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A SIMPLICIAL ALGORITHM FOR COMPUTING AN INTEGER ZERO POINT OF A MAPPING WITH THE DIRECTION PRESERVING PROPERTY *1)

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Abstract

A mapping $f : Z^n \rightarrow R^n$ is said to possess the direction preserving property if $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points x and y with $\|x - y\|_\infty \leq 1$. In this paper, a simplicial algorithm is developed for computing an integer zero point of a mapping with the direction preserving property. We assume that there is an integer point x^0 with $c \leq x^0 \leq d$ satisfying that $\max_{1 \leq i \leq n}(x_i - x_i^0)f_i(x) > 0$ for any integer point x with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c - e \leq x \leq d + e\}$, where c and d are two finite integer points with $c \leq d$ and $e = (1, 1, \cdots, 1)^\top \in R^n$. This assumption is implied by one of two conditions for the existence of an integer zero point of a mapping with the preserving property in van der Laan et al. (2004). Under this assumption, starting at x^0, the algorithm follows a finite simplicial path and terminates at an integer zero point of the mapping. This result has applications in general economic equilibrium models with indivisible commodities.

Mathematics subject classification: 90C49.
Key words: Integer Zero Point, Direction Preserving, Simplicial Algorithm, Triangulation, Existence.

1. Introduction

The problem we consider in this paper is to compute an integer zero point of a mapping $f : Z^n \rightarrow R^n$. The interests in integer zero points or fixed points of a mapping have been inspired by the work in Iimura (2003) though the statement of the existence of a discrete fixed point in Iimura (2003) is incorrect and a corrected statement was given in Iimura et al. (2004) after an application of the integrally convex set defined in Favati and Tardella (1990). A brief introduction to the applications of discrete fixed points of a mapping in economics can be found in Iimura (2003) and references therein.

Following the definition in Iimura (2003), we say that $f(x)$ satisfies the direction preserving property if $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points x and y with $\|x - y\|_\infty \leq 1$. We assume throughout this paper that $f(x)$ satisfies the direction preserving property. Recently, under two different conditions, based on the $2n$-ray algorithm in van der Laan and Talman (1981), a constructive proof of the existence of an integer zero point of a mapping with the direction preserving property was obtained in van der Laan et al. (2004). Those two conditions can be stated as follows.

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Condition 1.1. *There exist integer vectors m, x^0, and M with $m + e < x^0 < M - e$ and $e = (1, 1, \cdots, 1)^\top$ such that $(x - x^0)^\top f(x) > 0$ for any integer point x on the boundary of $C = \{y \in R^n \mid m \leq y \leq M\}$.*

Condition 1.2. *There exists an integer vector u with $u > e$ such that $f_k(x)f_k(-y) \leq 0$, $k = 1, 2, \cdots, n$, for any two cell connected integer points x and y on the boundary of $U = \{z \in R^n \mid -u \leq z \leq u\}$.*

Given these two conditions, the following two theorems can be found in van der Laan et al. (2004).

Theorem 1.1. *If Condition 1.1 holds, there exists an integer point $x^* \in C$ such that $f(x^*) = 0$.*

Theorem 1.2. *If Condition 1.2 holds, there exists an integer point $x^* \in U$ such that $f(x^*) = 0$.*

In Dang (2005), a new condition for the existence of an integer zero point of the mapping was introduced, which is as follows.

Condition 1.3. *There is an integer point x^0 with $c \leq x^0 \leq d$ satisfying that $\max_{1 \leq i \leq n}(x_i - x_i^0)f_i(x) > 0$ for any integer point x with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c - e \leq x \leq d + e\}$, where c and d are two finite integer points with $c \leq d$ and $e = (1, 1, \cdots, 1)^\top$.*

For Conditions 1.1, 1.2 and 1.3, the following lemma was proved in Dang (2005).

Lemma 1.1. *Condition 1.1 implies Condition 1.3. However, Condition 1.3 implies neither Condition 1.1 nor Condition 1.2.*

Given Condition 3, based on the $(n+1)$-ray algorithm proposed in van der Laan and Talman (1979), the following theorem was proved in Dang (2005).

Theorem 1.3. *If Condition 1.3 holds, there exists an integer point $x^* \in H$ such that $f(x^*) = 0$.*

In this paper, based on the 2-ray algorithm given in Yamamoto (1984), we will develop a simplicial algorithm for computing an integer zero point of the mapping satisfying Condition 3. The 2-ray algorithm is one of simplicial methods for computing a fixed point of a continuous mapping. The simplical methods were originated in Scarf (1967), and have been substantially developed after Scarf's seminal work (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Dang and Maaren, 1998; Eaves, 1972; Eaves and Saigal, 1972; Forster, 1995; Kojima and Yamamoto, 1982; Kuhn, 1968; van der laan and Talman, 1979, 1981; Merrill, 1972; Scarf, 1973, 1981; Todd, 1976; Yamamoto, 1983; etc.). The basic idea of the algorithm is as follows. It assigns to each integer point of H an integer label and subdivides H into integer simplices. Starting at x^0, the algorithm follows a finite simplicial path and terminates at an integer zero point of the mapping. An advantage of the 2-ray algorithm over the $(n+1)$-ray algorithm is that some more superior triangulations of R^n can be its underlying triangulations without any modifications.

The rest of this paper is organized as follows. An integer labeling rule is introduced in Section 2. The algorithm is given in Section 3.

2. Integer Labeling

Let $N = \{1, 2, \cdots, n\}$, $N_0 = \{1, 2, \cdots, n+1\}$, and u^i be the ith unit vector of R^n for $i = 1, 2, \cdots, n$. Let $I(0) = \emptyset$ and $I(h) = \{1, 2, \cdots, h\}$ for any $h \in N$. For $\alpha \in \{-1, 1\}$ and $h \in N$, let

$$X(x^0, h, \alpha) = \{x \in R^n \mid \alpha(x_h - x_h^0) \geq 0 \text{ and } x_j = x_j^0,\ j \in N \backslash I(h)\}.$$

To obtain a simplicial algorithm for computing an integer zero point of f, we need a triangulation of H that subdivides every integer unit cube contained in H into integer simplices. Here, an integer unit cube is a unit cube having only integer vertices and an integer simplex is a simplex having only integer vertices. There are several triangulations suitable for this purpose, which include the K_1-triangulation in Freudenthal (1942), the J_1-triangulation in Todd (1976), the D_1-triangulation in Dang (1991), etc. A specific choice of a triangulation plays however no dominant role at all in this paper though efficiency of simplicial methods depends critically on the underlying triangulation. We will choose the D_1-triangulation as an underlying triangulation of the algorithm. For completeness of the following discussions, we introduce the D_1-triangulation here.

A simplex of the D_1-triangulation of R^n is the convex hull of $n+1$ integer vectors, y^0, y^1, $\ldots$, y^n, which are given as follows.

$$\begin{aligned}
&\text{If } p = 0,\ y^0 = y \text{ and}\\
&y^k = y + s_{\pi(k)} u^{\pi(k)},\ k = 1, 2, \ldots, n, \text{ and}\\
&\text{if } p \geq 1,\ y^0 = y + s,\\
&y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)},\ k = 1, 2, \cdots, p-1, \text{ and}\\
&y^k = y + s_{\pi(k)} u^{\pi(k)},\ k = p, p+1, \ldots, n,
\end{aligned}$$

where y is an integer point of R^n with every component of $y - x^0$ being an even number, p an integer with $0 \leq p \leq n-1$, $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ a permutation of elements of $N = \{1, 2, \ldots, n\}$, and $s = (s_1, s_2, \cdots, s_n)^\top$ a sign vector with $s_i \in \{-1, 1\}$. Let D_1 be the set of all such simplices. Since a simplex of the D_1-triangulation is determined by y, π, s and p, we use $D_1(y, \pi, s, p)$ to denote it.

We say that two simplices of D_1 are adjacent if they have a common facet. We show how to generate all the adjacent simplices of a simplex of the D_1-triangulation of R^n in the following. For a given simplex $\sigma = D_1(y, \pi, s, p)$ with vertices y^0, y^1, $\ldots$, y^n, its adjacent simplex opposite to a vertex, say y^i, is given by $D_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p})$, where $\bar{y}$, $\bar{\pi}$, $\bar{s}$, $\bar{p}$ are generated according to the following table.

Let $\mathcal{D}_1$ be the set of faces of simplices of D_1. A q-dimensional simplex of $\mathcal{D}_1$ with vertices y^0, y^1, $\ldots$, y^q is denoted by $< y^0, y^1, \ldots, y^q >$. The restriction of $\mathcal{D}_1$ on $X(x^0, h, \alpha)$ for any $h \in N$ and $\alpha \in \{-1, 1\}$ is given by

$$\mathcal{D}_1 | X(x^0, h, \alpha) = \{\sigma \in \mathcal{D}_1 \mid \sigma \subset X(x^0, h, \alpha) \text{ and } \dim(\sigma) = h\},$$

where $\dim(\cdot)$ stands for the dimension of a set. Obviously, $\mathcal{D}_1 | X(x^0, h, \alpha)$ is a triangulation of $X(x^0, h, \alpha)$.

Pivot Rules of the D_1-Triangulation

i			$\bar{y}$	$\bar{\pi}$	$\bar{\pi}$	$\bar{p}$
0	$n=1$		$y+2s_{\pi(1)}u^{\pi(1)}$	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p
	$n\geq 2$	$p=0$	y	s	π	1
		$p=1$	y	s	π	0
		$p\geq 2$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p
$i\geq 1$		$p=0$	y	$s-2s_{\pi(i)}u^{\pi(i)}$	π	p
		$i<p-1$	y	s	π^a	p
		$i=p-1$	y	s	π	$p-1$
	$i>p-1$	$1\leq p<n-1$	y	s	π^b	$p+1$
	$i=n-1$	$1\leq p=n-1$	$y+2s_{\pi(n)}u^{\pi(n)}$	$s-2s_{\pi(n)}u^{\pi(n)}$	π	p
	$i=n$	$1\leq p=n-1$	$y+2s_{\pi(n-1)}u^{\pi(n-1)}$	$s-2s_{\pi(n-1)}u^{\pi(n-1)}$	π	p

$$\pi^a=(\pi(1),\cdots,\pi(i+1),\pi(i),\cdots,\pi(n))$$
$$\pi^b=(\pi(1),\cdots,\pi(p-1),\pi(i),\pi(p),\cdots,\pi(i-1),\pi(i+1),\cdots,\pi(n))$$

For $\sigma\in\mathcal{D}_1$, let $\text{grid}(\sigma)=\max\{\|x-y\|_\infty \mid x\in\sigma \text{ and } y\in\sigma\}$. We define $\text{mesh}(D_1)=\max_{\sigma\in\mathcal{D}_1}\text{grid}(\sigma)$. Clearly, $\text{grid}(\sigma)=1$ for any $\sigma\in\mathcal{D}_1$, and $\text{mesh}(D_1)=1$.

In our simplicial algorithm, we need an integer labeling rule that assigns an integer label to each integer point of H. Such an integer labeling rule is given in the following definition.

Definition 2.1. *For $x\in Z^n$, we assign to x an integer label $l(x)$ given by $l(x)=0$ if $f(x)=0$, and*

$$l(x)=\begin{cases}\min\{k\mid f_k(x)>0\} & \text{if } f_j(x)>0 \text{ for some } j\in N,\\ \\ n+1 & \text{if } f(x)\leq 0 \text{ and } f(x)\neq 0.\end{cases}$$

Definition 2.2. *For $h=0,1,\cdots,n$, a simplex $\sigma=<y^0,y^1,\cdots,y^h>$ of $\mathcal{D}_1$ is h-complete if σ carries $h+1$ different nonzero integer labels and h of these labels are contained in $I(h)$.*

Definition 2.3.

1. *A q-dimensional simplex $\sigma=<y^0,y^1,\ldots,y^q>$ of $\mathcal{D}_1$ is complete if $l(y^i)\neq l(y^j)$ for $0\leq i<j\leq q$, and $l(y^k)\neq 0$, $k=0,1,\ldots,q$.*
2. *A q-dimensional simplex $\sigma=<y^0,y^1,\ldots,y^q>$ of $\mathcal{D}_1$ is 0-complete if $l(y^i)\neq l(y^j)$ for $0\leq i<j\leq q$, and there is some k satisfying that $l(y^k)=0$.*
3. *For $h=1,2,\cdots,n$, a simplex $\sigma=<y^0,y^1,\cdots,y^h>$ of $\mathcal{D}_1$ is almost h-complete if σ carries either only all the integer labels in $I(h)$, or all the integer labels in $I(h-1)$, no integer labels 0 and h, and at least one integer label in $N\backslash I(h)$.*

As a direct result of Definition 2.3, we have

Lemma 2.1. *Every almost h-complete simplex has exactly two $(h-1)$-complete facets.*

Let ∂H denote the boundary of H. Then, according to the assumption, for any integer point $x\in\partial H$ with $f(x)\neq 0$,

$$\max_{1\leq i\leq n}(x_i-x_i^0)f_i(x)>0.$$

Lemma 2.2. *For any $h \in N$, there is no complete $(h-1)$-dimensional simplex in $X(x^0, h, -1) \cap \partial H$ carrying only integer labels in $I(h)$, and there is no complete $(h-1)$-dimensional simplex in $X(x^0, h, 1) \cap \partial H$ carrying all the integer labels in $I(h-1)$ and one integer label in $N_0 \backslash I(h)$.*

Proof. Suppose that there is a complete $(h-1)$-dimensional simplex in $X(x^0, h, -1) \cap \partial H$ carrying only integer labels in $I(h)$. Let $\sigma =< y^1, y^2, \cdots, y^h >$ be such a complete simplex.

Without loss of generality, we assume that $l(y^i) = i$, $i = 1, 2, \cdots, h$. Since $f(x)$ satisfies the direction preserving property and grid$(\sigma) = 1$, hence, for any i, $f_k(y^i) \geq 0$, $k = 1, 2, \cdots, h$. From $l(y^h) = h$ and Definition 2.1, we derive $f_k(y^h) \leq 0$, $k = 1, 2, \cdots, h-1$, and $f_h(y^h) > 0$. Then, $f_k(y^h) = 0$, $k = 1, 2, \cdots, h-1$. From the definition of $X(x^0, h, -1)$, we obtain that, for any $x \in X(x^0, h, -1)$, $x_h - x^0_h \leq 0$, and $x_i - x^0_i = 0$, $i \notin I(h)$. Thus,

$$(y^h_j - x^0_j) f_j(y^h) \leq 0, \ j = 1, 2, \cdots, n.$$

Therefore,

$$\max_{1 \leq j \leq n} (y^h_j - x^0_j) f_j(y^h) \leq 0.$$

Since $y^h \in \partial H$ and $f(y^h) \neq 0$, hence,

$$\max_{1 \leq j \leq n} (y^h_j - x^0_j) f_j(y^h) > 0.$$

A contradiction occurs. The first part of the lemma follows.

Suppose that there is a complete $(h-1)$-dimensional simplex in $X(x^0, h, 1) \cap \partial H$ carrying all the integer labels in $I(h-1)$ and one integer label in $N_0 \backslash I(h)$. Let $\sigma =< y^1, y^2, \cdots, y^h >$ be such a complete simplex.

Without loss of generality, we assume that $l(y^i) = i$, $i = 1, 2, \cdots, h-1$, and $l(y^h) \in N_0 \backslash I(h)$. Since $f(x)$ satisfies the direction preserving property and grid$(\sigma) = 1$, hence, for any i, $f_k(y^i) \geq 0$, $k = 1, 2, \cdots, h-1$. From $l(y^h) \in N_0 \backslash I(h)$ and Definition 2.1, we derive $f_k(y^h) \leq 0$, $k = 1, 2, \cdots, h$. Then, $f_k(y^h) = 0$, $k = 1, 2, \cdots, h-1$. From the definition of $X(x^0, h, 1)$, we obtain that, for any $x \in X(x^0, h, 1)$, $x_h - x^0_h \geq 0$, and $x_i - x^0_i = 0$, $i \notin I(h)$. Thus,

$$(y^h_j - x^0_j) f_j(y^h) \leq 0, \ j = 1, 2, \cdots, n.$$

Therefore,

$$\max_{1 \leq j \leq n} (y^h_j - x^0_j) f_j(y^h) \leq 0.$$

Since $y^h \in \partial H$ and $f(y^h) \neq 0$, hence,

$$\max_{1 \leq j \leq n} (y^h_j - x^0_j) f_j(y^h) > 0.$$

A contradiction occurs. The second part of the lemma follows.

As a result of the direction preserving property and mesh$(D_1) = 1$, we have

Lemma 2.3. *There is no complete n-dimensional simplex contained in H.*

Proof. Suppose that there is a complete n-dimensional simplex contained in H. Let $\sigma =< y^0, y^1, \cdots, y^n >$ be such a complete simplex. Without loss of generality, we assume $l(y^i) = i$, $i = 1, 2, \cdots, n$, and $l(y^0) = n+1$. Then,

$$f_i(y^i) > 0, \ i = 1, 2, \cdots, n.$$

Since $f(x)$ satisfies the direction preserving property and $\|y^0 - y^i\|_\infty = 1$, hence, $f_i(y^0) \geq 0$, $i = 1, 2, \cdots, n$. From $l(y^0) = n+1$, we obtain that $f(y^0) \leq 0$ and $f(y^0) \neq 0$. A contradiction occurs. The lemma follows.

3. The Algorithm

In this section, based on the 2-ray algorithm given in Yamamoto (1984), the integer labeling rule in Definition 2.1, and the results in Lemma 2.2 and Lemma 2.3, a simplicial algorithm for computing an integer zero point of f is obtained, which is as follows.

Initialization: Let $y^0 = x^0$ and compute $l(y^0)$. If $l(y^0) = 0$, the algorithm terminates and an integer zero point of f has been found. Otherwise, let $maxK = l(y^0)$, $h = 1$,

$$\alpha = \begin{cases} -1 & \text{if } h = maxK, \\ 1 & \text{otherwise,} \end{cases}$$

$\tau_0 =< y^0 >$, σ_0 be the unique h-dimensional simplex in $X(x^0, h, \alpha)$ having τ_0 as a facet, y^+ be the vertex of σ_0 opposite to τ_0, and $k = 0$. Go to Step 1.

Step 1: Compute $l(y^+)$. If $l(y^+) = 0$, the algorithm terminates and an integer zero point of f has been found. If either $l(y^+) = h$ and $maxK > h$ or $l(y^+) > h$ and $maxK = h$, then σ_k is h-complete and go to Step 3. Otherwise, proceed as follows. Let y^- be the unique vertex of τ_k such that

$$l(y^-) = \begin{cases} l(y^+) & \text{if } l(y^+) \leq h, \\ maxK & \text{otherwise,} \end{cases}$$

and τ_{k+1} the facet of σ_k opposite to y^-. Let $maxK = l(y^+)$ if $l(y^+) > h$, and go to Step 2.

Step 2: If $\tau_{k+1} \subset X(x^0, h-1, \alpha)$ for some $\alpha \in \{-1, 1\}$, go to Step 4. Otherwise, proceed as follows. Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet. Let y^+ be the vertex of σ_{k+1} opposite to τ_{k+1} and $k = k+1$. Go to Step 1.

Step 3: Let $maxK = l(y^+)$ if $l(y^+) > h$. Let $h = h+1$, $\tau_{k+1} = \sigma_k$, and

$$\alpha = \begin{cases} -1 & \text{if } h = maxK, \\ 1 & \text{otherwise.} \end{cases}$$

Let σ_{k+1} be the unique h-dimensional simplex in $X(x^0, h, \alpha)$ having τ_{k+1} as a facet, and y^+ be the vertex of σ_{k+1} opposite to τ_{k+1}. Let $k = k+1$, and go to Step 1.

Step 4: Let $\sigma_{k+1} = \tau_{k+1}$, y^- be the unique vertex of σ_{k+1} such that

$$l(y^-) = \begin{cases} h-1 & \text{if } \alpha = 1, \\ maxK & \text{otherwise,} \end{cases}$$

and τ_{k+2} the facet of σ_{k+1} opposite to y^-. Let $maxK = h-1$ if $\alpha = -1$. Let $h = h-1$ and $k = k+1$, and go to Step 2.

Theorem 3.1. *If Condition 1.3 holds, the algorithm will terminate within a finite number of iterations at an integer point $x^* \in H$ with $f(x^*) = 0$.*

Proof. Lemma 2.2 implies that all the simplices generated by the algorithm are contained in H. Applying Lemma 2.3 and following an standard argument given in Todd (1976), one can derive that the algorithm will never cycle. Since H is bounded, hence, there is a finite number of simplices in H and the algorithm will terminate within a finite number of iterations. From Lemma 2.3, we know that there is no complete n-dimensional simplex in H. This result implies that the algorithm will terminate at an integer point $x^* \in H$ with $f(x^*) = 0$. The theorem follows.

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