Journal of Computational Mathematics, Vol.24, No.5, 2006, 657–664.

PRECONDITIONING HIGHER ORDER FINITE ELEMENT SYSTEMS BY ALGEBRAIC MULTIGRID METHOD OF LINEAR ELEMENTS *1)

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Abstract

We present and analyze a robust preconditioned conjugate gradient method for the higher order Lagrangian finite element systems of a class of elliptic problems. An auxiliary linear element stiffness matrix is chosen to be the preconditioner for higher order finite elements. Then an algebraic multigrid method of linear finite element is applied for solving the preconditioner. The optimal condition number which is independent of the mesh size is obtained. Numerical experiments confirm the efficiency of the algorithm.

Mathematics subject classification: 65N30, 65N55.
Key words: Finite element, Algebraic multigrid methods, Preconditioned Conjugate Gradient, Condition number.

1. Introduction

Multigrid method is one of the most efficient methods for solving large scale algebraic systems arising from the discretizations of partial differential equations(c.f. [1, 2, 9, 8, 10, 11]). The mesh size independent convergence rate can be achieved for geometric multigrid methods. For many practical problems, since the complexity of problems and solution domains, we have to use unstructured grids shown as in the Figure 2 and 3 for examples. The algebraic multigrid methods (AMG) are more suitable for the unstructured grids than geometric multigrid methods. A typical algebraic multigrid algorithm is like the algorithm 2.1, where the matrix B_h is the stiff matrix of the k order Lagrangian finite element. In the algebraic multigrid procedure, the coarsening of the grids is the most important issue but it is not easy to control the number of coarse grid degrees of freedom. The known AMG methods for finite element systems are designed mainly based on the linear element[3, 1]. Whether the convergence rate depends on mesh size or not is still open. The numerical examples show the dependence, see Table 2.1.

Lagrangian finite elements are important class of finite elements family in practical applications, which includes the linear and high order Lagrangian finite elements (see the Figure 1). The matrix structural of higher order finite element system is much complicated than the linear ones. The direct application of AMG algorithm for linear element to the higher order finite element yields the reduction of the efficiency (see the Table 2.1 and Table 2.2 for details). The more robust and efficient AMG algorithms need to be designed carefully.

* Received March 28, 2005; Final revised November 3, 2005.
1) This work was subsidized by the National Basic Research Program of China under the grant 2005CB321701, 2005CB321702, the Research Fund for the Doctoral program of Chinese Ministry of Education, and NSAF(10376031).

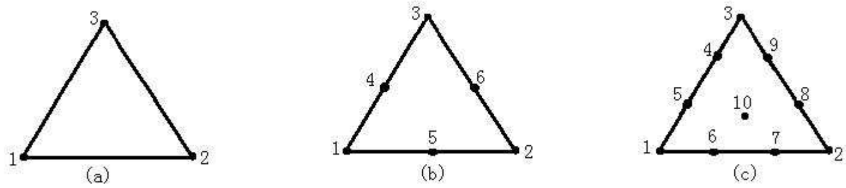

Figure 1: (a) The linear element. (b) The quadric element. (c) The cubic element.

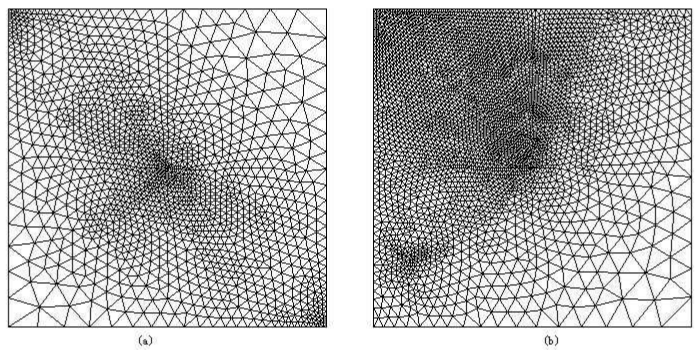

Figure 2: (a) The grid 1 with 2776 elements. (b) The grid 2 with 6427 elements.

Let T^h be a partition of Ω for a higher order Lagrangian finite element discretization, then we can introduce a refined grid T_h^l by refining the grid T^h through connecting the nodes, for example, the Figure 4 shows the grid T_h corresponding to quadric Lagrangian finite element and the refining grid T_h^l. Based on the new partition T_h^1 we can construct the stiffness matrix B_h of the linear Lagrangian finite element. This matrix B_h is supposed to be the preconditioner for the conjugate gradient algorithm for solving the discretization systems of high order Lagrangian finite element. The condition number of the preconditioned conjugate gradient methods is shown to be bounded independently on the mesh size. The numerical experiments confirm our theoretical results. The rigorous proof is given for the quadratic element and it can be extended to the higher order elements easily.

The rest of the paper is organized as follows. In section 2, we introduce the typical algebraic multigrid algorithm and give some comments. In section 3, for high order lagrangian finite elements, we give PCG methods based on algebraic multigrid method of linear finite element and provide some numerical experiments. Finally in section 4, we give a rigorous theoretical analysis for our PCG methods.

2. The Algebraic Multigrid Algorithm

For simplicity, we consider the following model problem

$$\begin{cases} -\nabla(a(x)\nabla u) = f, \quad x := (x_1, x_2) \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $c_0 \le a(x) \le c_1$ and $c_0,\ c_1$ are positive constants.

Let T^h be the triangular partition of the domain Ω, and $\mathcal{P}_k$ be the set of polynomials of degree no more than k, where h is the maximal diameter of all the partition elements in T^h. We introduce the following Lagrangian finite element space.

Definition 2.1 $V_h^k = \{v_h^k(x) : v_h^k(x) \in C(\bar{\Omega}), v_h^k|_T \in \mathcal{P}_k, \forall T \in T^h\}$ *is called a k order La-*

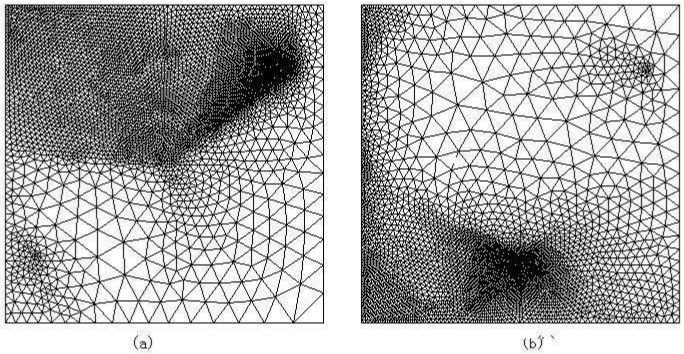

Figure 3: (a) The grid 3 with 8615 elements. (b) The grid 4 with 6181 elements.

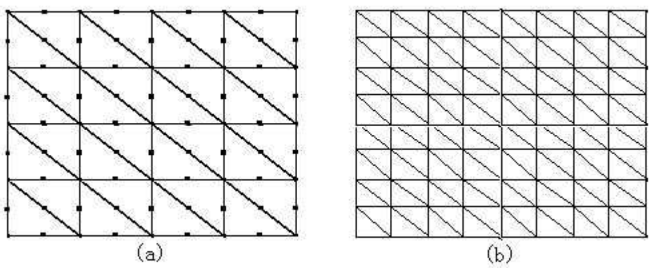

Figure 4: (a) The grid T^h corresponding to the quadric finite element. (b) The refining grid T_h^l of the grid T^h corresponding to the linear finite element.

grangian finite element space and the functions in V_h^k *are called* k *order Lagrangian finite element functions.*

Especially we call V_h^k the linear, quadratic and cubic Lagrangian finite element space for k=1, 2 and 3 respectively(see the Figure 2). The finite element solution of equation (2.1) $u_h^k \in V_h^k$ satisfies

$$a(u_h^k, v_h^k) = (f, v_h^k), \qquad \forall v_h^k \in V_h^k \cap H_0^1(\Omega), \tag{2.2}$$

where

$$a(u,v) = \int_\Omega a(x)(u_{x_1}v_{x_1} + u_{x_2}v_{x_2})\,dx,$$

$$(f,u) = \int_\Omega f u dx,$$

and the Sobolev space $H^m(\Omega) = \{v|\partial^\alpha v \in L^2(\Omega), |\alpha| \le m\}$.

Write the the corresponding discretization systems of the equations (2.2) on the grid T^h in the matrix form

$$A_h u_h = f_h. \tag{2.3}$$

A typical algebraic Multigrid method consists of two components, smoothing and coarsening including restriction and prolongation. In AMG, the smoother is taken as Gauss-Seidel iteration usually. The most important thing to make AMG efficient is coarsening. Let's apply the coarsening algorithm introduced in [4, 6] to the linear element system, and denote the corresponding interpolation operator sequence as $\{P_{k-1}^k\}_{k=2}^J$ which are constructed in energy minimization sense. We describe the corresponding AMG method to solve the equations (2.3) as follows.

Algorithm 2.1 (V-Cycle)

$$
\begin{aligned}
& for\ \ k = J, J-1, \cdots, 2 \\
& \quad for\ j = 1, m_1 \\
& \qquad w_k = w_k + G_k(b_k - \tilde{B}_k w_k); \\
& \quad end\ for \\
& \quad r_k = b_k - \tilde{B}_k w_k; \\
& \quad b_{k-1} = (P^k_{k-1})^t r_k; \\
& end\ for \\
& w_1 = (\tilde{B}_1)^{-1} b_1; \\
& for\ \ k = 2, \cdots, J-1, J \\
& \quad w_k = w_k + P^k_{k-1} w_{k-1}; \\
& \quad for\ j = 1, m_2 \\
& \qquad w_k = w_k + G_k(b_k - \tilde{B}_k w_k); \\
& \quad end\ for \\
& end\ for
\end{aligned}
$$

where $w_J := u_h, b_J := f_h, \tilde{B}_J := A_h$, m_1, m_2 *are the numbers of pre-smoothing and post-smoothing iterations and* G_k *is a certain smoothing operator respectively.*

Assume that $a(x) = 1$ in Equation (2.1), we examine the efficiency of the algebraic multigrid Algorithm 2.1, by solving the equations (2.3) of k order Lagrangian finite element. The numerical results are shown as follows.

Table 2.1. The numerical results of AMG for linear, quadratic and cubic Lagrangian finite element on the structured grid 2(see the Figure 4).

Element	The linear element		The quadric element		The cubic element	
	Time(s)	Iter. times	Time(s)	Iter. times	Time(s)	Iter. times
16 × 16	0.05	5	0.16	13	0.93	44
32 × 32	0.06	7	0.49	14	3.35	50
64 × 64	0.28	10	1.92	16	14.23	55
128 × 128	1.10	12	8.30	18	68.11	62

Table 2.2. The numerical results of AMG for linear, quadratic and cubic Lagrangian finite element on the unstructured grid 1 ∼ grid 4(see the Figure 2 and 3)

Element	The linear element		The quadratic element		The cubic element	
	Time(s)	Iter. times	Time(s)	Iter. times	Time(s)	Iter. times
grid 1	0.16	11	0.82	16	4.34	44
grid 2	0.33	13	2.15	19	10.76	49
grid 3	0.44	13	2.58	18	15.10	46
grid 4	0.22	11	1.97	18	9.72	49

Table 2.1 and Table 2.2 show that the iteration number increases as the order of finite element increases. In this paper we shall propose an efficient preconditioned conjugate gradient methods based on AMG of linear finite element for solving the equations of high order finite element 2.3. The optimal condition number is obtained and numerical results are provided.

3. Preconditioning by Linear Element

In this section, we will give the analysis of convergence of our new algorithms. For simplicity, we only discuss the LPCG algorithm for solving the equations (2.3) of quadratic Lagrangian finite element. Analogous to the above section, let T_h be the regular triangular grid of quadratic Lagrangian finite element space V_h^2 and T_h^l be the refining grid of linear finite element space

V_h^1(see the Figure 4), N be degrees of freedom of the space V_h^2. The matrices A_h and B_h denote the stiff matrices of the equation (2.1) for the finite element space V_h^2 and V_h^1, respectively.

Firstly, we introduce the notation $\lesssim$, $\gtrsim$, $\eqsim$ as same as that in the paper [11], which means that when we write

$$x_1 \lesssim y_1,\ x_2 \gtrsim y_2 \ and\ x_3 \eqsim y_3$$

then there exist constants C_1, c_2, c_3, and C_3 such that

$$x_1 \leq C_1 y_1,\ x_2 \geq c_2 y_2 \text{ and } c_3 x_3 \leq y_3 \leq C_3 x_3.$$

The following Lemma presents the estimate of condition numbers for two SPD matrices A_h and B_h (see [11]).

Lemma 3.1 *For the above SPD matrices A_h and B_h, the condition number of matrix $B_h A_h$ $k(B_h A_h) \lesssim O(1)$ if and only if the following inequalities hold*

$$(B_h V, V) \lesssim (A_h V, V) \lesssim (B_h V, V),\ \forall V = (v_1, v_2, \cdots, v_N)^T \in R^N \tag{3.1}$$

where, $(\cdot,\cdot)$ is the inner product of R^N.

Let $\{x_j\}_{j=1}^N$ be the set of non-Dirichlet nodes of the refining grid T_h^l, we introduce the following interpolation base functions $\{\phi_i^2\}_{i=1}^N$ in V_h^2 and $\{\phi_i^1\}_{i=1}^N$ in V_h^1, respectively, which satisfy

$$\phi_i^l(x_j) = \delta_{i,j}\ i, j = 1(1)N,\ l = 1, 2. \tag{3.2}$$

Define the functions $v_h^2(x)$ in V_h^2 and $v_h^1(x)$ in V_h^1 as follows

$$v_h^l(x) = \sum_{j=1}^N v_j \phi_j^l(x),\ l = 1, 2.$$

Then the inequalities (3.1) are equivalent to

$$a(v_h^1(x), v_h^1(x)) \lesssim a(v_h^2(x), v_h^2(x)) \lesssim a(v_h^1(x), v_h^1(x)),\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N \tag{3.3}$$

where,

$$a(u, v) = \int_\Omega a(x)(u_{x_1} v_{x_1} + u_{x_2} v_{x_2})\ dx.$$

By the coerciveness condition of $a(\cdot,\cdot)$, we can rewriting the inequalities (3.3) as

$$\|v_h^1\|_1 \lesssim \|v_h^2\|_1 \lesssim \|v_h^1\|_1,\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N \tag{3.4}$$

where, $\|\cdot\|$ is the norm of Sobolev space $H^1(\Omega)$.

Theorem 3.1 *For any given regular triangular grid T_h of quadratic Lagrangian finite element space V_h^2, the inequalities (3.4) hold.*

Proof. By the definition of the norm $\|\cdot\|$, we need to prove that

$$\|v_h^1\|_0 \lesssim \|v_h^2\|_0 \lesssim \|v_h^1\|_0,\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N \tag{3.5}$$

and

$$|v_h^1|_1 \lesssim |v_h^2|_1 \lesssim |v_h^1|_1,\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N. \tag{3.6}$$

This can be achieved by proving the following stronger results.

$$\|v_h^1\|_{0,\tau} \lesssim \|v_h^2\|_{0,\tau} \lesssim \|v_h^1\|_{0,\tau},\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N \tag{3.7}$$

and

$$|v_h^1|_{1,\tau} \lesssim |v_h^2|_{1,\tau} \lesssim |v_h^1|_{1,\tau},\ \forall (v_1, v_2, \cdots, v_N)^T \in R^N. \tag{3.8}$$

where, $\tau \in T_h$ is any triangular element τ of quadratic finite element and $\tau_j^l(j = 1(1)4)$ are corresponding four refining triangular elements of linear finite element(see the Figure 6(q1) and (q2)in section 4.)

Let the mesh size h be the maximal diameter of all elements in T^h. For any triangular element τ , denote $x_1(\xi, \eta)$ and $x_2(\xi, \eta)$ as the linear mapping from τ into standard reference triangle $\hat{\tau}$(see the Figure 5), then we have

$$\|v_h^l\|_{0,\tau}^2 \eqsim h^2\|\hat{v}_h^l\|_{0,\hat{\tau}}^2, \quad |v_h^l|_{1,\tau}^2 \eqsim |\hat{v}_h^l|_{1,\hat{\tau}}^2, \ \forall v_h^l \in V_h^l, \ l = 1, 2, \tag{3.9}$$

where, $\hat{v}_h^l(\xi, \eta) := v_h^l(x_1(\xi, \eta), x_2(\xi, \eta)), \ (\xi, \eta) \in \hat{\tau}$.
By (3.9), we need only to prove that the equations (3.7) and (3.8) hold in standard triangular element $\hat{\tau}$.

Since $\hat{v}_h^1(\xi, \eta)$ and $\hat{v}_h^2(\xi, \eta)$ have the same freedoms in $\hat{\tau}$, thus it's obvious under the norm $\|\cdot\|_0$ that

$$\|\hat{v}_h^2\|_{0,\hat{\tau}} \eqsim \|\hat{v}_h^1\|_{0,\hat{\tau}}, \tag{3.10}$$

In the following, we will prove under the semi-norm $|\cdot|_1$ that

$$|\hat{v}_h^2|_{1,\hat{\tau}} \eqsim |\hat{v}_h^1|_{1,\hat{\tau}}, \tag{3.11}$$

For any $p_0 \in \mathcal{P}_0$, we have by equivalence of norms for finite dimensional space

$$|\hat{v}_h^1|_{1,\hat{\tau}} = |\hat{v}_h^1 + p_0|_{1,\hat{\tau}} \le \|\hat{v}_h^1 + p_0\|_{1,\hat{\tau}} \le \|\hat{v}_h^2 + p_0\|_{1,\hat{\tau}},$$

thus, by Bramble-Hilbert Lemma,

$$|\hat{v}_h^1|_{1,\hat{\tau}} \le \inf_{\forall p_0 \in \mathcal{P}_0} \|\hat{v}_h^2 + p_0\|_{1,\hat{\tau}} \lesssim |\hat{v}_h^2|_{1,\hat{\tau}}. \tag{3.12}$$

Analogously, we can prove that

$$|\hat{v}_h^2|_{1,\hat{\tau}} \le \inf_{\forall p_0 \in \mathcal{P}_0} \|\hat{v}_h^1 + p_0\|_{1,\hat{\tau}} \lesssim |\hat{v}_h^1|_{1,\hat{\tau}}. \tag{3.13}$$

Combining (3.12) and (3.13), we get (3.11), which completes the proof of the Theorem 3.1.

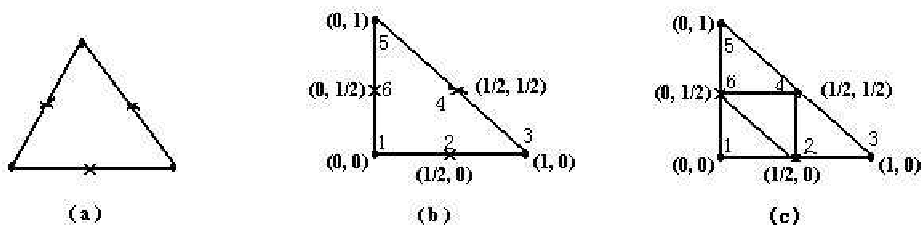

Figure 5: (a) A triangular element τ of quadratic finite element. (b) The corresponding standard triangular element $\hat{\tau}$. (c) The refining mesh $\hat{\tau}_l$ of standard triangular element $\hat{\tau}$

By the Theorem 3.1, we know that the condition number of our preconditioned conjugate gradient method is independent of the equations size, which verify the numerical experiment results above.
Remark. The idea of algorithm and the theoretical analysis we provide in this paper can be extended to the general high order triangular and isoparametric quadrilateral finite elements.

4. A PCG Algorithm Based on AMG

To design an efficient PCG method, we need to construct an appropriate preconditioner for the equations (2.3) of the high order Lagrangian finite element. Let N_k be the freedoms of k

order Lagrangian finite element space V_h^k, by refining the grid T^h, we can get a grid T_h^l so that the total numbers of nodes in T_h^l equal to N_k. For example, the Figure 6 show how to refine an element corresponding to quadratic and cubic Lagrangian finite element into the elements corresponding to linear finite element.

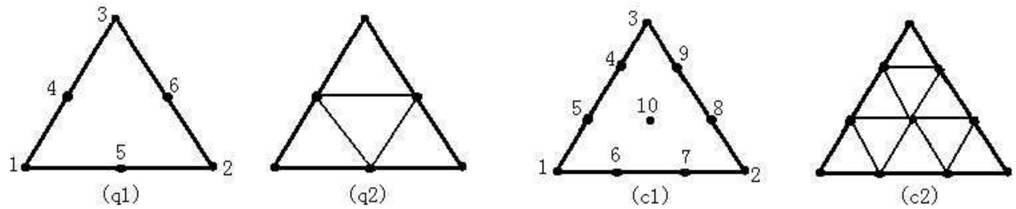

Figure 6: The Figure (q1) and (q2) express a element τ of quadratic finite element and corresponding four refining elements $\tau_j^l(j = 1(1)4)$ of linear finite element. The Figure (c1) and (c2) express a element τ of cubic finite element and corresponding nine refining elements $\tau_j^l(j = 1(1)9)$ of linear finite element.

For the refining grid T_h^l, we introduce the linear finite element space $V_h^1 = \{v_h^1(x) : v_h^1(x) \in C(\bar{\Omega}), v_h^1|_T \in \mathcal{P}_1, \forall T \in T_h^l\}$, then we can obtain the stiff matrix B_h of the linear Lagrangian finite element on a grid T_h^l by geometric approach. Choosing the approximate inverse matrix $\tilde{B}_h^{-1}$ of the matrix B_h as the preconditioner, we provided a preconditioned conjugate gradient methods for the discretization systems (2.3). In this paper, we define $\tilde{B}_h^{-1}$ as follows. For given vector $r_h \in R^{N_k}$, then $w_h = \tilde{B}_h^{-1} r_h$ is obtained by calling some $V - Cycles$ of the algorithm 2.1 to solving the following equations

$$B_h w_h = r_h, \tag{4.1}$$

We call the above preconditioned conjugate gradient methods as the LPCG algorithm. The Table 4.1 and 4.2 show that the iteration numbers is independent of the size of the equations, thus comparing with the method in which we apply the ordinary AMG method to solve the systems (2.3) directly (see The Table 2.1 and 2.2), our algorithm is more robust and efficient.

Table 4.1. The numerical results of LPCG algorithm for quadratic and cubic element with structured grids(see the Figure 4)

	quadratic element		cubic element	
Element	Time(s)	Iter. times	Time(s)	Iter. times
16 × 16	0.17	7	0.22	8
32 × 32	0.28	7	0.83	9
64 × 64	1.27	7	3.24	9
128 × 128	4.01	8	13.57	10

Table 4.2. The numerical results of LPCG algorithms for quadratic and cubic element with unstructured grids (see the Figure 2 and 3)

	quadratic element		cubic element	
Element	Time(s)	Iter. times	Time(s)	Iter. times
grid3	0.55	8	1.48	9
grid4	1.54	8	3.24	11
grid5	2.20	9	4.45	11
grid6	1.49	8	3.18	11

Next, using popular ILU-PCG algorithms where the preconditioner is the incomplete LU decompose matrix of the high order finite element matrix A_h, we make some numerical experiments. The Table 4.3 and 4.4 show the advantages of our LPCG algorithms.

Table 4.3. The numerical results of ILU-PCG algorithms for quadratic and cubic finite element with structured grids(see the Figure 4)

Element	quardic element		cubic element	
	Time(s)	Iter. times	Time(s)	Iter. times
16×16	0.22	23	0.44	41
32×32	0.49	40	1.60	71
64×64	1.59	62	7.20	114
128×128	8.40	102	37.57	166

Table 4.4. The numerical results of ILU-PCG algorithms for quadratic and cubic finite element with unstructured grids(see the Figure 2 and 3)

Element	quardic element		cubic element	
	Time(s)	Iter. times	Time(s)	Iter. times
grid3	1.27	98	4.51	169
grid4	2.80	146	12.47	255
grid5	4.67	187	24.83	325
grid6	3.07	145	12.57	249

Acknowledgement. We would like to express our sincerely thanks to Jinchao Xu and the referee(s) for the helpful suggestions.

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