01	0.00E+00	0.00E+00
9	2	1	1	7	6	7	6	-0.50000000E+00	-0.50000000E+00	0.22E-15	0.35E-09
10	2	0	1	13	12	15	12	-0.10000000E+01	-0.10000000E+01	0.13E-14	0.27E-14
11	2	0	1	9	8	9	8	-0.84984642E+01	-0.84984642E+01	0.36E-14	0.18E-14
12	2	0	1	11	8	11	8	-0.30000000E+02	-0.30000000E+02	0.00E+00	0.18E-14
13	2	0	1	34	34	35	34	0.10000000E+01	0.10000031E+01	0.31E-05	0.00E+00
14	2	1	2	7	6	7	6	0.13934650E+01	0.13934650E+01	0.44E-15	0.22E-15
15	2	0	2	4	3	4	3	0.30650001E+01	0.30650000E+01	0.57E-07	0.00E+00
18	2	0	2	11	7	11	7	0.50000000E+01	0.49991497E+01	0.85E-03	0.43E-02
19	2	0	2	18	17	18	17	-0.69618137E+00	-0.69618137E+00	0.28E-14	0.12E-12
20	2	0	3	5	4	5	4	0.38198730E+02	0.38198730E+02	0.71E-14	0.11E-15
21	2	0	1	18	17	18	17	-0.99959998E+00	-0.99959998E+00	0.89E-11	0.00E+00
22	2	0	2	6	5	6	5	0.10000000E+01	0.10000000E+01	0.00E+00	0.11E-15
23	2	0	5	7	6	7	6	0.20000000E+01	0.20000000E+01	0.53E-13	0.00E+00
24	2	0	3	14	13	14	13	-0.10000000E+01	-0.10000000E+01	0.22E-15	0.00E+00
26	3	1	1	101	95	101	95	0.00000000E+00	0.10526016E-07	0.11E-07	0.00E+00
27	3	1	1	48	31	48	31	0.40000000E+01	0.44401929E+01	0.44E+00	0.00E+00
28	3	1	1	8	7	8	7	0.00000000E+00	0.52520811E-17	0.53E-17	0.00E+00
29	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.14E-11	0.20E-11
30	3	0	1	19	18	19	18	0.10000000E+01	0.10000000E+01	0.58E-10	0.00E+00
31	3	0	1	76	43	76	43	0.60000000E+01	0.60906429E+01	0.91E-01	0.68E-10
32	3	1	2	5	4	5	4	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
34	3	0	2	9	8	9	8	-0.83403245E+00	-0.83403245E+00	0.11E-15	0.22E-14
35	3	0	1	8	7	8	7	0.11111111E+00	0.11111111E+00	0.17E-15	0.00E+00
36	3	0	1	3	2	3	2	-0.33000000E+04	-0.33000000E+04	0.00E+00	0.00E+00
37	3	0	2	11	10	11	10	-0.34560000E+04	-0.34560000E+04	0.45E-12	0.00E+00
39	4	2	2	23	22	23	22	-0.10000000E+01	-0.10000000E+01	0.78E-12	0.15E-13
40	4	3	3	26	25	26	25	-0.25000000E+00	-0.25000000E+00	0.61E-11	0.27E-13
41	4	1	1	9	8	9	8	0.19259259E+01	0.19259259E+01	0.22E-15	0.18E-12
42	4	2	2	21	14	21	14	0.13857864E+02	0.13857864E+02	0.18E-14	0.28E-12
43	4	0	3	16	13	16	13	-0.44000000E+02	-0.44000000E+02	0.17E-09	0.16E-11
44	4	0	6	9	8	9	8	-0.15000000E+02	-0.15000000E+02	0.00E+00	0.00E+00
46	5	2	2	51	27	51	27	0.00000000E+00	0.29482983E-03	0.29E-03	0.00E+00
48	5	2	2	11	10	11	10	0.00000000E+00	0.15430416E-15	0.15E-15	0.00E+00
49	5	2	2	51	32	51	32	0.00000000E+00	0.41189117E-01	0.41E-01	0.00E+00
50	5	3	3	51	50	51	50	0.00000000E+00	0.15175060E-07	0.15E-07	0.00E+00
51	5	3	3	8	7	8	7	0.00000000E+00	0.13946477E-17	0.14E-17	0.00E+00
53	5	3	3	41	37	41	37	0.40930233E+01	0.40930233E+01	0.13E-10	0.00E+00
56	7	4	4	101	100	101	100	-0.34560000E+01	-0.34559868E+01	0.13E-04	0.11E-09
59	2	0	3	23	24	25	24	-0.78042263E+01	-0.78042360E+01	0.96E-05	0.11E-12
60	3	1	1	12	11	12	11	0.32568200E-01	0.32568200E-01	0.38E-11	0.00E+00
62	3	1	1	29	28	29	28	-0.26272514E+05	-0.26272514E+05	0.18E-07	0.00E+00
64	3	0	1	35	33	37	33	0.62998424E+04	0.62998424E+04	0.29E-06	0.11E-15
65	3	0	1	24	18	24	18	0.95352886E+00	0.97168149E+00	0.18E-01	0.00E+00
66	3	0	2	14	13	14	13	0.51816327E+00	0.51816327E+00	0.23E-10	0.86E-13
71	4	1	2	8	7	8	7	0.17014017E+02	0.17014017E+02	0.34E-09	0.13E-13
72	4	0	2	49	48	49	48	0.72767938E+03	0.72767936E+03	0.18E-04	0.14E-16
73	4	1	3	6	5	6	5	0.29894378E+02	0.29894378E+02	0.18E-08	0.89E-15
76	4	0	3	10	9	10	9	-0.46818182E+01	-0.46818182E+01	0.18E-11	0.00E+00
78	5	3	3	9	8	9	8	-0.29197004E+01	-0.29197004E+01	0.15E-09	0.00E+00
79	5	3	3	20	19	20	19	0.78776821E-01	0.78776821E-01	0.17E-10	0.00E+00

80	5	3	3	8	7	8	7	0.53949848E-01	0.53949848E-01	0.79E-11	0.00E+00
81	5	3	3	12	11	12	11	0.53949848E-01	0.53949848E-01	0.46E-11	0.00E+00
83	5	0	6	6	5	6	5	-0.30665539E+05	-0.30665539E+05	0.83E-07	0.00E+00
84	5	0	6	22	18	22	18	-0.52803351E+02	-0.52803351E+02	0.15E-08	0.58E-10
88	2	0	1	22	20	22	20	0.13626568E+01	0.13626907E+01	0.34E-04	0.21E-14
89	3	0	1	32	31	32	31	0.13626568E+01	0.13626907E+01	0.34E-04	0.24E-14
90	4	0	1	39	28	39	28	0.13626568E+01	0.13626907E+01	0.34E-04	0.00E+00
91	5	0	1	44	34	49	34	0.13626568E+01	0.13626907E+01	0.34E-04	0.83E-15
92	6	0	1	38	28	38	28	0.13626568E+01	0.13626907E+01	0.34E-04	0.26E-14
95	6	0	4	3	2	3	2	0.15619514E-01	0.15619525E-01	0.11E-07	0.00E+00
96	6	0	4	3	2	3	2	0.15619513E-01	0.15619525E-01	0.12E-07	0.00E+00
104	8	0	6	101	100	101	100	0.39511634E+01	0.39512151E+01	0.52E-04	0.49E-09
111	10	3	3	101	100	101	100	-0.47761090E+02	-0.47761071E+02	0.19E-04	0.00E+00
112	10	3	3	32	31	32	31	-0.47761086E+00	-0.47761090E+00	0.38E-07	0.00E+00
113	10	0	8	76	75	76	75	0.24306209E+02	0.24306209E+02	0.42E-08	0.12E-12
215	2	0	1	8	7	8	7	0.00000000E+00	-0.33832439E-17	0.34E-17	0.34E-17
216	2	1	1	18	17	18	17	0.10000000E+01	0.99937529E+00	0.62E-03	0.00E+00
217	2	1	2	9	8	9	8	-0.80000000E+00	-0.80000000E+00	0.22E-13	0.89E-13
221	2	0	1	22	21	22	21	-0.10000000E+01	-0.10000000E+01	0.00E+00	0.00E+00
222	2	0	1	7	6	7	6	-0.15000000E+01	-0.15000000E+01	0.00E+00	0.00E+00
223	2	0	2	9	8	9	8	-0.83403245E+00	-0.83403245E+00	0.43E-12	0.10E-10
224	2	0	4	38	37	38	37	-0.30400000E+03	-0.30400000E+03	0.88E-11	0.00E+00
225	2	0	5	7	6	7	6	0.20000000E+01	0.20000000E+01	0.53E-13	0.00E+00
226	2	0	2	11	10	11	10	-0.50000000E+00	-0.50000000E+00	0.24E-12	0.69E-12
227	2	0	2	8	7	8	7	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
228	2	0	2	8	7	8	7	-0.30000000E+01	-0.30000000E+01	0.00E+00	0.00E+00
230	2	0	2	23	21	23	21	0.37500000E+00	0.37500000E+00	0.00E+00	0.00E+00
232	2	0	3	8	7	8	7	-0.10000000E+01	-0.10000000E+01	0.22E-15	0.00E+00
234	2	0	1	101	100	101	100	-0.80000000E+00	-0.80000000E+00	0.40E-10	0.00E+00
237	2	0	3	19	20	21	20	-0.58903436E+02	-0.58903436E+02	0.90E-08	0.00E+00
238	2	0	3	17	18	19	18	-0.58903436E+02	-0.58903436E+02	0.90E-08	0.00E+00
239	2	0	1	21	21	22	21	-0.58903435E+02	-0.58903436E+02	0.13E-05	0.00E+00
248	3	1	2	15	14	15	14	-0.80000000E+00	-0.80000000E+00	0.85E-13	0.35E-12
249	3	0	1	19	18	19	18	0.10000000E+01	0.10000000E+01	0.58E-10	0.00E+00
250	3	0	2	3	2	3	2	-0.33000000E+04	-0.33000000E+04	0.00E+00	0.00E+00
251	3	0	1	11	10	11	10	-0.34560000E+04	-0.34560000E+04	0.45E-12	0.00E+00
252	3	1	1	37	36	37	36	0.40000000E-01	0.40000000E-01	0.00E+00	0.00E+00
253	3	0	1	9	8	9	8	0.69282032E+02	0.69282032E+02	0.30E-06	0.00E+00
254	3	2	2	19	18	19	18	-0.17320508E+01	-0.17320508E+01	0.22E-15	0.00E+00
262	4	1	4	7	6	7	6	-0.10000000E+02	-0.10000000E+02	0.00E+00	0.30E+01
263	4	2	4	35	34	35	34	-0.10000000E+01	-0.10000000E+01	0.44E-12	0.97E-12
264	4	0	3	20	19	20	19	-0.43999999E+00	-0.43999999E+00	0.37E-13	0.22E-11
269	5	3	3	40	39	40	39	0.40930233E+01	0.40930233E+01	0.98E-11	0.31E+00
270	5	0	1	9	8	9	8	-0.10000000E+01	-0.10000000E+01	0.36E-14	0.32E-13
277	4	0	4	101	100	101	100	0.50761905E+01	0.50744526E+01	0.17E-02	0.47E-02
278	6	0	6	101	100	101	100	0.78385281E+01	0.78850365E+01	0.47E-01	0.57E-02
280	10	0	10	5	5	5	5	0.13375428E+02	0.13376555E+02	0.11E-02	0.00E+00
315	2	0	3	11	10	11	10	-0.80000000E+00	-0.80000000E+00	0.00E+00	0.22E-15
322	2	1	1	24	24	25	24	0.49996001E+03	0.49996001E+03	0.20E-05	0.10E+01
323	2	0	2	93	92	93	92	0.37989446E+01	0.37989446E+01	0.48E-07	0.77E-12
325	2	1	3	7	6	7	6	0.37913415E+01	0.37913414E+01	0.51E-07	0.00E+00
326	2	0	2	9	8	9	8	-0.79807821E+02	-0.79807821E+02	0.15E-06	0.56E-14
327	2	0	1	101	100	101	100	0.30646306E-01	0.30646302E-01	0.38E-08	0.00E+00
329	2	0	3	7	6	7	6	-0.69618139E+04	-0.69618139E+04	0.24E-04	0.68E-13

330	2	0	1	11	10	11	10	0.16205833E+01	0.16205833E+01	0.22E-07	0.67E-14
331	2	0	1	101	100	101	100	0.42580000E+01	0.43156000E+01	0.58E-01	0.00E+00
336	3	2	2	26	27	51	35	-0.33789573E+00	-0.33848594E+00	0.59E-03	0.00E+00
337	3	0	1	12	11	12	11	0.60000000E+01	0.60000000E+01	0.89E-15	0.44E-15
339	3	0	1	13	12	13	12	0.33616797E+01	0.33616797E+01	0.11E-07	0.00E+00
340	3	0	1	12	11	12	11	-0.54000000E-01	-0.54000000E-01	0.13E-13	0.11E-15
341	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.20E-08	0.20E-11
342	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.20E-08	0.20E-11
343	3	0	2	6	5	6	5	-0.56847825E+01	-0.56847825E+01	0.36E-14	0.23E-12
344	3	1	1	12	11	12	11	0.32568200E-01	0.32568200E-01	0.26E-09	0.11E-15
345	3	1	1	24	17	24	17	0.32568200E-01	0.32568200E-01	0.26E-09	0.00E+00
346	3	0	2	6	5	6	5	-0.56847825E+01	-0.56847825E+01	0.36E-14	0.23E-12
347	3	0	1	3	2	3	2	0.17374625E+05	0.17374625E+05	0.27E-03	0.00E+00
353	4	1	3	3	2	3	2	-0.39933673E+02	-0.39933673E+02	0.47E-06	0.00E+00
361	5	0	6	31	30	31	30	-0.77641212E+06	-0.77641212E+06	0.47E-02	0.00E+00
375	10	9	9	35	34	48	46	-0.15160000E+02	-0.15623819E+02	0.46E+00	0.00E+00
396	2	1	1	4	4	6	5	0.00000000E+00	0.00000000E+00	0.00E+00	0.00E+00
397	2	1	1	18	18	21	20	0.00000000E+00	0.22015327E-11	0.22E-11	0.00E+00

Table 5.3 (Testing results about NLP problems by using Fletcher's SQP-Filter method)

TP	N	ME	M	NF	NDF	NG	NDG	FEX	F	DFX	DGX
6	2	1	1	13	12	13	12	0.00000000E+00	0.12263679E-14	0.12E-14	0.00E+00
7	2	1	1	10	9	10	9	-0.17320508E+01	-0.17320508E+01	0.00E+00	0.00E+00
8	2	2	2	7	6	7	6	-0.10000000E+01	-0.10000000E+01	0.00E+00	0.00E+00
9	2	1	1	7	6	7	6	-0.50000000E+00	-0.50000000E+00	0.22E-15	0.61E-10
10	2	0	1	13	12	13	12	-0.10000000E+01	-0.10000000E+01	0.36E-14	0.71E-14
11	2	0	1	9	8	9	8	-0.84984642E+01	-0.84984642E+01	0.36E-14	0.18E-14
12	2	0	1	10	9	10	9	-0.30000000E+02	-0.30000000E+02	0.27E-12	0.53E-12
13	2	0	1	34	34	35	34	0.10000000E+01	0.10000031E+01	0.31E-05	0.00E+00
14	2	1	2	7	6	7	6	0.13934650E+01	0.13934650E+01	0.44E-15	0.22E-15
15	2	0	2	4	3	4	3	0.30650001E+01	0.30650000E+01	0.57E-07	0.00E+00
18	2	0	2	11	10	11	10	0.50000000E+01	0.50000000E+01	0.00E+00	0.00E+00
19	2	0	2	18	17	18	17	-0.69618137E+00	-0.69618137E+00	0.28E-14	0.12E-12
20	2	0	3	5	4	5	4	0.38198730E+02	0.38198730E+02	0.71E-14	0.11E-15
21	2	0	1	18	17	18	17	-0.99959998E+00	-0.99959998E+00	0.89E-11	0.00E+00
22	2	0	2	6	5	6	5	0.10000000E+01	0.10000000E+01	0.00E+00	0.11E-15
23	2	0	5	7	6	7	6	0.20000000E+01	0.20000000E+01	0.53E-13	0.00E+00
24	2	0	3	14	13	14	13	-0.10000000E+01	-0.10000000E+01	0.22E-15	0.00E+00
26	3	1	1	101	98	101	98	0.00000000E+00	0.82782422E-03	0.83E-03	0.00E+00
27	3	1	1	47	46	47	46	0.40000000E+01	0.41002731E+01	0.10E+00	0.00E+00
28	3	1	1	8	7	8	7	0.00000000E+00	0.52520811E-17	0.53E-17	0.00E+00
29	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.14E-11	0.20E-11
30	3	0	1	19	18	19	18	0.10000000E+01	0.10000000E+01	0.58E-10	0.00E+00
31	3	0	1	13	12	13	12	0.60000000E+01	0.60000000E+01	0.44E-14	0.78E-15
32	3	1	2	5	4	5	4	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
34	3	0	2	9	8	9	8	-0.83403245E+00	-0.83403245E+00	0.11E-15	0.22E-14
35	3	0	1	8	7	8	7	0.11111111E+00	0.11111111E+00	0.17E-15	0.00E+00
36	3	0	1	3	2	3	2	-0.33000000E+04	-0.33000000E+04	0.00E+00	0.00E+00
37	3	0	2	11	10	11	10	-0.34560000E+04	-0.34560000E+04	0.45E-12	0.00E+00
39	4	2	2	23	22	23	22	-0.10000000E+01	-0.10000000E+01	0.78E-12	0.15E-13

40	4	3	3	26	25	26	25	-0.25000000E+00	-0.25000000E+00	0.61E-11	0.27E-13
41	4	1	1	9	8	9	8	0.19259259E+01	0.19259259E+01	0.22E-15	0.18E-12
42	4	2	2	13	12	13	12	0.13857864E+02	0.13857864E+02	0.20E-11	0.28E-12
43	4	0	3	18	17	18	17	-0.44000000E+02	-0.44000000E+02	0.82E-12	0.54E-12
44	4	0	6	9	8	9	8	-0.15000000E+02	-0.15000000E+02	0.00E+00	0.00E+00
46	5	2	2	101	98	101	98	0.00000000E+00	0.87462215E-03	0.87E-03	0.40E+01
48	5	2	2	11	10	11	10	0.00000000E+00	0.15430416E-15	0.15E-15	0.00E+00
49	5	2	2	101	100	101	100	0.00000000E+00	0.25433534E-06	0.25E-06	0.00E+00
50	5	3	3	64	63	64	63	0.00000000E+00	0.60718749E-10	0.61E-10	0.00E+00
51	5	3	3	8	7	8	7	0.00000000E+00	0.13946477E-17	0.14E-17	0.00E+00
52	5	3	3	101	100	101	100	0.53266476E+01	0.53276119E+01	0.96E-03	0.00E+00
53	5	3	3	40	39	40	39	0.40930233E+01	0.40930233E+01	0.98E-11	0.00E+00
56	7	4	4	101	100	101	100	-0.34560000E+01	-0.34559868E+01	0.13E-04	0.11E-09
59	2	0	3	23	24	25	24	-0.78042263E+01	-0.78042360E+01	0.96E-05	0.11E-12
60	3	1	1	12	11	12	11	0.32568200E-01	0.32568200E-01	0.38E-11	0.00E+00
62	3	1	1	29	28	29	28	-0.26272514E+05	-0.26272514E+05	0.18E-07	0.00E+00
64	3	0	1	8	5	8	5	0.62998424E+04	0.36192633E+05	0.30E+05	0.15E+02
65	3	0	1	19	18	19	18	0.95352886E+00	0.95352886E+00	0.48E-10	0.14E-13
66	3	0	2	14	13	14	13	0.51816327E+00	0.51816327E+00	0.23E-10	0.86E-13
71	4	1	2	8	7	8	7	0.17014017E+02	0.17014017E+02	0.34E-09	0.13E-13
72	4	0	2	49	48	49	48	0.72767938E+03	0.72767936E+03	0.18E-04	0.14E-16
73	4	1	3	6	5	6	5	0.29894378E+02	0.29894378E+02	0.18E-08	0.89E-15
76	4	0	3	10	9	10	9	-0.46818182E+01	-0.46818182E+01	0.18E-11	0.00E+00
78	5	3	3	9	8	9	8	-0.29197004E+01	-0.29197004E+01	0.15E-09	0.00E+00
79	5	3	3	20	19	20	19	0.78776821E-01	0.78776821E-01	0.17E-10	0.00E+00
80	5	3	3	8	7	8	7	0.53949848E-01	0.53949848E-01	0.79E-11	0.00E+00
81	5	3	3	12	11	12	11	0.53949848E-01	0.53949848E-01	0.46E-11	0.00E+00
83	5	0	6	6	5	6	5	-0.30665539E+05	-0.30665539E+05	0.83E-07	0.00E+00
84	5	0	6	3	1	3	1	-0.52803351E+02	-0.23512435E+02	0.29E+02	0.00E+00
88	2	0	1	24	23	24	23	0.13626568E+01	0.13626907E+01	0.34E-04	0.19E-14
89	3	0	1	32	31	32	31	0.13626568E+01	0.13626907E+01	0.34E-04	0.24E-14
95	6	0	4	3	2	3	2	0.15619514E-01	0.15619525E-01	0.11E-07	0.00E+00
96	6	0	4	3	2	3	2	0.15619513E-01	0.15619525E-01	0.12E-07	0.00E+00
104	8	0	6	101	100	101	100	0.39511634E+01	0.39512151E+01	0.52E-04	0.49E-09
111	10	3	3	101	100	101	100	-0.47761090E+02	-0.47761071E+02	0.19E-04	0.00E+00
112	10	3	3	32	31	32	31	-0.47761086E+00	-0.47761090E+00	0.38E-07	0.00E+00
113	10	0	8	76	75	76	75	0.24306209E+02	0.24306209E+02	0.42E-08	0.12E-12
215	2	0	1	8	7	8	7	0.00000000E+00	-0.33832439E-17	0.34E-17	0.34E-17
216	2	1	1	18	17	18	17	0.10000000E+01	0.99937529E+00	0.62E-03	0.00E+00
217	2	1	2	9	8	9	8	-0.80000000E+00	-0.80000000E+00	0.22E-13	0.89E-13
221	2	0	1	22	21	22	21	-0.10000000E+01	-0.10000000E+01	0.00E+00	0.00E+00
222	2	0	1	7	6	7	6	-0.15000000E+01	-0.15000000E+01	0.00E+00	0.00E+00
223	2	0	2	9	8	9	8	-0.83403245E+00	-0.83403245E+00	0.43E-12	0.10E-10
224	2	0	4	38	37	38	37	-0.30400000E+03	-0.30400000E+03	0.88E-11	0.00E+00
225	2	0	5	7	6	7	6	0.20000000E+01	0.20000000E+01	0.53E-13	0.00E+00
226	2	0	2	11	10	11	10	-0.50000000E+00	-0.50000000E+00	0.24E-12	0.69E-12
227	2	0	2	8	7	8	7	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
228	2	0	2	8	7	8	7	-0.30000000E+01	-0.30000000E+01	0.00E+00	0.00E+00
230	2	0	2	101	100	101	100	0.37500000E+00	-0.25978641E-19	0.38E+00	0.10E+01
232	2	0	3	8	7	8	7	-0.10000000E+01	-0.10000000E+01	0.22E-15	0.00E+00
234	2	0	1	101	100	101	100	-0.80000000E+00	-0.80000000E+00	0.40E-10	0.00E+00
237	2	0	3	19	20	21	20	-0.58903436E+02	-0.58903436E+02	0.90E-08	0.00E+00
238	2	0	3	17	18	19	18	-0.58903436E+02	-0.58903436E+02	0.90E-08	0.00E+00

239	2	0	1	21	21	22	21	-0.58903435E+02	-0.58903436E+02	0.13E-05	0.00E+00
248	3	1	2	15	14	15	14	-0.80000000E+00	-0.80000000E+00	0.85E-13	0.35E-12
249	3	0	1	19	18	19	18	0.10000000E+01	0.10000000E+01	0.58E-10	0.00E+00
250	3	0	2	3	2	3	2	-0.33000000E+04	-0.33000000E+04	0.00E+00	0.00E+00
251	3	0	1	11	10	11	10	-0.34560000E+04	-0.34560000E+04	0.45E-12	0.00E+00
252	3	1	1	37	36	37	36	0.40000000E-01	0.40000000E-01	0.00E+00	0.00E+00
253	3	0	1	9	8	9	8	0.69282032E+02	0.69282032E+02	0.30E-06	0.00E+00
254	3	2	2	19	18	19	18	-0.17320508E+01	-0.17320508E+01	0.22E-15	0.00E+00
262	4	1	4	7	6	7	6	-0.10000000E+02	-0.10000000E+02	0.00E+00	0.30E+01
263	4	2	4	35	34	35	34	-0.10000000E+01	-0.10000000E+01	0.44E-12	0.97E-12
264	4	0	3	20	19	20	19	-0.43999999E+00	-0.43999999E+00	0.37E-13	0.22E-11
269	5	3	3	40	39	40	39	0.40930233E+01	0.40930233E+01	0.98E-11	0.53E-14
270	5	0	1	9	8	9	8	-0.10000000E+01	-0.10000000E+01	0.36E-14	0.32E-13
277	4	0	4	101	100	101	100	0.50761905E+01	0.50744526E+01	0.17E-02	0.47E-02
278	6	0	6	101	100	101	100	0.78385281E+01	0.78850365E+01	0.47E-01	0.57E-02
280	10	0	10	5	5	5	5	0.13375428E+02	0.13376555E+02	0.11E-02	0.00E+00
315	2	0	3	11	10	11	10	-0.80000000E+00	-0.80000000E+00	0.00E+00	0.22E-15
322	2	1	1	24	24	25	24	0.49996001E+03	0.49996001E+03	0.20E-05	0.10E+01
323	2	0	2	93	92	93	92	0.37989446E+01	0.37989446E+01	0.48E-07	0.77E-12
325	2	1	3	7	6	7	6	0.37913415E+01	0.37913414E+01	0.51E-07	0.00E+00
326	2	0	2	9	8	9	8	-0.79807821E+02	-0.79807821E+02	0.15E-06	0.56E-14
327	2	0	1	101	100	101	100	0.30646306E-01	0.30646302E-01	0.38E-08	0.00E+00
329	2	0	3	7	6	7	6	-0.69618139E+04	-0.69618139E+04	0.24E-04	0.68E-13
330	2	0	1	11	10	11	10	0.16205833E+01	0.16205833E+01	0.22E-07	0.67E-14
331	2	0	1	101	100	101	100	0.42580000E+01	0.43156000E+01	0.58E-01	0.00E+00
336	3	2	2	31	7	37	7	-0.33789573E+00	0.41999998E+02	0.42E+02	0.00E+00
337	3	0	1	12	11	12	11	0.60000000E+01	0.60000000E+01	0.89E-15	0.44E-15
339	3	0	1	13	12	13	12	0.33616797E+01	0.33616797E+01	0.11E-07	0.00E+00
340	3	0	1	12	11	12	11	-0.54000000E-01	-0.54000000E-01	0.13E-13	0.11E-15
341	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.20E-08	0.20E-11
342	3	0	1	13	12	13	12	-0.22627417E+02	-0.22627417E+02	0.20E-08	0.20E-11
343	3	0	2	6	5	6	5	-0.56847825E+01	-0.56847825E+01	0.36E-14	0.23E-12
344	3	1	1	12	11	12	11	0.32568200E-01	0.32568200E-01	0.26E-09	0.11E-15
345	3	1	1	27	23	27	23	0.32568200E-01	0.32568200E-01	0.26E-09	0.00E+00
346	3	0	2	6	5	6	5	-0.56847825E+01	-0.56847825E+01	0.36E-14	0.23E-12
347	3	0	1	3	2	3	2	0.17374625E+05	0.17374625E+05	0.27E-03	0.00E+00
353	4	1	3	3	2	3	2	-0.39933673E+02	-0.39933673E+02	0.47E-06	0.00E+00
361	5	0	6	31	30	31	30	-0.77641212E+06	-0.77641212E+06	0.47E-02	0.00E+00
375	10	9	9	35	34	35	34	-0.15160000E+02	-0.15161044E+02	0.10E-02	0.00E+00
396	2	1	1	5	7	8	7	0.00000000E+00	0.20173323E-53	0.20E-53	0.00E+00
397	2	1	1	35	40	41	40	0.00000000E+00	0.29145294E-23	0.29E-23	0.00E+00

Remarks.

(1) Notations about some expressions in Table 5.2 and 5.3 are given as follows.
ME—number of equality constraints, NG—number of constraint function evaluations,
NDG—number of gradient evaluations of constraint function.

(2) TP397 is the example that we have given in section 3 and it is a classical example to illustrate the Marotos effect. Our new SQP-Filter method can overcome the Marotos effect more effectively than Fletcher's. Furthermore, 31 test problems which our new method fails to solve are not given in Table 5.2 and 34 test problems which Fletcher's method fails to solve are not given in Table 5.3(mainly caused by the failure of restoration algorithm).

The majority(98 test problems) of all 130 test problems which have been successfully solved(including TP90, TP91 and TP92 which have successfully been solved by our new method but not Fletcher's method) have the same numerical behavior. The main reason for this may lie in the distance between the initial points and the optimal points is moderate, i.e., not too long

or too short. 5 test problems show almost the same numerical behavior about using these two methods, i.e., the number of function evaluations and gradient evaluations is almost the same, although they are not completely uniform. 17 test problems show much difference in numerical and the reason may be as follows.
(1) When the initial points are far from the optimal points as our new method gives an estimation for that in advance, our new method will exclude much more points and this would fasten the global convergence;
(2) For some test problems, the initial points are near to the optimal point. But our method still reject much more points and this could lead to a slow convergence.
(3) Some test problems converge slowly when the iterative points are nearer to optimal points. As our method still use *two-step criterion* to judge whether points could be accepted by the filter, it will make the convergence slower.
(4) For test problems which are typical examples to illustrate Marotos effect, as we use *two-step criterion* to judge whether points could be accepted by the filter, it will prohibit this effect.

In the following we will give a table to illustrate the compositive behavior of the two methods which are used to solve NLP test problems in [8](see Table 5.4).

In conclusion, the results of this paper show that our new SQP-Filter method behaves at least the same as the one proposed by Fletcher and Leyffer in [4]. We not only give some results of convergence but also test these two methods on a large test set presented by W.Hock and K.Schittkowski in [8]. And the numerical results also show that our method is effective in solving nonlinear programming problems.

Table 5.4

our new method behaves slightly better	18, 88
Fletcher's method behaves slightly better	7, 8, 42
our new method behaves obviously better	27, 43, 46, 49, 50, 64, 84, 90, 91, 92, 230, 336, 345, 396, 397
Fletcher's method behaves obviously better	31, 65
two methods behave almost the same	the remaining 98 test problems

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