MULTIGRID ALGORITHM FOR THE COUPLING SYSTEM OF NATURAL BOUNDARY ELEMENT METHOD AND FINITE ELEMENT METHOD FOR UNBOUNDED DOMAIN PROBLEMS[*1)]

Sheng Zhang

(*School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China*
Email: szhang@nankai.edu.cn)

Dehao Yu

(*LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences,*
Beijing 100080, China
Email: ydh@lsec.cc.ac.cn)

Abstract

In this paper, some V-cycle multigrid algorithms are presented for the coupling system arising from the discretization of the Dirichlet exterior problem by coupling the natural boundary element method and finite element method. The convergence of these multigrid algorithms is obtained even with only one smoothing on all levels. The rate of convergence is found uniformly bounded independent of the number of levels and the mesh sizes of all levels, which indicates that these multigrid algorithms are optimal. Some numerical results are also reported.

Mathematics subject classification: 65N30, 65N38, 65N22, 65F10.
Key words: Multigird algorithm, Finite element method, Boundary element method, Coupling, Unbounded domain problem.

1. Introduction

In many fields of scientific and engineering computing, it is necessary to solve boundary values problems of partial differential equations over unbounded domains. The standard techniques such as the finite element method, which is effective for problems in bounded domain, may meet some difficulties for unbounded domain problems and in particular the corresponding computing cost may be very high. So for problems of this kind, it is a good choice to use the method that combines the boundary element method and finite element method. This treatment enables us to combine the advantages of boundary element method for treating domains extended to infinity with those of finite element method in treating the complicated bounded domain problems. Research in this direction is of great importance in both theory and practical computation.

The procedure of this kind of coupling can be briefly described as follows. The unbounded domain is divided into two subregions, i.e., a bounded inner one and an unbounded outer one, by introducing an artificial common boundary. Then, the problem is reduced to an equivalent one in the bounded region. There are many approaches to accomplish this reduction (refer to [6, 7, 8, 10, 11, 12, 14, 15, 17, 19, 20, 24, 26] and references therein). Natural boundary reduction method is one of them.

Natural boundary reduction method and its coupling with finite element method, which is also known as the exact artificial boundary condition method, were suggested and developed first by Feng in 1980, Yu in 1982 and Han in 1985. And a very similar method, the so-called DtN method, was devised by Keller and Givoli in 1989. In this reduction, the problem over unbounded exterior domain is reduced to an bounded problem with a hyper-singular integral equation on the artificial boundary by using a Green function to get the exact artificial boundary condition with hyper-singular integrals. It is fully compatible with the variational principle over the domain, and the boundary elements are also fully compatible with the domain elements. This coupling is natural and direct. Moreover, the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form. As a result, the analysis of the discrete problem is simplified, and also the error estimates and the numerical stability are restored (see [11, 23, 24]). In this paper, we follow this approach.

With a discretization scheme, the construction of efficient algorithms for solving the resulting discrete system is of great importance. So, our goal is to construct efficient algorithms for the discrete system obtained from the coupling of natural boundary element method and finite element method.

It is well known that multigrid algorithms are among the most efficient methods for solving discretization equations arising from various finite element approximations of boundary value problem on bounded domain (for multigrid method, refer to [1, 2, 3, 4, 13, 21] and references therein). During the last three decades, there has been intensive research toward multigrid methods. The purpose of this paper is to construct multigrid algorithms for discretization equations arising from the coupling of the natural boundary element method and finite element method for the Dirichlet exterior problem and to investigate their convergence.

In the following sections, some V-cycle multigrid algorithms are constructed. We will investigate the convergence of these multigrid algorithms even with only one smoothing on all levels. The rate of convergence is shown to be uniformly bounded independent of the number of levels and the mesh sizes of all levels, which indicates that the proposed multigrid algorithms are optimal.

The remainder of this paper is organized as follows: In section 2, we present our model problem and introduce the natural boundary reduction method. Multigrid algorithm is described and analyzed in section 3. And some numerical results are reported in section 4.

2. Model Problem and Natural Boundary Reduction

We adopt the standard notations for Sobolev space, with their norms and semi-norms as presented in [5, 9]. Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$, $\Omega^c = \mathbb{R}^2 \setminus (\Omega \cup \partial\Omega)$, $f \in L^2(\Omega^c)$ be a given compactly supported function. We consider the following model problem

$$\begin{cases} -\triangle u & = f, \quad in \ \ \Omega^c, \\ \quad u & = 0, \quad on \ \ \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r), \qquad\qquad |\nabla u(x,y)| = O(1/r^2),$$

as $r = \sqrt{x^2 + y^2} \to \infty$ where α is a constant. Define

$$H^1_\triangle(\Omega^c) = \{v | \ \ \frac{v}{\sqrt{r^2 + 1}\ln(r^2 + 2)} , \frac{\partial v}{\partial x} , \frac{\partial v}{\partial y} \in L^2(\Omega^c) , \ v|_{\partial\Omega} = 0\}$$

and

$$a(w,v) = \int\int_{\Omega^c} \nabla w \cdot \nabla v dxdy , \qquad \forall w, v \in H^1_\triangle(\Omega^c) .$$
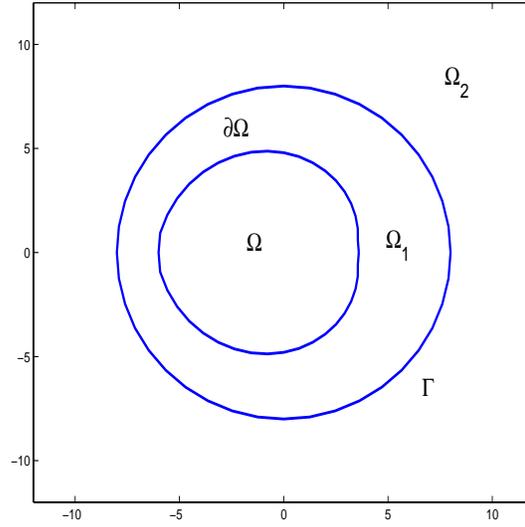
Figure 1: Artificial boundary

Then the corresponding variational form of (2.1) can be written as: Find $u \in H^1_{\triangle}(\Omega^c)$ such that

$$a(u, v) = (f, v) \; , \qquad\qquad \forall v \in H^1_{\triangle}(\Omega^c) \; . \qquad\qquad (2.2)$$

According to the hypothesis on f, we choose a circle disc Ω_0 containing $\bar{\Omega}$ and *supp* f. Let $\Omega_1 = \Omega^c \cap \Omega_0$, $\Omega_2 = \Omega_0^c = \mathbb{R}^2 \setminus (\Omega_0 \cup \partial \Omega_0)$ and $\Gamma = \partial \Omega_0$ (see Figure 1). Then we have

$$a(u, v) = a_1(u, v) + a_2(u, v), \qquad\qquad (2.3)$$

where $a_i(u, v) = \int \int_{\Omega_i} \nabla u \cdot \nabla v \, dx dy, \;\; i = 1, 2$.

Next, we introduce the natural boundary reduction method and derive a coupled variational form equivalent to (2.3). From Green's formula on Ω_2, we have

$$a_2(u, v) = \int_{\Gamma} \frac{\partial}{\partial n} u(z) \cdot v(z) dz + \int \int_{\Omega_2} f v \, dx dy \; . \qquad\qquad (2.4)$$

Let $V(z, z')$ be the Green's function for the Laplace operator on the domain Ω_2, which satisfies

$$\begin{cases} -\triangle V(z, z') &= \delta(z - z'), \quad \forall z, z' \in \Omega_2, \\ V(z, z')|_{z \in \Gamma} &= 0, \qquad\qquad \forall z' \in \Omega_2, \end{cases}$$

subject to the same asymptotic conditions as u. By taking $w = V(z, z'), v = u$ in Green's second formula

$$\int \int_{\Omega_2} (w \triangle v - v \triangle w) dz' = \int_{\Gamma} (w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n}) dz' \; ,$$

we get (refer to [23, 24])

$$u(z) = \int \int_{\Omega_2} f(z') V(z, z') dz' - \int_{\Gamma} \frac{\partial}{\partial n'} V(z, z') u(z') dz' \; , \qquad\qquad \forall z \in \Omega_2 \; ,$$

where n and n' denote the exterior normal vectors on Γ (viewed as the boundary of Ω_2) at the respective points z and z'. Thus we obtain

$$\frac{\partial u}{\partial n}(z) = \int \int_{\Omega_2} f(z') \frac{\partial}{\partial n} V(z, z') dz' - \int_{\Gamma} \frac{\partial^2}{\partial n \partial n'} V(z, z') u(z') dz' \; , \qquad\qquad \forall z \in \Gamma \; . \qquad (2.5)$$

Let

$$Ku(z) = -\int_\Gamma \frac{\partial^2}{\partial n \partial n'} V(z, z') u(z') dz' , \qquad z \in \Gamma . \qquad (2.6)$$

Then, it follows from (2.4)-(2.6) and the fact $supp\ f \subset \Omega_0$ that

$$a_2(u, v) = \int_\Gamma Ku(z) \cdot v(z) dz. \qquad (2.7)$$

Define $H^1_*(\Omega_1) = \{v|\ v \in H^1(\Omega_1),\ v|_{\partial\Omega} = 0\}$ and

$$b(u, v) = a_1(u, v) + < Ku, v >_\Gamma, \qquad (2.8)$$

where $< \cdot, \cdot >_\Gamma$ denotes the L^2 inner product on Γ. With (2.3) and (2.7), we can rewrite the variation form (2.2) as: Find $u \in H^1_*(\Omega_1)$ such that

$$b(u, v) = \int\int_{\Omega_1} fv dx dy, \qquad \forall v \in H^1_*(\Omega_1). \qquad (2.9)$$

Remark 2.1. The operator $K : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is shown to be just the Dirichlet-Neumann operator (Steklov-Poincaré operator) for Ω_2 in [22]. So, it is symmetric and semi-positive definite with respect to the inner product $< \cdot, \cdot >_\Gamma$ (see [23, 24]), which indicates that $b(\cdot, \cdot)$ is symmetric, bounded and coercive in $H^1_*(\Omega_1)$. Thus, it follows from the well known Lax-Milgram Theorem that the variational problem (2.9) has unique solution $u \in H^1_*(\Omega_1)$.

Remark 2.2. As Γ is a circle, the Green's function $V(z, z')$ can be expressed explicitly. For example, in the case that the center of the circle Γ is the origin and its radius is R,

$$V(z, z') = \frac{1}{4\pi} \ln \frac{R^4 + r^2 r'^2 - 2R^2 rr' \cos(\theta - \theta')}{R^2(r^2 + r'^2 - 2rr' \cos(\theta - \theta'))} , \qquad z = (r, \theta),\ \ z' = (r', \theta') \in \Omega_2.$$

Moreover, we have (refer to [23, 24])

$$\frac{\partial^2}{\partial n \partial n'} V(z, z') = \frac{1}{4\pi \sin^2((\theta - \theta')/2)} , \qquad z = (r, \theta),\ \ z' = (r', \theta') \in \Gamma.$$

It is worth pointing out that these explicit expressions ensure the practical use of the natural boundary reduction method. Moreover these expressions also imply another advantage of the natural boundary reduction method compared over many other approaches: we need not to solve any boundary integral equation associated with the unbounded subdomain. Instead only calculation of certain singular integrations is needed.

Remark 2.3. In order to show how to calculate the singular integrations involved in the bilinear form, we divide the artificial boundary Γ into m circular arcs with the same length. Let $\{\phi_i\}_{i=1}^m$ be the set of the nodal basis functions on Γ. Noticing that, in polar coordinates (r, θ), the nodal basis functions associated with Γ are piecewise linear with respect to the variable θ, we can obtain (refer to [23, 24])

$$\begin{aligned} < K\phi_i, \phi_j >_\Gamma &= -\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\phi_i(\theta)\phi_j(\theta')}{\sin^2((\theta-\theta')/2)} d\theta d\theta' \\ &= \frac{4m^2}{\pi^3} \sum_{k=1}^\infty \frac{1}{k^3} \sin^4 \frac{k\pi}{m} \cos \frac{2k(i-j)\pi}{m} , \qquad i, j = 1, \cdots, m . \end{aligned}$$

From this expression, we can easily find that the stiffness matrix of K is symmetric and circulant, which also enhances efficiency and implies only small memory storage is needed for the stiffness matrix. Moreover, since the series converges quickly, a suitable short finite sum can be used to simplify the calculation.

3. Multigrid Algorithm

In this section, we introduce a multigrid algorithm and analyze its convergence.

First, we introduce some multi-level triangulations and notations. Because of the appearance of curved triangles, if we use the usual approach to refine mesh by obtaining the $k+1$ level triangulation $\mathcal{T}_{k+1}$ by dividing the triangle in the k level into four by connecting the midpoint of each edge and construct the corresponding finite element space, then, unfortunately, the resulting spaces are not nested. And obviously non-nested spaces will cause some additional difficulty and trouble for the analysis of the convergence of the multigrid algorithm. In order to avoid this additional difficulty caused by non-nested spaces, we do not use the usual approach to refine mesh. Instead, we introduce another approach by employing the initial triangulation as a parametrization of Ω_1 and obtain the refinement step from subdividing the reference triangle which leads to a sequence of nested spaces(refer to [18]).

More precisely, let Γ be parameterized by a 1-periodic function $\psi : [0,1] \to \Gamma$ such that

$$\beta(z) := |\psi'(z)| > 0$$

for all $z \in [0,1]$. And let $0 = z_0^{(1)} < z_1^{(1)} < \cdots < z_{N_1}^{(1)} = 1$, $N_1 \in \mathbb{N}$, be a uniform partition of $[0,1]$ with $z_i^{(1)} - z_{i-1}^{(1)} = h_1 := 1/N_1$, $i = 1,2,\cdots,N_1$. We denote by Ω_{h1} the polygonal domain whose vertices on Γ are $\psi(z_0^{(1)}), \psi(z_1^{(1)}), \cdots, \psi(z_{N_1}^{(1)})$. Let $\mathcal{T}_1$ be a regular triangulation of $\bar{\Omega}_{h1}$ by triangles of diameter satisfying $diam\ \tau_i \leq h_1 \sup_{z \in [0,1]} \beta(z)$ for all $\tau_i \in \mathcal{T}_1$. Then there exists affine mapping G_i such that $G_i(\hat{\tau}) = \tau_i$ for each $\tau_i \in \mathcal{T}_1$ where $\hat{\tau} = \triangle((0,0),(1,0),(0,1))$ is the reference triangle. Next, we replace each triangle $\tau_i \in \mathcal{T}_1$ with two vertices on Γ by the corresponding curved triangle. Without loss of generality we may suppose that the vertices p_0, p_1, p_2 of a curved triangle τ_i satisfy $p_1 = \psi(z_j^{(1)})$, $p_2 = \psi(z_{j+1}^{(1)})$, respectively. Then, a C^∞-mapping $\tilde{G}_i$ with $\tilde{G}_i(\hat{\tau}) = \tau_i$ is given by (refer to [25])

$$\tilde{G}_i = G_i + U_i$$

with

$$U_i(t) = \frac{t_1}{1-t_2}[\psi((1-t_2)z_j^{(1)} + t_2 z_{j+1}^{(1)}) - (1-t_2)\psi(z_j^{(1)}) - t_2\psi(z_{j+1}^{(1)})] \ .$$

We denote this initial triangulation with non-curved and curved triangles by $\tilde{\mathcal{T}}_1$. Subdividing in the usual way the reference triangle $\hat{\tau}$ in 4, 16, 64, $\cdots$ triangles yields a sequence of meshes

$$\tilde{\mathcal{T}}_1 \subset \tilde{\mathcal{T}}_2 \subset \tilde{\mathcal{T}}_3 \subset \cdots$$

with step width $diam\ \tau_i \leq h_j \sup_{z \in [0,1]} \beta(z)$ for all $\tau_i \in \tilde{\mathcal{T}}_j$, where $h_j = 2^{-(j-1)}h_1$.

The finite element spaces on these meshes are considered to be piecewise linear and continuous. Let the degrees of freedom on the mesh $\tilde{\mathcal{T}}_j$ be $\tilde{N}_j$ and denote the corresponding nodal basis function by $\phi_{j,k}$, $k = 1,2,\cdots,\tilde{N}_j$. With the notation $W_j = span\ \{\phi_{j,k},\ k = 1,\cdots,\tilde{N}_j\}$ for $j = 1,2,\cdots,J$, we obtain

$$W_1 \subset W_2 \subset \cdots \subset W_J \subset H_*^1(\Omega_1) \ .$$

Then the corresponding J-level discrete variational problem of (2.9) is: Find $u_J \in W_J$ such that

$$b(u_J, v) = \int\int_{\Omega_1} fv dx dy \ , \qquad\qquad \forall v \in W_J \ . \qquad\qquad (3.1)$$

It is not difficult to see that (3.1) has a unique solution (see [23, 24]). Moreover the corresponding error estimates in H^1, L^2 and L^∞ norm can also be found in [23, 24].

In what follows, we denote c or C with or without subscript a generic positive constant, which can take different values in different occurrences but always be independent of the mesh size and the number of levels.

Define operators $A_k : W_k \mapsto W_k$, $\hat{A}_k : W_k \mapsto W_k$, $S_k : W_J \mapsto W_k$, $\hat{S}_k : W_J \mapsto W_k$ and $T_k : W_J \mapsto W_k$, $k = 1, 2, \cdots, J$, by

$$(A_k w, v) = b(w, v), \qquad \forall w, v \in W_k, \tag{3.2}$$

$$(\hat{A}_k w, v) = a_1(w, v), \qquad \forall w, v \in W_k, \tag{3.3}$$

$$b(S_k w, v) = b(w, v), \qquad \forall w \in W_J, \ v \in W_k, \tag{3.4}$$

$$a_1(\hat{S}_k w, v) = a_1(w, v), \qquad \forall w \in W_J, \ v \in W_k, \tag{3.5}$$

$$(T_k w, v) = (w, v), \qquad \forall w \in W_J, \ v \in W_k. \tag{3.6}$$

From above definitions, we can easily obtain

$$T_k A_J = A_k S_k, \tag{3.7}$$

and

$$T_k \hat{A}_J = \hat{A}_k \hat{S}_k. \tag{3.8}$$

Let Q_k be a certain smoother, then the V-cycle multigrid algorithm can be described as follows:

Algorithm 3.1.

Set $B_1 = A_1^{-1}$. For $k > 1$ define $B_k : W_k \mapsto W_k$ in terms of B_{k-1} as follows:

Let $g \in W_k$,

1. Set $x_0 = 0$.
2. Define x_i for $i = 1, 2, \cdots, m(k)$ by

$$x_i = x_{i-1} + Q_k^t(g - A_k x_{i-1}).$$

3. Set $y_{m(k)} = x_{m(k)} + q_k$, where q_k is defined by

$$q_k = B_{k-1} T_{k-1}(g - A_k x_{m(k)}).$$

4. Define y_i for $i = m(k) + 1, m(k) + 2, \cdots, 2m(k)$ by

$$y_i = y_{i-1} + Q_k(g - A_k y_{i-1}).$$

5. Set $B_k g = y_{2m(k)}$.

In step 2 above, Q_k^t denotes the adjoint of Q_k with respect to the inner product $(\cdot, \cdot)$ and we take $m(k) = 1$ for all k which is sufficient in our analysis. The case $m(k) > 1$ and the cases with only pre-smoothing or post-smoothing can be analyzed similarly.

Let $P_k = I - Q_k A_k$, $k = 1, \cdots, J$, $D_k = Q_k A_k S_k$ for $k > 1$ and $D_1 = S_1$. Then it is easy to check that the error operator associated with the discretization equation

$$A_k u = f \tag{3.9}$$

is given by

$$\tilde{E}_k = I - B_k A_k S_k = E_k E_k^*, \tag{3.10}$$

where the superscript $*$ denotes the adjoint with respect to $b(\cdot, \cdot)$ and

$$E_k = (I - D_k)(I - D_{k-1}) \cdots (I - D_1). \tag{3.11}$$

In order to analyze the convergence of the multigrid algorithm, we make some assumptions, which will be verified later. Let $\tilde{D}_k = A_k S_k / \lambda_k = T_k A_J / \lambda_k$ for $k > 1$ and $\tilde{D}_1 = S_1$, where λ_k denotes the largest eigenvalue of A_k.

(A1)   There exists a constant $C_b > 0$ independent of k such that

$$b(v, v) \le C_b \sum_{k=0}^{J} b(\tilde{D}_k v, v), \qquad \forall v \in W_J. \tag{3.12}$$

(A2)   There exist $0 < \zeta < 1$ and $\widetilde{C} > 0$ independent of k such that

$$b(\tilde{D}_k w, w) \leq (\widetilde{C}\zeta^{k-j})^2 b(w, w), \qquad \forall w \in W_j, j \leq k. \tag{3.13}$$

For the smoother Q_k, we assume the following two condition are satisfied.

(A3)   There exits a constant $C_Q \geq 1$ independent of k such that

$$\frac{(v, v)}{\lambda_k} \leq C_Q(\bar{Q}_k v, v), \qquad \forall v \in W_k, \tag{3.14}$$

where $\bar{Q}_k = (I - P_k^* P_k)A_k^{-1}$.

(A4)   There exists a positive constant $\sigma < 2$ independent of k such that

$$b(D_k v, D_k v) \leq \sigma b(D_k v, v), \qquad \forall v \in W_J. \tag{3.15}$$

With these assumptions, we can obtain the convergence theorem of multigrid algorithm by following the frame work of [3]. For the self-containedness of this paper, we still provide a proof here.

Theorem 3.1. *If the assumptions* (A1)-(A4) *are satisfied, then there exists a positive constant* $\delta < 1$ *independent of h and J such that*

$$0 \leq b((I - B_J A_J)v, v) \leq \delta b(v, v), \qquad \forall v \in W_J. \tag{3.16}$$

Proof. From (3.10), it is obvious that the lower inequality holds since

$$b((I - B_J A_J)v, v) = b(E_J^* v, E_J^* v) := |||E_J^* v|||^2 \geq 0 .$$

And by the fact that $|||E_J^* v||| = |||E_J v|||$, we only need to estimate $|||E_J v|||$ for the upper inequality. From (3.11), we get

$$E_k = (I - D_k)E_{k-1}, \tag{3.17}$$

from which yields

$$b(E_k v, E_k v) = b(E_{k-1} v, E_{k-1} v) - 2b(E_{k-1} v, D_k E_{k-1} v) + b(D_k E_{k-1} v, D_k E_{k-1} v),$$

or equivalently,

$$b(E_{k-1} v, E_{k-1} v) - b(E_k v, E_k v) = b((2I - D_k)E_{k-1} v, D_k E_{k-1} v). \tag{3.18}$$

Let $E_0 = I$. Then it follows from (3.18) that

$$b(v, v) - b(E_J v, E_J v) = \sum_{i=1}^{J} b((2I - D_k)E_{k-1} v, D_k E_{k-1} v). \tag{3.19}$$

Define $\bar{D}_k = \bar{Q}_k A_k S_k = (I - P_k^* P_k)S_k$ for $k > 1$ and $\bar{D}_k = S_1$. From $P_k = I - Q_k A_k$ and the definition of P_k^*, it is easy to check that $P_k^* = I - Q_k^t A_k$. Combining this with (3.2), (3.4) and the definition of D_k, we have

$$
\begin{aligned}
&b(\bar{D}_k E_{k-1} v, E_{k-1} v) \\
&= b((I - (I - Q_k^t A_k)(I - Q_k A_k))S_k E_{k-1} v, E_{k-1} v) \\
&= b((Q_k^t + Q_k)A_k S_k E_{k-1} v, E_{k-1} v) - b(Q_k^t A_k Q_k A_k S_k E_{k-1} v, E_{k-1} v) \\
&= b((Q_k^t + Q_k)A_k S_k E_{k-1} v, S_k E_{k-1} v) - b(Q_k^t A_k Q_k A_k S_k E_{k-1} v, S_k E_{k-1} v) \\
&= ((Q_k^t + Q_k)A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) - (Q_k^t A_k Q_k A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) \\
&= 2(A_k S_k E_{k-1} v, Q_k A_k S_k E_{k-1} v) - (A_k Q_k A_k S_k E_{k-1} v, Q_k A_k S_k E_{k-1} v) \\
&= b((2I - D_k)E_{k-1} v, D_k E_{k-1} v).
\end{aligned}
\tag{3.20}
$$

Thus, (3.19) and (3.20) imply

$$b(v, v) - b(E_J v, E_J v) = \sum_{i=1}^{J} b(\bar{D}_k E_{k-1} v, E_{k-1} v). \tag{3.21}$$

From (A1), (3.2), (3.4), the triangle inequality, the fact $\tilde{D}_1 = \bar{D}_1 = S_1$ and $E_0 = I$, we have

$$
\begin{aligned}
b(v,v) &\leq C_b \sum_{i=1}^{J} b(\tilde{D}_k v, v) \\
&= C_b[b(\tilde{D}_1 v, v) + \sum_{i=2}^{J} b(A_k S_k v, v)/\lambda_k] = C_b[b(\tilde{D}_1 v, v) + \sum_{i=2}^{J} \|A_k S_k v\|_0^2/\lambda_k] \\
&\leq C_b[b(\tilde{D}_1 v, v) + 2\sum_{i=2}^{J}(\|A_k S_k E_{k-1} v\|_0^2 + \|A_k S_k(I - E_{k-1})v\|_0^2)/\lambda_k] \\
&= C_b[b(\bar{D}_1 E_0 v, E_0 v) + 2\sum_{i=2}^{J}\|A_k S_k E_{k-1} v\|_0^2/\lambda_k + 2\sum_{i=2}^{J} b(A_k S_k(I - E_{k-1})v, (I - E_{k-1})v)/\lambda_k] \\
&\leq 2C_b[b(\bar{D}_1 E_0 v, E_0 v) + \sum_{i=2}^{J}\|A_k S_k E_{k-1} v\|_0^2/\lambda_k + \sum_{i=2}^{J} b(\tilde{D}_k(I - E_{k-1})v, (I - E_{k-1})v)].
\end{aligned}
\tag{3.22}
$$

For $k = 2, 3, \cdots, J$, (A3), (3.2) and (3.4) imply that

$$
\begin{aligned}
\sum_{i=2}^{J}\|A_k S_k E_{k-1} v\|_0^2/\lambda_k &\leq C_Q \sum_{i=2}^{J}(\bar{Q}_k A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) \\
&= C_Q \sum_{i=2}^{J} b(\bar{Q}_k A_k S_k E_{k-1} v, E_{k-1} v) = C_Q \sum_{i=2}^{J} b(\bar{D}_k E_{k-1} v, E_{k-1} v).
\end{aligned}
\tag{3.23}
$$

Let $\tilde{v} = (I - E_{k-1})v$. Noting (3.17), we get $E_{i-1} - E_i = D_i E_{i-1}$, from which follows

$$
I - E_k = \sum_{i=1}^{k} D_i E_{i-1} .
\tag{3.24}
$$

Let $w_i = D_i E_{i-1} v$. By (3.24), (3.2), (3.4), Cauchy-Schwarz inequality, (A2) and (A4), it follows

$$
\begin{aligned}
\sum_{k=2}^{J} b(\tilde{D}_k \tilde{v}, \tilde{v}) &\leq \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} b(\tilde{D}_k D_i E_{i-1} v, D_j E_{j-1} v) \\
&= \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} (A_k S_k D_i E_{i-1} v, A_k S_k D_j E_{j-1} v)/\lambda_k \\
&\leq \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \|A_k S_k D_i E_{i-1} v\|_0 \|A_k S_k D_j E_{j-1} v\|_0/\lambda_k \\
&= \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} b(\tilde{D}_k w_i, w_i)^{1/2} b(\tilde{D}_k w_j, w_j)^{1/2} \\
&\leq \tilde{C}^2 \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \zeta^{2k-i-j} b(w_i, w_i)^{1/2} b(w_j, w_j)^{1/2} \\
&\leq \tilde{C}^2 \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \zeta^{2k-i-j}[b(w_i, w_i) + b(w_j, w_j)]/2 \\
&= \tilde{C}^2 \sum_{k=2}^{J}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \zeta^{k-j}\zeta^{k-i} b(w_i, w_i) \leq \frac{\zeta\tilde{C}^2}{1-\zeta} \sum_{k=2}^{J}\sum_{i=1}^{k-1} \zeta^{k-i} b(w_i, w_i) \\
&= \frac{\zeta\tilde{C}^2}{1-\zeta} \sum_{i=1}^{J-1}\sum_{k=i+1}^{J} \zeta^{k-i} b(w_i, w_i) \leq \frac{\zeta^2\tilde{C}^2}{(1-\zeta)^2} \sum_{i=1}^{J-1} b(D_i E_{i-1} v, D_i E_{i-1} v) \\
&\leq \frac{\sigma\zeta^2\tilde{C}^2}{(1-\zeta)^2} \sum_{i=1}^{J-1} b(D_i E_{i-1} v, E_{i-1} v) .
\end{aligned}
\tag{3.25}
$$

On the other hand, from (3.20) and (A4), we have

$$
\begin{aligned}
b(\bar{D}_k E_{i-1} v, E_{i-1} v) &= b((2I - D_k)E_{i-1} v, D_k E_{i-1} v) \\
&\geq (2 - \sigma)b(D_k E_{i-1} v, E_{i-1} v) .
\end{aligned}
\tag{3.26}
$$

Note $b(\bar{D}_J E_{J-1} v, E_{J-1} v) = (\bar{Q}_J A_J E_{J-1} v, A_J E_{J-1} v) \geq 0$. This, together with (3.25) and

(3.26), yields

$$
\begin{aligned}
\sum_{k=2}^{J} b(\tilde{D}_k \tilde{v}, \tilde{v}) \quad &\leq \frac{\sigma \zeta^2 \tilde{C}^2}{(1-\zeta)^2} \sum_{i=1}^{J-1} b(D_i E_{i-1} v, E_{i-1} v) \\
&\leq \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2 \tilde{C}^2 \sum_{k=1}^{J-1} b(\bar{D}_k E_{k-1} v, E_{k-1} v) \\
&\leq \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2 \tilde{C}^2 \sum_{k=1}^{J} b(\bar{D}_k E_{k-1} v, E_{k-1} v) .
\end{aligned}
\tag{3.27}
$$

Thus, from (3.22), (3.23), (3.27) and (3.21), it follows

$$
b(v, v) \leq C_m \sum_{k=1}^{J} b(\bar{D}_k E_{k-1} v, E_{k-1} v) = C_m [b(v, v) - b(E_J v, E_J v)],
$$

where $C_m = 2C_b[C_Q + \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2 \tilde{C}^2]$. Let $\delta = 1 - 1/C_m < 1$. Then

$$
b(E_J v, E_J v) \leq (1 - 1/C_m) b(v, v) = \delta b(v, v) .
$$

This completes the proof.

With Theorem 3.1, it is obvious that we only need to verify (A1)-(A4) to achieve the convergence of the multigrid algorithm. To this end, we still need some more notations. Let $\hat{D}_k = \hat{A}_k \hat{S}_k / \hat{\lambda}_k = T_k \hat{A}_J / \hat{\lambda}_k$ for $k > 1$ and $\hat{D}_1 = \hat{S}_1$, where $\hat{\lambda}_k$ denotes the largest eigenvalue of $\hat{A}_k$. It is well known that (A1) and (A2) hold for $a_1(\cdot, \cdot)$ which are denoted as (A1D) and (A2D) respectively here (refer to [3]).

(A1D)   There exists a constant $C_a > 0$ independent of k such that

$$
a_1(v, v) \leq C_a \sum_{k=0}^{J} a_1(\hat{D}_k v, v), \qquad \forall v \in W_J,
\tag{3.28}
$$

(A2D)   There exist $0 < \zeta_a < 1$ and $\tilde{C}_a > 0$ independent of k such that

$$
a_1(\hat{D}_k w, w) \leq (\tilde{C}_a \zeta_a^{k-j})^2 a_1(w, w), \qquad \forall w \in W_j, j \leq k.
\tag{3.29}
$$

Next, we show that (A.1) and (A.2) also hold for $b(\cdot, \cdot)$. First, some lemmas are needed. Let $\hat{w}$ denote the discrete harmonic extension of $w|_\Gamma$, which is defined by

$$
\begin{cases}
a_1(\hat{w}, \hat{v}) = 0, & \forall \hat{v} \in W_J^0, \\
\hat{w}|_{\partial \Omega} = 0, & \hat{w}|_\Gamma = w|_\Gamma,
\end{cases}
$$

where $W_J^0 = \{v| \; v \in W_J , v|_\Gamma = 0\}$.

Lemma 3.1. *[16] For any $w \in W_J$, we have*

$$
< Kw, w >_\Gamma \leq |\hat{w}|_{1, \Omega_1}.
\tag{3.30}
$$

With this lemma, we can obtain the following result.

Lemma 3.2. *For any $w \in W_J$, we have*

$$
a_1(w, w) \leq b(w, w) \leq C_0 a_1(w, w).
\tag{3.31}
$$

Proof. Since the lower inequality is obvious, we only need to prove the upper one. To this end, let $\tilde{w} \in W_J$ be the discrete harmonic extension of $w|_\Gamma$, which is defined by

$$
\begin{cases}
a_1(\tilde{w}, \psi) = 0, & \forall \psi \in W_J^0, \\
\tilde{w}|_{\partial \Omega} = 0, & \tilde{w}|_\Gamma = w|_\Gamma.
\end{cases}
$$

Set $w_1 = w - \tilde{w} \in W_J^0$. Then we get

$$a_1(w, w) = |w_1|_{1,\Omega_1}^2 + |\tilde{w}|_{1,\Omega_1}^2, \tag{3.32}$$

and

$$b(w, w) = |w_1|_{1,\Omega_1}^2 + |\tilde{w}|_{1,\Omega_1}^2 + <Kw, w>_\Gamma . \tag{3.33}$$

With (3.32), (3.33) and Lemma 3.1, we obtain

$$
\begin{aligned}
b(w, w) &= |w_1|_{1,\Omega_1}^2 + |\tilde{w}|_{1,\Omega_1}^2 + <Kw, w>_\Gamma \\
&\leq |w_1|_{1,\Omega_1}^2 + 2|\tilde{w}|_{1,\Omega_1}^2 \leq C_0 a_1(w, w).
\end{aligned}
$$

This completes the proof.

The following two lemmas can be found in [3].

Lemma 3.3. *Suppose $\bar{A}$ and $\tilde{A}$ are two symmetric positive definite operators on W_J. Then for all $w \in W_J$,*

$$C_1(\bar{A}w, w) \leq (\tilde{A}w, w) \leq C_2(\bar{A}w, w) \tag{3.34}$$

if and only if

$$C_1(\tilde{A}^{-1}w, w) \leq (\bar{A}^{-1}w, w) \leq C_2(\tilde{A}^{-1}w, w), \tag{3.35}$$

where C_1 and C_2 are the same constants in both inequalities.

Lemma 3.4. *Assume that two symmetric positive definite operators $\bar{A}$ and $\tilde{A}$ on W_J with corresponding bilinear forms $\bar{a}(\cdot, \cdot)$ and $\tilde{a}(\cdot, \cdot)$ sstisfy*

$$C_1\tilde{a}(w, w) \leq \bar{a}(w, w) \leq C_2\tilde{a}(w, w) \qquad \forall w \in W_J . \tag{3.36}$$

Then (A1) holds for $\tilde{A}$ if and only if (A1) holds for $\bar{A}$.

With the help of Lemma 3.2, Lemma 3.4 and (A1D), we obtain

Theorem 3.2. *(A1) holds for $b(\cdot, \cdot)$.*

Next, we show that the assumption (A2) also holds for $b(\cdot, \cdot)$.

Theorem 3.3. *(A2) holds for $b(\cdot, \cdot)$.*

Proof. For $k = 1$, there is nothing to prove. For $k > 1$, with (3.2)-(3.5), (3.7) and (3.8), we have

$$a_1(\hat{D}_k w, w) = \hat{\lambda}_k^{-1} \|\hat{A}_k \hat{S}_k w\|_0^2 = \hat{\lambda}_k^{-1} \|T_k \hat{A}_J w\|_0^2, \tag{3.37}$$

and

$$b(\tilde{D}_k w, w) = \lambda_k^{-1} \|A_k S_k w\|_0^2 = \lambda_k^{-1} \|T_k A_J w\|_0^2 . \tag{3.38}$$

Then, from (3.37) and (A2D), we have

$$
\begin{aligned}
\hat{\lambda}_k^{-1} \|T_k \hat{A}_J w\|_0^2 &= a_1(\hat{D}_k w, w) \leq (\tilde{C}_a \zeta_a^{k-j})^2 a_1(w, w) \\
&= (\tilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} \hat{A}_J w, \hat{A}_J w) .
\end{aligned} \tag{3.39}
$$

Set $v = \hat{A}_J w$. Then, it follows

$$\hat{\lambda}_k^{-1} \|T_k v\|_0^2 \leq (\tilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} v, v) .$$

From the above inequality, Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
\lambda_k^{-1} \|T_k v\|_0^2 &\leq \hat{\lambda}_k^{-1} \|T_k v\|_0^2 \leq (\tilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} v, v) \\
&\leq C_2 (\tilde{C}_a \zeta_a^{k-j})^2 (A_J^{-1} v, v) .
\end{aligned}
$$

Setting $\widetilde{C} = C_2^{1/2}\widetilde{C}_a$, $\zeta = \zeta_a$ and $v = A_J w$ in the above inequality, we obtain from (3.38) that

$$b(\tilde{D}_k w, w) = \lambda_k^{-1}\|T_k A_J w\|_0^2 \leq (\widetilde{C}\zeta^{k-j})^2 b(w,w).$$

This completes the proof.

Next, we introduce some smoothers such that (A3) and (A4) are satisfied. Due to the appearance of the term $< Kw, w >_\Gamma$, it makes the nodal basis function on the artificial boundary does not have local support. As a result, smoothers constructed in [3] can not be used directly because they may not satisfy (A3) or (A4) any more. Therefore, the smoothers have to be choosen and checked carefully to overcome this difficulty.

Before presenting some smoothers satisfying (A3) and (A4), we make some observations for smoothers of the form $Q_k = \frac{\mu}{\lambda_k}I$, where μ is a parameter. In the following, we will discuss the condition under which (A3) and (A4) are satisfied by this kind of smoothers.

First, we check the assumption (A3). Noting that for this kind of smoother, we have $P_k = I - \frac{\mu}{\lambda_k}A_k$ and

$$
\begin{aligned}
b((I - P_k^* P_k)v, v) &= b(v,v) - b(P_k v, P_k v) \\
&= b(v,v) - [b(v,v) - \frac{2\mu}{\lambda_k}b(A_k v, v) + \frac{\mu^2}{\lambda_k^2}b(A_k v, A_k v)] \\
&= \frac{2\mu}{\lambda_k}\|A_k v\|_0^2 - \frac{\mu^2}{\lambda_k^2}b(A_k v, A_k v) .
\end{aligned}
\tag{3.40}
$$

It follows from the fact $b(A_k v, A_k v) \leq \lambda_k \|A_k v\|_0^2$ that

$$\frac{\mu^2}{\lambda_k^2}b(A_k v, A_k v) \leq \frac{\mu^2}{\lambda_k^2}\lambda_k\|A_k v\|_0^2 = \frac{\mu^2}{\lambda_k}\|A_k v\|_0^2 .\tag{3.41}$$

Thus, with (3.40) and (3.41), we obtain

$$b((I - P_k^* P_k)v, v) \geq \frac{\mu(2-\mu)}{\lambda_k}\|A_k v\|_0^2 .\tag{3.42}$$

For $0 < \mu < 2$, we chose $C_Q = 1/(\mu(2-\mu)) \geq 1$. Then, it follows from (3.42) that

$$
\begin{aligned}
C_Q(\bar{Q}A_k v, A_k v) &= C_Q((I - P_k^* P_k)A_k^{-1}A_k v, A_k v) = C_Q b((I - P_k^* P_k)v, v) \\
&\geq C_Q \frac{\mu(2-\mu)}{\lambda_k}\|A_k v\|_0^2 = \|A_k v\|_0^2/\lambda_k .
\end{aligned}
$$

Setting $w = A_k v$, we have (A3) holds for this kind of smoother.

To check (A4), we notice that for the smoother of the form $Q_k = \frac{\mu}{\lambda_k}I$, it follows from (3.2) and (3.4) that

$$b(D_k v, D_k v) = \frac{\mu^2}{\lambda_k^2}b(A_k S_k v, A_k S_k v),\tag{3.43}$$

$$
\begin{aligned}
b(D_k v, v) &= \mu b(A_k S_k v, v)/\lambda_k = \mu b(A_k S_k v, S_k v)/\lambda_k \\
&= \mu\|A_k S_k v\|_0^2/\lambda_k .
\end{aligned}
\tag{3.44}
$$

Since $b(A_k S_k v, A_k S_k v) \leq \lambda_k \|A_k S_k v\|_0^2$, we obtain from (3.43) and (3.44) that

$$
\begin{aligned}
b(D_k v, D_k v) &= \frac{\mu^2}{\lambda_k^2}b(A_k S_k v, A_k S_k v) \\
&\leq \frac{\mu^2}{\lambda_k^2}\lambda_k\|A_k S_k v\|_0^2 = \mu b(D_k v, v) .
\end{aligned}
$$

Taking $\sigma = \mu$, for $0 < \mu < 2$, we get that (A4) holds.

Thus, we obtain

Theorem 3.4. *For $0 < \mu < 2$, the smoothers of the form $Q_k = \frac{\mu}{\lambda_k}I$ satisfy (A3) and (A4).*

Remark 3.1. The largest eigenvalue of the matrix is involved in the construction of this kind of smoothers, which is not easy to obtain in practical computation. This gives some difficulty in using this kind of smoothers directly in practical computation. However this theorem is still important and useful in constructing smoothers and for providing us with a better understanding of the role the smoother plays in the convergence of the multigrid algorithm (see the analysis below).

Let us consider smoothers of the form

$$Q_k = \frac{1}{\eta} I \ . \tag{3.45}$$

To make Q_k above satisfy (A3) and (A4), η should satisfy some conditions. Next, we will give some conditions based on Theorem 3.4.

Two cases are considered. The first one is the case $\eta \geq \lambda_k$. In this case, it is obvious that there exists a positive constant $\mu \leq 1$ such that $\frac{1}{\eta} = \frac{\mu}{\lambda_k}$. From Theorem 3.4, it follows the desired smoothers. This case is of great importance for the practical computation and many practical smoothers can be obtained from this case. As mentioned above, it is not easy to get the largest eigenvalue of the matrix, but a upper bound of the largest eigenvalue of the matrix can be easily obtained using many different methods. All these upper bounds can be used to construct smoothers of the form (3.45) for the purpose of the practical computation.

The second case is $\frac{\lambda_k}{2} < \eta \leq \lambda_k$. In this case, we can see that there exists a constant $1 \leq \mu < 2$ such that $\frac{1}{\eta} = \frac{\mu}{\lambda_k}$. Also, for this case, we still can obtain desired smoothers. This case indicates that the multigrid algorithm is still convergent for any upper bounds η of $\lambda_k/2$ even if $\eta < \lambda_k$. This seems an interesting result, which may provide us a better understanding of the role the smoother plays in the convergence of the multigrid algorithm. On the other hand, it also implies that if some upper bounds of $\lambda_k/2$ can be obtained, then all these bounds can also be used to construct smoothers of the form (3.45) for practical use. The convergence of the multigrid algorithm is still ensured in this case.

To conclude, we obtain the following theorem

Theorem 3.5. Let $\frac{\lambda_k}{2} < \eta$. Then the smoothers of the form (3.45) satisfy (A3) and (A4).

With Theorems 3.1-3.3 and 3.5, we complete the construction and analysis of the proposed multigrid algorithm.

4. Numerical Results

Let us consider the following model problem for our numerical experiment

$$\begin{cases} -\triangle u &= f, \quad in \ \ \Omega^c, \\ u &= 0, \quad on \ \ \partial\Omega, \end{cases} \tag{4.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r) \ , \qquad |\nabla u(x,y)| = O(1/r^2) \ , \qquad r = \sqrt{x^2 + y^2} \to \infty \ ,$$

where Ω is unit circle disc, $\alpha = 1$, and

$$f = \begin{cases} \frac{4}{(x^2+y^2)^2} \ , & 1 < x^2 + y^2 < \frac{9}{4}, \\ 0 \ , & \frac{9}{4} \leq x^2 + y^2. \end{cases}$$

We make the coupling at the circle Γ with radius 2. The exact solution of the model problem and the computational solution of the finest level are denoted as u and u_h respectively. The discrete norm $\| \cdot \|_D$ is defined as

$$\|w\|_D = h_J \sqrt{\sum_i w(x_i)^2},$$

Table 1: Numerical results for $J = 3$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
128	2048	2.9060e-3	10
256	8192	7.4122e-4	10
512	32768	1.8700e-4	9
1024	131072	4.6633e-5	8

Table 2: Numerical results for $J = 4$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
128	2048	2.9354e-3	12
256	8192	7.5595e-4	11
512	32768	1.9390e-4	10
1024	131072	4.9087e-5	9
2048	524288	1.2272e-5	8

Table 3: Numerical results for $J = 5$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
256	8192	8.9339e-4	11
512	32768	2.1598e-4	11
1024	131072	5.8905e-5	10
2048	524288	1.3499e-5	10
4096	2097152	3.0680e-6	10

where the sum is taken over all nodes x_i of the finest level finite element space U_J. It is well known that this discrete norm is equivalent to the standard L^2 norm.

In what follows, the number of circular arcs that Γ is divided into on the finest level and the number of unknowns on the finest level are denoted as m and $\mathcal{N}$, respectively. ITn stands for the number of iterations needed to achieve the corresponding error $\|u - u_h\|_D$. In all our numerical experiments, the multigrid algorithm with only pre-smoothing is used and only one smoothing is taken at each level.

The results for the cases $J = 3$, $J = 4$ and $J = 5$ are presented in Table 1, Table 2 and Table 3 respectively.

From all these tables, we find that the proposed multigrid algorithm performs well even if the number of unknowns is very large (for example, the case of more than 2 million unknowns) and the number of iterations needed to achieve the corresponding accuracy is independent of the number of levels and the mesh sizes used, which supports our theoretical predictions.

References

[1] R. Bank and T. Dupont, An optimal order process for solving finite element equations, *Math. Comp.*, **36** (1981), 35-51.

[2] D. Braess and W. Hachbush, A new convergence proof for the multigrid method including the V-cycle, *SIAM J. Numer. Anal.*, **20** (1983), 967-975.

[3] J. Bramble, Multigrid Methods, Pitman Research Notes in Mathematical series, Longman Scientific and Technical, New York, 1993.

[4] J. Bramble and J. Pasciak, New estimate for multigrid algorithms including the V-cycle, *Math. Comp.*, **60** (1993), 447-471.

[5] S. Brenner and L. Scott, The mathematical theory of finite element method, Springer, New York, 1994.

[6] F. Brezzi and C. Johnson, On the coupling of boundary integral and finite element methods, Calcolo, **16** (1979), 189-201.

[7] U. Brink and E. Stephan, Convergence rates for the coupling of FEM and BEM, *IMA J. Numer. Anal.*, **16** (1996), 93-110.

[8] M. Costabel, Symmetric methods for the coupling of finite elements and boundary elements, in: Boundary Elements IV, Vol. 1, Comput. Mechanics Publications, Southampton, 1987, 411-420.

[9] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland Publishers, Amsterdam, 1978.

[10] K. Feng, Finite element method and natural boundary reduction, Proceedings of the International Congress of Mathematicians, Warszawa, 1983, 1439-1453.

[11] K. Feng and D. Yu, Canonical integral equation of elliptic boundary value problems and their numerical solution, in: Proc. of China-France Symp. on FEM, Beijing, 1982, Science Press, Beijing, 1983, 221-252 .

[12] G. Gatica and W. Wendland, Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems, Preprint 95-17, Mathematisches Institute A, Univ. Stuttgart, 1995.

[13] W. Hackbusch, MultiGrid Method and Applications, Spring-Verlag, Berlin, New York, 1985.

[14] H. Han, A new class of variational formulations for the coupling of finite element and boundary element methods, *J. Comp. Math.*, **8** (1990), 223-232.

[15] G. Hsiao, The coupling of BEM and FEM – a brief review, *Boundary Element X,* **1** (1988), 431-446.

[16] Q. Hu and D. Yu, Solving singularity problems in unbounded domains by coupling of natural BEM and composite grid FEM, *Appl. Numer. Math.*, **37** (2001), 127-143.

[17] C. Johnson and J. Nédélec, On the coupling of boundary integral and finite element methods, *Math. Comp.*, **35** (1980), 1063-1079.

[18] S. Meddahi, An optimal iterative process for the Johnson-Nédélec method of coupling boundary and finite elements, *SIAM J. Numer. Anal.*, **35** (1998), 1393-1415.

[19] S. Meddahi, J. Valdés, O. Menéndez and P. Pérez, On the coupling of boundary integral and mixed finite element methods, *J. Comput. Appl. Math.*, **69** (1996), 113-124.

[20] W. Wendland, On asymptotic error estimates for the combined BEM and FEM, *Innovative Numerical Methods in Engrg.*, **88** (1986), 55-70.

[21] J. Xu, Iterative methods by space decomposition and subspace correction, *SIAM Review*, **34** (1992), 581-613.

[22] D. Yu, The relation between the Steklov-Poincaré operator, the natural integral operator and Green functions, *Chinese J. Numer. Math. Appl.* **17** (1995), 95-106.

[23] D. Yu, The Mathematical Theory of the Natural Boundary Element Method, Science Press, Beijing, 1993. (in Chinese)

[24] D. Yu, The Natural Boundary Integral Method and Its Applications, Kluwer Academic Publisher/Science Press, Beijing, 2002.

[25] A. Zenisek, Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations, Academic Press, London, 1990.

[26] O. Zienkiewicz, D. Kelly and P. Bettess, The coupling of the finite element method and boundary solution procedures, *Int. J. for Numer. Methods in Engrg.*, **11** (1977), 355-375.