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ANISOTROPIC POLARIZATION TENSORS FOR ELLIPSES AND ELLIPSOIDS *1)

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Abstract

In this paper we present a systematic way of computing the polarization tensors, anisotropic as well as isotropic, based on the boundary integral method. We then use this method to compute the anisotropic polarization tensor for ellipses and ellipsoids. The computation reveals the pair of anisotropy and ellipses which produce the same polarization tensors.

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1. Introduction

Consider a field ∇H in $\mathbb{R}^d$, $d = 2, 3$, where H is a harmonic function in $\mathbb{R}^d$. The most important such a field to consider is given by $H(x) = x_j$, $j = 1, \ldots, d$. The field is disturbed by the presence of an inclusion B which is a bounded Lipschitz domain in $\mathbb{R}^d$. Let ∇u be the perturbed field. Then the perturbation admits the multipole expansion

$$(u - H)(x) = \sum_{|\alpha|,|\beta|=1}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial_x^\alpha \Gamma(x) M_{\alpha\beta} \partial^\beta H(0), \quad |x| \to \infty, \tag{1.1}$$

where Γ is the fundamental solution for the Laplacian, and α, β are multi-indices. See [3]. The quantities $M_{\alpha\beta}$, which describe the perturbation of the field completely, are called the generalized polarization tensors (PT). In particular, when $|\alpha| = |\beta| = 1$, then $M = (M_{\alpha\beta})$ is called the first order polarization tensor.

The notion of the polarization tensor can be extended to include the case when the conductivity of the inclusion and that of the matrix (background) are anisotropic. Suppose that the conductivity of the background $\mathbb{R}^d \setminus \overline{B}$ is $\widetilde{\gamma}$, while that of the inclusion B is γ, where $\widetilde{\gamma}$ and γ are positive definite symmetric $d \times d$ constant matrices. After an obvious change of variables we may assume that $\widetilde{\gamma} = I$, the $d \times d$ identity matrix. The matrix γ represents an (anisotropic) material property of the inclusion B. Thus the conductivity profile here is given by $\gamma_B = \chi(\mathbb{R}^d \setminus \overline{B})I + \chi(B)\gamma$, where $\chi(B)$ denotes the characteristic function of B. Here and throughout this paper we assume that $\gamma - I$ is either positive or negative definite. Even in this case the perturbation $(u - H)(x)$ has multipole expansions (1.1) at ∞. We denote the anisotropic polarization tensors (APT) associated with this conductivity distribution γ_B by $M_{\alpha\beta} = M_{\alpha\beta}(I, \gamma; B)$.

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The purpose of this paper is to present a simple method to compute PTs anisotropic as well as isotropic using the layer potential techniques. The method of this paper reduces the computation of PTs to the computation of the boundary integral operator $\mathcal{K}_B$ on ∂B which is defined by (2.6). The method also provides us with a systematic way to compute the PTs numerically. We then proceed to compute anisotropic PTs associated with ellipses and ellipsoids. It has been known that the first order PTs can be realized in terms of ellipses in a unique way if the conductivities of the inclusion and the matrix are isotropic [12, 6]. However, the material property of the inclusion, namely, anisotropic conductivity, cannot be extracted by means of the first order PT. The computations of this paper completely characterize the pairs of anisotropy and an ellipse which yield the same PTs. It should be noted that the first order PTs for the ellipses and ellipsoids are known (see [26]), and the higher order PTs for disks, ellipses, and balls were computed in [24]. The method of this paper has been applied to the computation of the elatstic moment tensor [7].

The notion of the (first order) PT appears in various contexts such as the inverse problems to detect unknown small inclusions, the theory of composite materials, and low frequency asymptotics to name a few. It was Friedman and Vogelius who first used the first order PT for the inverse problem of identifying the location of small inclusions [18]. Since then, various non-iterative direct methods to identify small inclusions using the PT have been proposed and tested numerically [14, 10, 12, 11, 6, 20, 3]. The PT also naturally appears in the asymptotic expansions of effective properties of dilute composite materials [5, 8, 9, 15, 19, 26] and low frequency asymptotics [16, 23]. For a derivation of higher order terms of the asymptotic expansion and extensions to anisotropic and elastic composites, and for extensive references, we refer to a recent book [4]. The notion of PT has been generalized to include all the higher order terms and various important properties of the generalized PT have been obtained [1, 2, 3]. Among them are symmetry, positive-definiteness, the Hashin-Strikman bounds, and the fact that the inclusion is completely determined by all the generalized PT even if the conductivity matrix γ is anisotropic. It should be mentioned that the Hashin-Strikman bounds for the first order PT, which is optimal, was obtained by Lipton [25] and Capdeboscq-Vogelius [13]. The notion of the PT was also used in the study of potential flows by Pólya, Szegö, and Schiffer [27, 28]

This paper is organized as follows. In section 2, we derive a general way to compute PTs using boundary integrals. In section 3, we derive formula for PTs on ellipses. Section 4 is for ellipsoids.

2. Layer Potential Method for Computation of PT

A fundamental solution to the Laplacian is given by

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ -\dfrac{1}{4\pi} \dfrac{1}{|x|}, & d = 3. \end{cases} \tag{2.1}$$

With this fundamental solution, the single and double layer potentials are defined to be

$$\mathcal{S}_B\phi(x) := \int_{\partial B} \Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d, \tag{2.2}$$

$$\mathcal{D}_B\phi(x) := \int_{\partial B} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial B, \tag{2.3}$$

where ν is the unit outward normal to ∂B. Then the following jump formula holds on ∂B when B is a bounded Lipschitz domain [29]:

$$(\mathcal{D}_B\phi)\big|_{\pm}(x) = \left(\mp\frac{1}{2}I + \mathcal{K}_B\right)\phi(x) \quad \text{a.e. } x \in \partial B, \tag{2.4}$$

$$\frac{\partial}{\partial\nu}\mathcal{S}_B\phi\bigg|_{\pm}(x) = \left(\pm\frac{1}{2}I + \mathcal{K}_B^*\right)\phi(x) \quad \text{a.e. } x \in \partial B, \tag{2.5}$$

where the operator $\mathcal{K}_B$ on ∂B is defined by

$$\mathcal{K}_B\phi(x) = \frac{1}{\omega_d}\text{p.v.}\int_{\partial B}\frac{\langle y-x,\nu_y\rangle}{|x-y|^d}\phi(y)\,d\sigma(y) \tag{2.6}$$

and $\mathcal{K}_B^*$ is the L^2-adjoint of $\mathcal{K}_B$. The subscripts $+$ and $-$ denote the limit from outside and inside B, respectively.

Let γ be a $d\times d$ symmetric positive definite constant matrix given by

$$\gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{2.7}$$

Let γ_* be the positive definite symmetric matrix such that $\gamma_*^2 = \gamma^{-1}$. Then the fundamental solution $\Gamma^\gamma(x)$ of the differential operator $\nabla\cdot(\gamma\nabla)$ is given by

$$\Gamma^\gamma(x) = \frac{1}{\sqrt{|\gamma|}}\Gamma(\gamma_* x), \quad x \neq 0, \tag{2.8}$$

where $|\gamma|$ is the determinant of γ. The single layer potentials associated with the matrix γ of $\phi \in L^2(\partial B)$ on B is denote by $\mathcal{S}_B^\gamma\phi$, namely,

$$\mathcal{S}_B^\gamma\phi(x) = \int_{\partial B}\Gamma^\gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d\,. \tag{2.9}$$

Note that $u = \mathcal{S}_B^\gamma\phi$ is γ-harmonic, i.e., $\nabla\cdot(\gamma\nabla u) = 0$ in $B \cup (\mathbb{R}\setminus\bar{B})$.

We will use the standard notations for multi-indices: for the multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$, $x^\alpha := x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$, etc. Let $(\psi_\alpha,\varphi_\alpha) \in L^2(\partial B)\times L^2(\partial B)$ be the unique solution of the system of integral equations

$$\begin{cases} \mathcal{S}_B^\gamma\psi_\alpha - \mathcal{S}_B\varphi_\alpha = x^\alpha \\ \langle\gamma\nabla\mathcal{S}_B^\gamma\psi_\alpha,\nu\rangle|_- - \langle\nabla\mathcal{S}_B\varphi_\alpha,\nu\rangle|_+ = \langle\nabla x^\alpha,\nu\rangle \end{cases} \quad \text{on } \partial B. \tag{2.10}$$

The unique solvability of (2.10) was proved in [17]. Then for a pair of multi-indices α and β the anisotropic polarization tensors (APT) associated with the conductivity $\gamma_B = \chi(\mathbb{R}^d\setminus\overline{B})I + \chi(B)\gamma$ are proved to be given by

$$M_{\alpha\beta} = \int_{\partial B} y^\beta\varphi_\alpha(y)d\sigma(y). \tag{2.11}$$

See [20, 4]. When $\alpha = \mathsf{e}_i$ and $\beta = \mathsf{e}_j$ for $i,j = 1,\ldots,d$, where e_i is the standard basis for $\mathbb{R}^d$, we denote $M_{\alpha\beta}$ by M_{ij}, i.e.,

$$M_{ij} = \int_{\partial B} y_j\varphi_i(y)ds(y), \tag{2.12}$$

where $\psi_i = \psi_\alpha$ and $\varphi_i = \varphi_\alpha$ with $\alpha = \mathsf{e}_i$.

Let I, J be the finite sets of multi-indices and $\{a_\alpha | \alpha \in I\}$, $\{b_\beta | \beta \in J\}$ be such that $\sum_{\alpha\in I} a_\alpha x^\alpha$ and $\sum_{\beta\in J} b_\beta x^\beta$ are harmonic polynomials. Put

$$\psi := \sum_{\alpha\in I} a_\alpha \psi_\alpha, \quad \phi := \sum_{\alpha\in I} a_\alpha \varphi_\alpha, \quad v := \sum_{\alpha\in I} a_\alpha x^\alpha, \quad w := \sum_{\beta\in J} b_\beta x^\beta.$$

Then, one can easily see from the linearity of the integral equation (2.10) that (ψ, ϕ) is the solution to

$$\begin{cases} \mathcal{S}_B^\gamma \psi - \mathcal{S}_B \phi = v \\ \langle \gamma \nabla \mathcal{S}_B^\gamma \psi, \nu\rangle|_- - \langle \nabla \mathcal{S}_B \phi, \nu \rangle|_+ = \langle \nabla v, \nu\rangle \end{cases} \quad \text{on } \partial B. \tag{2.13}$$

We have

$$\sum_{\alpha\in I}\sum_{\beta\in J} a_\alpha b_\beta M_{\alpha\beta} = \int_{\partial B} w(y)\phi(y) d\sigma(y). \tag{2.14}$$

Because of the jump formula (2.5), we get

$$\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_+ - \langle \nabla \mathcal{S}_B \phi, \nu\rangle|_- = \phi, \quad \text{on } \partial B.$$

It thus follows from (2.13) and the divergence theorem that

$$\begin{aligned} \sum_{\alpha\in I}\sum_{\beta\in J} a_\alpha b_\beta M_{\alpha\beta} &= \int_{\partial B} w\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_+ - w\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_- \\ &= \int_{\partial B} w\langle \gamma \nabla \mathcal{S}_B^\gamma \psi, \nu\rangle|_- - w\langle \nabla v, \nu\rangle - w\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_- . \end{aligned} \tag{2.15}$$

Observe that

$$\int_{\partial B} w\langle \nabla v, \nu\rangle = \int_B \nabla v \cdot \nabla w = \sum_{\alpha\in I}\sum_{\beta\in J} a_\alpha b_\beta \int_B \nabla y^\alpha \cdot \nabla y^\beta. \tag{2.16}$$

We now compute the term

$$\int_{\partial B} w\langle \gamma \nabla \mathcal{S}_B^\gamma \psi, \nu\rangle|_- - w\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_- .$$

For that we consider isotropic and anisotropic case separately.

(1) **Isotropic case**. Suppose that γ is an isotropic matrix, say $\gamma = \sigma I$ for some positive constant σ. In this case $\Delta w = 0$ and hence we have

$$\begin{aligned} &\int_{\partial B} w\langle \gamma \nabla \mathcal{S}_B^\gamma \psi, \nu\rangle|_- - w\langle \nabla \mathcal{S}_B \phi, \nu\rangle|_- \\ &= \int_{\partial B} (\mathcal{S}_B \phi)\langle (\gamma - I)\nabla w, \nu\rangle + \int_{\partial B} v\langle \gamma \nabla w, \nu\rangle \\ &= (\sigma - 1)\int_{\partial B} \langle \nabla \mathcal{S}_B \phi, \nu\rangle\Big|_- w + \int_B \nabla v \cdot \gamma \nabla w \\ &= -\frac{\sigma-1}{2}\int_{\partial B} \phi w + (\sigma - 1)\int_{\partial B} \mathcal{K}_B^*(\phi) w + \sum_{\beta\in J} a_\alpha b_\beta \int_B \nabla y^\alpha \cdot \gamma \nabla y^\beta, \end{aligned}$$

where the last equality comes from (2.5). It then follows from (2.14) that

$$\begin{aligned}\int_{\partial B}(\mathcal{S}_B\phi)\langle(\gamma-I)\nabla w,\nu\rangle &= -\frac{\sigma-1}{2}\sum_{\alpha\in I}\sum_{\beta\in J}a_\alpha b_\beta M_{\alpha\beta}+(\sigma-1)\int_{\partial B}\phi\mathcal{K}_B(w)\\ &= -\frac{\sigma-1}{2}\sum_{\alpha\in I}\sum_{\beta\in J}a_\alpha b_\beta M_{\alpha\beta}+(\sigma-1)\sum_{\alpha\in I}\sum_{\beta\in J}a_\alpha b_\beta\int_{\partial B}\varphi_\alpha\mathcal{K}_B(y^\beta).\end{aligned}$$

Thus we finally get from (2.15) that

$$\sum_\alpha\sum_\beta a_\alpha b_\beta\left[\frac{\sigma+1}{2(\sigma-1)}M_{\alpha\beta}-\int_{\partial B}\varphi_\alpha\mathcal{K}_B(y^\beta)-\int_B\nabla y^\alpha\cdot\nabla y^\beta\right]=0.$$

Therefore if we solve

$$\frac{\sigma+1}{2(\sigma-1)}M_{\alpha\beta}-\int_{\partial B}\varphi_\alpha\mathcal{K}_B(y^\beta)=\int_B\nabla y^\alpha\cdot\nabla y^\beta, \tag{2.17}$$

then we obtain get $\sum_\alpha\sum_\beta a_\alpha b_\beta M_{\alpha\beta}$.

Then significance of the formula (2.17) lies in the fact that the computation of PTs is reduced to the matter of computing $\mathcal{K}_B(y^\beta)$ on ∂B. For example, if there exist coefficients $C_{\beta\delta}$ such that

$$\mathcal{K}_B(y^\beta)=\sum_{|\delta|\le n}C_{\beta\delta}y^\delta \quad \text{on } \partial B, \quad \text{for all } |\beta|\le n, \tag{2.18}$$

which is the case when B is a disk or an ellipse, then we get

$$\frac{\sigma+1}{2(\sigma-1)}M_{\alpha\beta}-\sum_{|\delta|\le n}C_{\beta\delta}M_{\alpha\delta}=\int_B\nabla y^\alpha\cdot\nabla y^\beta, \quad |\beta|\le n. \tag{2.19}$$

By solving this system of linear equations for $(M_{\alpha\beta})_{|\alpha|\le n,|\beta|\le n}$, one can get PTs of all orders. We note that PTs for isotropic disks and ellipses have been computed in [24] using elliptic coordinates (see also [3]).

Before completing the discussion for the isotropic case, let us make a remark on the condition (2.18). It depends on the geometry of ∂B and may not hold for general domains. It would be interesting to see what kinds of shape guarantee (2.18) for some n. Recently, it is proved by Kang and Milton [21, 22] that if (2.18) holds for $n=1$, then B is an ellipse and ellipsoid, which leads them to the proof of the Pólya-Szegö conjecture.

(2) **Anisotropic case**. If γ is anisotropic, then because of the discrepancy between harmonic functions and γ-harmonic functions, the argument for the isotropic case does not go through. However, if $|\beta|=1$ then y^β is harmonic as well as γ-harmonic, and hence we can get formula for $M_{\alpha\beta}$ when $|\beta|=1$.

Suppose that $|\beta|=1$, say $\beta=\mathtt{e}_j$ for some $j=1,\ldots,d$, where $\mathtt{e}_j$ is the standard basis. Let $w(y)=y^\beta=y_j$. In this case, since

$$(\gamma-I)\nabla w=\nabla\langle(\gamma-I)\mathtt{e}_j,y\rangle,$$

we obtain by the same argument used for the isotropic case,

$$
\begin{aligned}
&\int_{\partial B} w\langle\gamma\nabla\mathcal{S}_B^\gamma\psi,\nu\rangle|_- - w\langle\nabla\mathcal{S}_B\phi,\nu\rangle|_- \\
&= \int_{\partial B} (\mathcal{S}_B\phi)\langle(\gamma-I)\nabla w,\nu\rangle + \int_{\partial B} v\langle\gamma\nabla w,\nu\rangle \\
&= \int_{\partial B} \langle\nabla\mathcal{S}_B\phi,\nu\rangle\Big|_- \langle(\gamma-I)\mathsf{e}_j,y\rangle + \int_B \nabla v\cdot\gamma\nabla w \\
&= -\frac{1}{2}\int_{\partial B}\phi\langle(\gamma-I)\mathsf{e}_j,y\rangle + \int_{\partial B}\mathcal{K}_B^*(\phi)\langle(\gamma-I)\mathsf{e}_j,y\rangle + \int_B \nabla v\cdot\gamma\mathsf{e}_j \\
&= -\sum_\alpha a_\alpha \sum_{i=1}^d (\gamma-I)_{ij}\left[\frac{1}{2}\int_{\partial B}\varphi_\alpha y_i - \int_{\partial B}\varphi_\alpha\mathcal{K}_B(y_i)\right] + \sum_\alpha a_\alpha\int_B \nabla y^\alpha\cdot\gamma\mathsf{e}_j \\
&= -\sum_\alpha a_\alpha \sum_{i=1}^d (\gamma-I)_{ij}\left[\frac{1}{2}M_{\alpha i} - \int_{\partial B}\varphi_\alpha\mathcal{K}_B(y_i)\right] + \sum_\alpha a_\alpha\int_B \nabla y^\alpha\cdot\gamma\mathsf{e}_j.
\end{aligned}
$$

Here we use the notation $M_{\alpha j}$ for $M_{\alpha\beta}$ when $\beta = \mathsf{e}_j$. We then get from (2.15)

$$
\sum_\alpha a_\alpha M_{\alpha j} + \sum_\alpha a_\alpha \sum_{i=1}^d (\gamma-I)_{ij}\left[\frac{1}{2}M_{\alpha i} - \int_{\partial B}\varphi_\alpha\mathcal{K}_B(y_i)\right] = \sum_\alpha a_\alpha\int_B \nabla y^\alpha\cdot(\gamma-I)\mathsf{e}_j.
$$

so the system of equation to solve is the following: for any multi-index α and $j = 1,\ldots,d$,

$$
M_{\alpha j} + \sum_{i=1}^d (\gamma-I)_{ij}\left[\frac{1}{2}M_{\alpha i} - \int_{\partial B}\varphi_\alpha\mathcal{K}_B(y_i)\right] = \int_B \nabla y^\alpha\cdot(\gamma-I)\mathsf{e}_j. \tag{2.20}
$$

Suppose that the domain B satisfies

$$
\mathcal{K}_B(y_i)(x) = \sum_{k=1}^d C_{ik}x_k, \quad i = 1,\ldots,d, \quad \text{or} \quad \mathcal{K}_B(y)(x) = C_B x, \quad \text{on } \partial B \tag{2.21}
$$

for some constant matrix $C_B = (C_{ik})$. Then we get

$$
M_{\alpha j} + \sum_{i=1}^d (\gamma-I)_{ij}\left[\frac{1}{2}M_{\alpha i} - \sum_{k=1}^d C_{ik}M_{\alpha k}\right] = X_{\alpha j},
$$

where $X_{\alpha j} := \int_B \nabla y^\alpha\cdot(\gamma-I)\mathsf{e}_j$. If we use the notations $M_\alpha := (M_{\alpha i},\ldots,M_{\alpha d})^T$, $X_\alpha := (X_{\alpha i},\ldots,X_{\alpha d})^T$, then we end up with

$$
M_\alpha + \frac{1}{2}(\gamma-I)M_\alpha - (\gamma-I)C_B M_\alpha = X_\alpha.
$$

In short, we obtain the following theorem:

Theorem 2.1. *Suppose that the domain B satisfies* (2.21). *Let $M_\alpha := (M_{\alpha i},\ldots,M_{\alpha d})^T$, $X_\alpha := (X_{\alpha i},\ldots,X_{\alpha d})^T$ where*

$$
X_{\alpha j} := \int_B \nabla y^\alpha\cdot(\gamma-I)\mathsf{e}_j.
$$

Then,

$$M_\alpha = \Big[\frac{1}{2}(\gamma + I) - (\gamma - I)C_B\Big]^{-1} X_\alpha. \tag{2.22}$$

In particular, the Pólya-Szegö matrix $M = (M_{ij})$ is given by

$$M = |B|\Big[\frac{1}{2}(\gamma + I) - (\gamma - I)C_B\Big]^{-1}(\gamma - I), \tag{2.23}$$

where $|B|$ is the volume of B.

3. Anisotropic Polarization Tensors for Ellipses

We now compute the anisotropic PTs for ellipses. To this end it suffices to compute the matrix C_B given in (2.21).

Suppose that B is the ellipse defined by

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1. \tag{3.1}$$

Using the parametrization $X(t) = (p\cos t, q\sin t), 0 \le t \le 2\pi$, for the boundary of B, we obtain

$$\begin{aligned}\mathcal{K}_B\phi(x) &= \frac{1}{2\pi}\int_{\partial B}\frac{\langle y - x, \nu_y\rangle}{|x-y|^2}\phi(y)d\sigma(y)\\ &= \frac{pq}{2\pi(p^2+q^2)}\int_0^{2\pi}\frac{\phi(X(t))dt}{1 - Q\cos(t+\theta)},\end{aligned}$$

where $x = X(\theta)$, and $Q = \dfrac{p^2 - q^2}{p^2 + q^2}$. Note that

$$\int_0^{2\pi}\frac{\cos t}{1 - Q\cos t}dt = -\frac{2\pi}{Q} + \frac{2\pi}{Q\sqrt{1-Q^2}}\ , \qquad \int_0^{2\pi}\frac{\sin t}{1 - Q\cos t}dt = 0.$$

Thus, if $\phi(y) = y_j$, $j = 1, 2$, we get

$$\mathcal{K}_B(y_1)(x) = \frac{p-q}{2(p+q)}x_1, \qquad \mathcal{K}_B(y_2)(x) = -\frac{p-q}{2(p+q)}x_2. \tag{3.2}$$

In other words,

$$\mathcal{K}_B(y)(x) = C_B x, \quad x \in \partial B,$$

where

$$C_B = \frac{p-q}{2(p+q)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3.3}$$

More generally, if B is an ellipse such that $B = R(B')$ for some rotation R and an ellipse B' of the form (3.1), then by a simple change of variables one can see that

$$\mathcal{K}_B(y)(x) = \mathcal{K}_{B'}(Ry)(R^{-1}x),$$

and hence

$$\mathcal{K}_B(y)(x) = RC_{B'}R^T x, \quad x \in \partial B, \tag{3.4}$$

where $C_{B'}$ is the matrix given by (3.3). Thus we get the following theorem:

Theorem 3.1. *Let B be an ellipse such that $B = R(B')$ for some rotation R and an ellipse B' of the form* (3.1). *Let $m = \dfrac{p-q}{p+q}$ and $J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then,*

$$M_\alpha = 2\big[(\gamma + I) - m(\gamma - I)RJR^T\big]^{-1} X_\alpha. \tag{3.5}$$

The Pólya-Szegö matrix $M = (M_{ij})$ is given by

$$M = 2|B|\big[(\gamma + I) - m(\gamma - I)RJR^T\big]^{-1}(\gamma - I). \tag{3.6}$$

If $\gamma = \sigma I$ is isotropic and B is an ellipse of the form $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, then

$$M = (\sigma - 1)|B| \begin{pmatrix} \frac{p+q}{p+\sigma q} & 0 \\ 0 & \frac{p+q}{\sigma p+q} \end{pmatrix},$$

which coincides with the one derived in [12]. If B is a disk and γ is anisotropic, then $m = 0$ and hence

$$M = 2|B|(\gamma + I)^{-1}(\gamma - I). \tag{3.7}$$

Note that the Pólya-Szegö matrix $M = (M_{ij})$ is given by

$$M = 2|B|\big[(\gamma - I)^{-1}(\gamma + I) - mRJR^T\big]^{-1}. \tag{3.8}$$

Therefore the pair (γ, B) of a conductivity matrix and an ellipse yields the same Pólya-Szegö matrix if and only if $(\gamma - I)^{-1}(\gamma + I) - mRJR^T$ is the same. For example, the pair

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

satisfying

$$\frac{1}{a-1} - \frac{1}{b-1} = \frac{p-q}{p+q}$$

yields the same Pólya-Szegö matrix.

4. Anisotropic Polarization Tensors for Ellipsoids

We now compute the APTs for three dimensional ellipsoids. To this end we compute the matrix C_B defined in (2.21) when B is the ellipsoid given by

$$\frac{x_1^2}{p_1^2} + \frac{x_2^2}{p_2^2} + \frac{x_3^2}{p_3^2} = 1. \tag{4.1}$$

Let

$$\varphi_k(x) := \mathcal{S}_B(\nu_k)(x) = \int_{\partial B} \Gamma(x - y)\nu_k(y)d\sigma(y), \quad x \in B, \quad k = 1, 2, 3,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal to ∂B. Since $\Gamma(x-y) = \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha \Gamma(x) y^\alpha$ for $y \in \partial B$ and $|x|$ large, we have

$$\varphi_k(x) = -\frac{1}{4\pi} \int_{\partial B} y_k \nu_k(y) d\sigma \frac{x_k}{|x|^3} + O(|x|^{-3}) = -\frac{p_1 p_2 p_3}{3} \frac{x_k}{|x|^3} + O(|x|^{-3}). \tag{4.2}$$

Lemma 4.1. *Suppose that B is an ellipsoid of the form* (4.1). *Then*

$$\varphi_k(x) = s_k x_k, \quad k = 1, 2, 3, \tag{4.3}$$

where

$$s_k = -\frac{p_1 p_2 p_3}{2} \int_0^\infty \frac{1}{(p_k^2 + s)\sqrt{(p_1^2 + s)(p_2^2 + s)(p_3^2 + s)}} ds. \tag{4.4}$$

Proof. Let

$$u(x) := \begin{cases} x_k + \varphi_k(x), & x \in B, \\ \varphi_k(x), & x \in \mathbb{R}^3 \setminus B. \end{cases}$$

Then u is the unique solution to the following problem:

$$\begin{cases} \Delta u = 0, & \text{in } B \cup (\mathbb{R}^3 \setminus \overline{B}), \\ u|_+ - u_- = -x_k, & \text{on } \partial B, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0, & \text{on } \partial B, \\ u(x) = -\dfrac{p_1 p_2 p_3}{3} \dfrac{x_k}{|x|^3} + O(|x|^{-3}), & \text{as } |x| \to \infty. \end{cases} \tag{4.5}$$

The transmission conditions on ∂B in (4.5) come from the continuity of $\mathcal{S}_B(\nu_k)$ across ∂B and (2.5).

We now construct explicitly the solution of (4.5) using ellipsoidal coordinates. The confocal ellipsoidal coordinates are ρ, μ, ξ satisfying

$$\begin{aligned} \frac{x_1^2}{p_1^2 + \rho} + \frac{x_2^2}{p_2^2 + \rho} + \frac{x_3^2}{p_3^2 + \rho} &= 1 \\ \frac{x_1^2}{p_1^2 + \mu} + \frac{x_2^2}{p_2^2 + \mu} + \frac{x_3^2}{p_3^2 + \mu} &= 1 \\ \frac{x_1^2}{p_1^2 + \xi} + \frac{x_2^2}{p_2^2 + \xi} + \frac{x_3^2}{p_3^2 + \xi} &= 1, \end{aligned}$$

subject to the conditions $-p_3^2 < \xi < -p_2^2 < \mu < -p_1^2 < \rho$. Note that the boundary ∂B of B is the surface $\rho = 0$. It is proved in Section 7.7 of [26] that if $\varphi(\rho)$ is a solution of the ordinary differential equation

$$2\frac{d}{d\rho}\left[\sqrt{g(\rho)}\frac{d}{d\rho}\left(\varphi(\rho)\sqrt{p_k^2 + \rho}\right)\right] = \varphi(\rho)\frac{d}{d\rho}\sqrt{\frac{g(\rho)}{p_k^2 + \rho}}, \tag{4.6}$$

where

$$g(\rho) = (p_1^2 + \rho)(p_2^2 + \rho)(p_3^2 + \rho), \tag{4.7}$$

then $\varphi(\rho)x_k$ is harmonic. Straightforward computations show that (4.6) can be written as

$$2\frac{d}{d\rho}\left[\sqrt{(p_k^2 + \rho)g(\rho)}\frac{d\varphi}{d\rho}\right] + \sqrt{\frac{g(\rho)}{p_k^2 + \rho}}\frac{d\varphi}{d\rho} = 0. \tag{4.8}$$

Therefore we have an explicit formula for the solution φ:

$$\varphi(\rho) = C \int_\rho^\infty \frac{1}{(p_k^2 + s)\sqrt{g(s)}} ds \tag{4.9}$$

for some constant C.

Note that $\lim_{\rho\to\infty} \frac{|x|^2}{\rho} = 1$. In view of (4.2), we need to have

$$\lim_{\rho\to\infty} \rho^{3/2}\varphi(\rho) = -\frac{p_1p_2p_3}{3},$$

and hence $C = -\frac{p_1p_2p_3}{2}$. In conclusion, we obtain

$$\varphi(\rho) = -\frac{p_1p_2p_3}{2}\int_\rho^\infty \frac{1}{(p_k^2+s)\sqrt{g(s)}}ds. \tag{4.10}$$

It then follows that the function v defined by

$$v(x) := \begin{cases} (\varphi(0)+1)x_k, & x \in B\ (\rho<0), \\ \varphi(\rho)x_k, & x \in \mathbb{R}^3 \setminus B\ (\rho \geq 0), \end{cases}$$

is a solution of (4.5) and hence we get (4.3) with $s_k = \varphi(0)$. The transmission conditions in (4.5) hold because of the relation $\frac{\partial x_k}{\partial \rho} = \frac{x_k}{2(p_k^2+\rho)}$, a proof of which can be found in [26]. This completes the proof.

For $x \in B$, we have

$$\mathcal{D}_B(y_k)(x) = x_k + \mathcal{S}_B(\nu_k)(x),$$

and it follows from (2.4) that

$$\mathcal{K}_B(y_k)(x) = (\frac{1}{2}+s_k)x_k. \tag{4.11}$$

We finally get the following theorem from (2.22) and (2.23).

Theorem 4.2. *Suppose that B is an ellipsoid given by $B = R(B')$ for some unitary transformation R and an ellipsoid B' of the form (4.1). Then, for any muti-index α,*

$$M_\alpha = \big[\frac{1}{2}(\gamma+I) - (\gamma-I)RCR^T\big]^{-1}X_\alpha, \tag{4.12}$$

where

$$C = \begin{pmatrix} \frac{1}{2}+s_1 & 0 & 0 \\ 0 & \frac{1}{2}+s_2 & 0 \\ 0 & 0 & \frac{1}{2}+s_3 \end{pmatrix} \tag{4.13}$$

and s_k is given by (4.4). In particular, the Pólya-Szegö matrix $M = (M_{ij})$ is given by

$$M = |B|\big[\frac{1}{2}(\gamma+I) - (\gamma-I)RCR^T\big]^{-1}(\gamma-I), \tag{4.14}$$

where $|B|$ is the volume of B.

We note that the Pólya-Szegö matrix for isotropic ellipsoidal inclusion can be found in [26].

If B is the unit ball, then one can see that

$$s_k = -\frac{1}{2}\int_0^\infty (1+s)^{-5/2}ds = -\frac{1}{3}$$

for $k = 1, 2, 3$, and hence

$$M = |B|\big[\frac{1}{2}(\gamma+I) - \frac{1}{6}(\gamma-I)\big]^{-1}(\gamma-I). \tag{4.15}$$

In particular, if $\gamma = \sigma I$ for some positive constant σ, then

$$M = \frac{3(\sigma - 1)}{\sigma + 2}|B|I, \tag{4.16}$$

which is a classical formula of PT for balls. See [26, 3].

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