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RADIAL BASIS FUNCTION INTERPOLATION IN SOBOLEV SPACES AND ITS APPLICATIONS*

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Abstract

In this paper we study the method of interpolation by radial basis functions and give some error estimates in Sobolev space $H^k(\Omega)$ $(k \geq 1)$. With a special kind of radial basis function, we construct a basis in $H^k(\Omega)$ and derive a meshless method for solving elliptic partial differential equations. We also propose a method for computing the global data density.

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Key words: Sobolev space, Radial basis function, Global data density, Meshless method.

1. Introduction

For a smooth function u, known only at a set $X = \{x^{(1)}, \cdots, x^{(N)}\}$ consisting of pairwise distinct points in $\mathbb{R}^d$, the corresponding radial basis function approach is the interpolant

$$s_u(x) = \sum_{i=1}^{N} a_i \psi(x - x^{(i)}) + \sum_{l=1}^{Q} b_l p_l(x), \tag{1.1}$$

whose coefficients a_i and b_l are determined by the following linear system

$$\sum_{i=1}^{N} a_i \psi(x^{(j)} - x^{(i)}) + \sum_{l=1}^{Q} b_l p_l(x^{(j)}) = u(x^{(j)}), \qquad j = 1, \cdots, N, \tag{1.2}$$

$$\sum_{j=1}^{N} a_j p_i(x^{(j)}) = 0, \qquad i = 1, \cdots, Q, \tag{1.3}$$

where $Q = C_{q+d-1}^{d}$, $\psi(x) = \phi(|x|)$, $|.|$ is the Euclidean norm, and $p_1, \cdots, p_Q$ is a basis of $\mathbb{P}_q$, the space of polynomials defined on $\mathbb{R}^d$ with total order $< q$ $(q \geq 0)$. Especially, when $q = 0$, the interpolant (1.1) reads as

$$s_u(x) = \sum_{i=1}^{N} a_i \psi(x - x^{(i)}) \tag{1.4}$$

and the coefficients a_i are determined by

$$\sum_{i=1}^{N} a_i \psi(x^{(j)} - x^{(i)}) = u(x^{(j)}), \qquad j = 1, \cdots, N. \tag{1.5}$$

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Definition 1.1. ([2]) *A function $F : \mathbb{R}^d \to R$ is said to be conditionally positive definite (resp. strictly conditionally positive definite) of order q ($q \geq 0$), if for all finite subsets X in $\mathbb{R}^d$, the quadratic form*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j F(x^{(i)} - x^{(j)}) \tag{1.6}$$

is nonnegative (resp. positive) for all vectors (resp. nonzero vectors) $c \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} c_i p(x^{(i)}) = 0$ for any given $p \in \mathbb{P}_q$. If $q = 0$, F is called positive definite (resp. strictly positive definite).

The nonsingularity of system (1.2)-(1.3) for a wide choice of functions ψ and polynomials of order q is assured by

Theorem 1.1. *Let $N \geq Q$. Assume that $\psi(x) = \phi(|x|)$ is strictly conditionally positive definite of order q ($q \geq 0$). Assume furthermore that there exists a subset $X' \subset X$ containing Q points such that*

$$p|_{X'} = 0 \quad \text{for } p \in \mathbb{P}_q \qquad \text{implies} \quad p \equiv 0. \tag{1.7}$$

Then the interpolation system (1.2)-(1.3) *is always uniquely solvable.*

The proof of this theorem can be found in the appendix of this paper, and readers may refer to Madych [5] and Micchelli [6] for more details about conditionally positive definite functions. We give in Table 1.1 some frequently used radial basis functions, where $\lceil \beta/2 \rceil$ denotes the smallest integer greater than or equal to $\beta/2$, and $\lfloor d/2 \rfloor$ denotes the biggest integer less than or equal to $d/2$.

Table 1.1: Radial basis function

Name	$\psi(x) = \phi(r),\ r = \|x\|$	$\widehat{\psi}(\xi)$	q
Gaussians	$e^{-\beta r^2}, \quad \beta > 0$	$C(d,\beta)e^{-\|\xi\|^2/4\beta}$	0
Thin plate spline	$(-1)^{1+\beta/2} r^\beta \ln r, \quad \beta \in 2\mathbb{N}$ $(-1)^{\lceil \beta/2 \rceil} r^\beta,\ \beta \in \mathbb{R}_{>0} \setminus 2\mathbb{N}$	$C(d,\beta)\|\xi\|^{-d-\beta}$	$1+\beta/2$ $\lceil \beta/2 \rceil$
Sobolev spline	$K_{\beta-d/2}(r) r^{\beta-d/2},\ \beta > d/2$ K MacDonald's function	$C(d,\beta)(1+\|\xi\|^2)^{-\beta}$	0
compactly supported functions, C^{2l}	$(1-r)_+^\beta p(r)$ $\partial p = l,\ \beta = \lfloor d/2 \rfloor + 2l + 1$	$(1+\|\xi\|)^{-d-2l-1}$	0

There is a rapidly growing on list of literatures related to radial basis functions, and the accuracy of interpolation by radial basis functions often comes out very satisfactory. Yet, this satisfaction is based on the presumption that the approximand is reasonably smooth. At least, such interpolations require the pointwise value of the approximand. However, taking functions in H^1 for example, this pointwise value may not be well defined for a wide variety of functions. Although an interpolation approach by radial basis functions in Sobolev spaces is provided in [9, 10, 12], yet, their assumption $k > \frac{d}{2}$ restricts the approximand to be continuous in essential. The approximation of nonsmooth functions by using continuous piecewise polynomials is studied in [7], but regular grid data is required. In this paper, we propose an interpolation of scattered data by radial basis functions in Sobolev space $H^k(\Omega), k \geq 1$, and by means of such interpolation with a special kind of radial basis function, we derive a meshless method for solving elliptic partial differential equations.

The following notations are used throughout this paper. We denote the Schwartz space S of all $C^\infty(\mathbb{R}^d)$-functions that, together with all their derivatives, decay faster than any polynomial at the infinity. For $q \in \mathbb{N}_{\geq 0}$ the set of all functions $\gamma \in S$ satisfying $\gamma(\xi) = O(||\xi||^{2q})$ for $||\xi|| \to 0$ will be denoted by S_q. Recall that a function ψ is called slowly increasing if there exists an integer $n \in \mathbb{N}_{\geq 0}$ such that $|\psi(\xi)| = O(||\xi||^n)$ for $||\xi|| \to +\infty$. For any $u \in L^1(\mathbb{R}^d)$, its classical Fourier transform is defined as

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)dx. \tag{1.8}$$

Definition 1.2. ([8]) *Suppose $\psi : \mathbb{R}^d \to \mathbb{C}$ is continuous and slowly increasing. A continuous function $\widehat{\psi} : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is said to be the generalized Fourier transform of ψ if there exists a nonnegative integer q such that*

$$\int_{\mathbb{R}^d} \psi(x)\widehat{\gamma}(x)dx = \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\gamma(\xi)d\xi, \qquad \forall\ \gamma \in \mathcal{S}_q.$$

The smallest of such q is called the order of $\widehat{\psi}$.

In this paper, we assume that $u \in H^k(\Omega)$, $k \geq 1$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{k-1,1}$ boundary Γ. We always assume that $\psi(x) = \phi(|x|)$ is at least an C^m function with $m > d/2$ such that its generalized Fourier transform $\widehat{\psi}$ exists and satisfies

$$c_1(1 + \|\xi\|^2)^{-m} \leq \widehat{\psi}(\xi) \leq c_2(1 + \|\xi\|^2)^{-m}, \tag{1.9}$$

where c_1 and c_2 are two positive constants.

2. Computation of the Global Data Density

Definition 2.1. *For error estimation, we define the 'density' of a given set $X = \{x^{(1)}, \cdots, x^{(N)}\}$ consisting of pairwise distinct points in Ω by*

$$h \triangleq h(X;\Omega) = \sup_{x\in\Omega} \min_{x^{(j)}\in X} |x - x^{(j)}|. \tag{2.1}$$

For any given $x^{(j)}$ $(1 \leq j \leq N)$, set

$$\Omega_j \triangleq \Big\{x \mid x \in \Omega,\ |x - x^{(j)}| \leq |x - x^{(i)}|,\ i \neq j, 1 \leq i \leq N\Big\} \tag{2.2}$$

and

$$h_j \triangleq \sup_{x\in\Omega_j} |x - x^{(j)}|. \tag{2.3}$$

It is straightforward to show that

Theorem 2.1. *Let h be the 'density' defined in* (2.1)*. Then $h = \max\limits_{1\leq j\leq N} h_j$.*

3. Interpolation in Sobolev Space

For a nonsmooth function u, say $u \in H^1(\mathbb{R}^d)$, system (1.2)-(1.3) fails since u has no pointwise value in two or more dimensions. In this case, we must modify the former approach as follows: Introduce a mollifier and use the mollification u^h as the appproximand to replace u.

For the sake of error estimation, some restrictions should be added to the mollifier. We assume that

$$\widehat{\eta}(\xi) \in C_0^\infty(B(0,2)),\quad \widehat{\eta}(\xi) \equiv 1,\quad \text{when}\quad |\xi| \leq 1. \tag{3.1}$$

Set

$$\eta_h(x) = h^{-d}\eta(x/h). \tag{3.2}$$

For any given $u \in H^k(\mathbb{R}^d), k \geq 1$, define its mollification

$$u^h = \eta_h * u. \tag{3.3}$$

Then, $u^h \in C^\infty(\mathbb{R}^d)$.

For a function u with higher regularity, Wendland proved in [9] that

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^1 boundary. Assume that the generalized Fourier transform of $\psi(x) = \phi(|x|)$ exists and satisfies* (1.9). *Denote $s_u = \sum_{i=1}^N a_i\psi(x - x^{(i)}) + \sum_{l=1}^Q b_l p_l(x)$ the interpolant on $X = \{x^{(1)}, \cdots, x^{(N)}\} \subset \Omega$ to a function $u \in H^m(\Omega)$. Then there exists a constant $h_0 > 0$ such that for all X with $h \leq h_0$, where h is defined by* (2.1), *we have*

$$\|u - s_u\|_{H^s(\Omega)} \leq Ch^{m-s}\|u\|_{H^m(\Omega)}, \qquad 0 \leq s \leq m, \tag{3.4}$$

where, the positive constant C is independent of h and u.

More generally, if $u \in H^k(k \geq 1)$, we have

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{k-1,1} \cap C^1$ boundary and $u \in H^k(\Omega), 1 \leq k \leq m$. Assume that the generalized Fourier transform of $\psi(x) = \phi(|x|)$ exists and satisfies* (1.9). *Let $s_{u^h} = \sum_{i=1}^N a_i\psi(x - x^{(i)}) + \sum_{l=1}^Q b_l p_l(x)$ be the interpolant on $X = \{x^{(1)}, \cdots, x^{(N)}\} \subset \Omega$ to u^h. Then there exists a constant $h_0 > 0$ such that for all X with $h \leq h_0$, where h is defined by* (2.1), *we have*

$$\|u - s_{u^h}\|_{H^s(\Omega)} \leq Ch^{k-s}\|u\|_{H^k(\Omega)}, \qquad 0 \leq s < k, \tag{3.5}$$

where the positive constant C is independent of h and u.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. ([4]) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{k-1,1} \cap C^1$ boundary, $k \geq 1$. Then for any given $\Omega' \supset \Omega$, there exists a continuous linear extension operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ such that $Eu|_\Omega = u$ for all $u \in W^{k,p}(\Omega)$. Moreover, there exists a positive constant $C = C(k, \Omega, \Omega')$ such that*

$$\|Eu\|_{W^{k,p}(\Omega')} \leq C\|u\|_{W^{k,p}(\Omega)}. \tag{3.6}$$

Lemma 3.3. *Let $k \leq m$. Then for any $u \in H^k(\mathbb{R}^d)$, there exists a constant $C > 0$ (independent of u and h) such that u^h defined by* (3.3) *satisfies*

$$\|u^h\|_{H^m(\mathbb{R}^d)} \leq Ch^{k-m}\|u\|_{H^k(\mathbb{R}^d)}. \tag{3.7}$$

Proof. We note that, for any integer s, there exists a positive constant C such that

$$\frac{1}{C}\|u\|^2_{H^s(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d}(1 + |\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi \leq C\|u\|^2_{H^s(\mathbb{R}^d)}, \quad \forall u \in H^s(\mathbb{R}^d). \tag{3.8}$$

Observe also from (3.3) that

$$(\widehat{\eta_h * u})(\xi) = \widehat{\eta_h}(\xi)\widehat{u}(\xi) = \widehat{\eta}(h\xi)\widehat{u}(\xi). \tag{3.9}$$

Therefore, we have

$$\begin{aligned}\|u^h\|^2_{H^m(\mathbb{R}^d)} &\leq C\int_{\mathbb{R}^d}(1 + |\xi|^2)^m|(\widehat{\eta_h * u})(\xi)|^2 d\xi \\ &\leq C\int_{\mathbb{R}^d}(1 + |\xi|^2)^m|\widehat{\eta}(h\xi)\widehat{u}(\xi)|^2 d\xi. \end{aligned} \tag{3.10}$$

Since $\widehat{\eta}(\xi) \in C_0^\infty(B(0,2))$, we have $\widehat{\eta}(h\xi) \equiv 0$, if $|\xi| \geq \frac{2}{h}$. Consequently, (3.10) implies that

$$\begin{aligned}\|u^h\|^2_{H^m(\mathbb{R}^d)} &\leq C\int_{|\xi|\leq\frac{2}{h}}(1+|\xi|^2)^m|\widehat{\eta}(h\xi)\widehat{u}(\xi)|^2d\xi \\ &= C\int_{|\xi|\leq\frac{2}{h}}|\widehat{\eta}(h\xi)|^2(1+|\xi|^2)^{m-k}(1+|\xi|^2)^k|\widehat{u}(\xi)|^2d\xi \\ &\leq C\max_{|\xi|\leq\frac{2}{h}}|\widehat{\eta}(h\xi)|^2\Big(1+\frac{2^2}{h^2}\Big)^{m-k}\int_{|\xi|\leq\frac{2}{h}}(1+|\xi|^2)^k\widehat{u}(\xi)d\xi \\ &\leq Ch^{2(k-m)}\int_{\mathbb{R}^d}(1+|\xi|^2)^k\widehat{u}(\xi)d\xi \\ &= Ch^{2(k-m)}\|u\|^2_{H^k(\mathbb{R}^d)}. \end{aligned} \tag{3.11}$$

This completes the proof of this lemma.

Lemma 3.4. *Let $u \in H^k(\mathbb{R}^d)$, $k \geq 1$. Then there exists a positive constant C (independent of u and h) such that*

$$\|u-u^h\|_{H^s(\mathbb{R}^d)} \leq Ch^{k-s}\|u\|_{H^k(\mathbb{R}^d)}, \qquad 0 \leq s < k. \tag{3.12}$$

Proof. In view of (3.8) and (3.9), we obtain

$$\begin{aligned}\|u-u^h\|^2_{H^s(\mathbb{R}^d)} &\leq C\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\widehat{(u-u^h)}(\xi)|^2d\xi \\ &= C\int_{\mathbb{R}^d}(1+|\xi|^2)^s|1-\widehat{\eta}(h\xi)|^2|\widehat{u}(\xi)|^2d\xi \\ &= C\int_{\mathbb{R}^d}(1+|\xi|^2)^k|\widehat{u}(\xi)|^2\frac{|1-\widehat{\eta}(h\xi)|^2}{(1+|\xi|^2)^{k-s}}d\xi. \end{aligned} \tag{3.13}$$

It follows from (3.1) that the integrand in (3.13) vanishes outside $|\xi| \geq \frac{1}{h}$. Consequently,

$$\begin{aligned}\|u-u^h\|^2_{H^s(\mathbb{R}^d)} &\leq C\int_{|\xi|\geq\frac{1}{h}}(1+|\xi|^2)^k|\widehat{u}(\xi)|^2\frac{|1-\widehat{\eta}(h\xi)|^2}{(1+|\xi|^2)^{k-s}}d\xi \\ &\leq C\frac{1}{(1+h^{-2})^{k-s}}\int_{|\xi|\geq\frac{1}{h}}(1+|\xi|^2)^k|\widehat{u}(\xi)|^2d\xi \\ &\leq Ch^{2(k-s)}\|u\|^2_{H^k(\mathbb{R}^d)}. \end{aligned} \tag{3.14}$$

This completes the proof of Lemma 3.4.

Proof of Theorem 3.1. According to the $C^{k-1,1}$ smoothness of the boundary of Ω, by Lemma 3.2, there exists a continuous extension of u with compact support. Then by zero extension, we may extend u from Ω to $\mathbb{R}^d$. The extended function will be still denoted by u and

$$\|u\|_{H^k(\mathbb{R}^d)} \leq C\|u\|_{H^k(\Omega)}, \tag{3.15}$$

where $C = C(k,\Omega)$. Then, the combination of Lemma 3.1, Lemma 3.3, Lemma 3.4 and (3.15) gives

$$\begin{aligned}\|u-s_{u^h}\|_{H^s(\Omega)} &\leq \|u-u^h\|_{H^s(\Omega)}+\|u^h-s_{u^h}\|_{H^s(\Omega)} \\ &\leq \|u-u^h\|_{H^s(\mathbb{R}^d)}+Ch^{m-s}\|u^h\|_{H^m(\Omega)} \\ &\leq Ch^{k-s}\|u\|_{H^k(\mathbb{R}^d)}+Ch^{m-s}h^{k-m}\|u\|_{H^k(\mathbb{R}^d)} \\ &\leq Ch^{k-s}\|u\|_{H^k(\mathbb{R}^d)} \leq Ch^{k-s}\|u\|_{H^k(\Omega)}. \end{aligned} \tag{3.16}$$

This completes the proof of Theorem 3.1.

Example 1. From Table 1.1, there are two strictly positive definite radial basis functions

1. Sobolev spline with $\beta \geq m$; and
2. Compactly supported functions with $l \geq m - \frac{d+1}{2}$.

Both of them satisfy the condition (1.9) which is required by Theorem 3.1.

4. Construction of a Basis

Theorem 4.1. *Let $0 \le s < m$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^1 boundary. Assume that $\psi(x) = \phi(|x|)$ is strictly conditionally positive definite of order q ($q \ge 0$) such that its generalized Fourier transform exists and satisfies* (1.9). *Assume $\widetilde{X} \triangleq \{x^{(1)}, \cdots, x^{(N)}, \cdots\} \subset \Omega$ consists of pairwise distinct points. Then*

$$p_1(x), \cdots, p_Q(x), \psi(x - x^{(1)}), \cdots, \psi(x - x^{(N)}), \cdots$$

is a basis of $H^s(\Omega)$.

Proof. Step 1. We first demonstrate that $p_1(x), \cdots, p_Q(x), \psi(x - x^{(1)}), \cdots, \psi(x - x^{(N)})$ are linearly independent. If not, there exist $a = (a_1, \cdots, a_N)^T$ and $b = (b_1, \cdots, b_Q)^T$ with $a^2 + b^2 \ne 0$, such that

$$\sum_{j=1}^{N} a_j \psi(x - x^{(j)}) + \sum_{l=1}^{Q} b_l p_l(x) \equiv 0, \quad \forall x \in \Omega. \tag{4.1}$$

Since the total order of $p_l(x)$ is less than q, taking n-th order ($n > q$) derivative with respect to x_k ($1 \le k \le d$) in (4.1), we get

$$\sum_{j=1}^{N} a_j D^n_{x_k} \psi(x - x^{(j)}) \equiv 0, \quad \forall x \in \Omega. \tag{4.2}$$

Consequently,

$$\begin{aligned} 0 &\equiv \sum_{j=1}^{N} a_j D^n_{x_k} \psi(x - x^{(j)}) \\ &= \sum_{j=1}^{N} a_j D^n_{x_k} \Big(\int_{\mathbb{R}^d} e^{i<x - x^{(j)}, \xi>} \widehat{\psi}(\xi) d\xi \Big) \\ &= \int_{\mathbb{R}^d} (i\xi_k)^n e^{i<x,\xi>} \sum_{j=1}^{N} a_j e^{-i<x^{(j)},\xi>} \widehat{\psi}(\xi) d\xi, \quad \forall x \in \Omega. \end{aligned} \tag{4.3}$$

Setting $x = x^{(l)}$ in (4.3), we have

$$\int_{\mathbb{R}^d} (i\xi_k)^n e^{i<x^{(l)},\xi>} \sum_{j=1}^{N} a_j e^{-i<x^{(j)},\xi>} \widehat{\psi}(\xi) d\xi \equiv 0. \tag{4.4}$$

Multiplying both sides of (4.4) by a_l, and summing up over l, we get

$$\int_{\mathbb{R}^d} (i\xi_k)^n \big| \sum_{j=1}^{N} a_j e^{i<x^{(j)},\xi>} \big|^2 \widehat{\psi}(\xi) d\xi \equiv 0. \tag{4.5}$$

Noting that ψ is strictly conditionally positive definite of order q ($q \ge 0$), $\widehat{\psi}$ is nonnegative and non-vanishing(see [8]). Choose n to be even (to ensure that the sign of $(i\xi_k)^n$ in $\mathbb{R}^d$ does not change). Since $|\sum_{j=1}^{N} a_j e^{-i<x^{(j)},\xi>}|^2$ is continuous, (4.5) leads to

$$\sum_{j=1}^{N} a_j e^{-i<x^{(j)},\xi>} \equiv 0. \tag{4.6}$$

On the other hand, $e^{-i<x^{(1)},\xi>}, \cdots, e^{-i<x^{(N)},\xi>}$ is linearly independent for distinct $x^{(i)}$ ($i = 1, \cdots, N$). It follows from (4.6) that $a_j \equiv 0$ ($j = 1, \cdots, N$). Thus, (4.1) reads as

$$\sum_{l=1}^{Q} b_l p_l(x) \equiv 0, \quad \forall x \in \Omega. \tag{4.7}$$

The linear independence of $p_l(x)$ $(l = 1, \cdots, Q)$ implies that $b_l = 0$ $(l = 1, \cdots, Q)$, which gives a contradiction.

Step 2. For any given $u \in H^s(\Omega)$ and $\epsilon > 0$, since the boundary of Ω is C^1, there exists $v \in C^\infty(\overline{\Omega})$ such that

$$\|u - v\|_{H^s(\Omega)} \le \frac{\epsilon}{2}.$$

By Lemma 3.1, there is a constant $h_0 > 0$ such that if $h \le h_0$, then there exists an s_v of the form (1.1), such that

$$\|v - s_v\|_{H^s(\Omega)} \le \frac{\epsilon}{2}.$$

Hence

$$\|u - s_v\|_{H^s(\Omega)} \le \|u - v\|_{H^s(\Omega)} + \|v - s_v\|_{H^s(\Omega)} \le \epsilon.$$

This completes the proof of Theorem 4.1.

Example 2. Asuume that $\Omega \subset \mathbb{R}^d$ is bounded with C^1 boundary and $\widetilde{X} = \{x^{(1)}, \cdots, x^{(N)}, \cdots\}$ $\subset \Omega$ consists of pairwise distinct points. If the following strictly positive definite functions:

1. Sobolev spline with $\beta \ge m$,
2. Compactly supported functions with $l \ge m - \frac{d+1}{2}$

satisfy condition (1.9), then

$$\psi(x - x^{(1)}), \cdots, \psi(x - x^{(N)}), \cdots$$

is a basis of Sobolev space $H^s(\Omega)$ $(0 \le s < m - 1)$.

5. Applications

Consider the following model problem

$$-\triangle u = f \quad \text{in } \Omega\ , \tag{5.1}$$

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \Gamma, \tag{5.2}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^1 boundary Γ, $\frac{\partial u}{\partial n}$ is the outward unit normal derivative of u, $\alpha > 0$ is a constant and $f \in L^2(\Omega)$.

Write $V = H^1(\Omega)$. The corresponding weak problem reads: find $u \in V$ such that

$$a(u, v) = F(v), \quad \forall\ v \in V, \tag{5.3}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma \alpha u v dS, \tag{5.4}$$

$$F(v) = \int_\Omega f v dx. \tag{5.5}$$

It is easy to show that $a(\cdot, \cdot)$ is a continuous, coercive bilinear form on $V \times V$ and $F(\cdot)$ is a continuous linear functional on V. Therefore, by Lax-Milgram theorem, (5.3) has a unique solution.

In order to get the numerical solution of problem (5.1)-(5.2), we utilize Galerkin method. Choose

$$V_N = \text{span}\{p_1(x) \cdots, p_Q(x), \psi(x - x^{(1)}), \cdots, \psi(x - x^{(N)})\} \tag{5.6}$$

as a finite dimensional subspace of V. Assume ψ is strictly conditionally positive definite of order q $(q \ge 0)$ and its generalized Fourier trnaform $\widehat{\psi}$ exists and satisfies (1.9), and assume

$\{x^{(i)}\}_{i=1}^N$ is pairwise distinct. It follows from Theorem 4.1 that for any given $u_N \in V_N$

$$u_N(x) = \sum_{i=1}^{N} a_i \psi_i(x - x^{(i)}) + \sum_{l=1}^{Q} b_l p_l(x), \tag{5.7}$$

where a_i, b_l $(1 \le i \le N, 1 \le j \le Q)$ are some constants.

We now have a finite dimensional approximate problem: seek $u_N \in V_N$ such that

$$a(u_N, v_N) = F(v_N), \quad \forall v_N \in V_N. \tag{5.8}$$

Since $V_h \subset V$, the existence and uniqueness of (5.8) can be easily obtained.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{k-1,1} \cap C^1$ boundary. Assume that the generalized Fourier transform of $\psi(x) = \phi(|x|)$ exists and satisfies* (1.9) *with $m > \max(k, d/2)$. Let $u \in H^k(\Omega)$ $(k \ge 1)$ and u_N be the solutions to* (5.3) *and* (5.8)*, respectively. Then there exists a constant $h_0 > 0$ such that for all X with $h \le h_0$, where h is defined by* (2.1)*, we have*

$$\|u - u_N\|_{H^1(\Omega)} \le Ch^{k-1}\|u\|_{H^k(\Omega)}, \tag{5.9}$$

where C is a positive constant independent of h and u.

Proof. Applying Céa's Theorem (see [1]) and (3.5), we have

$$\|u - u_N\|_{H^1(\Omega)} \le C \inf_{v_N \in V_N} \|u - v_N\|_{H^1(\Omega)} \le C\|u - s_{u^h}\|_{H^1(\Omega)} \le Ch^{k-1}\|u\|_{H^k(\Omega)}. \tag{5.10}$$

The proof of this theorem is then complete.

If we only know that $u \in H^1(\Omega)$, i.e., $k = 1$, there is no approximate order of h in (5.9). However, from Theorem 4.1, $p_1(x), \cdots, p_Q(x), \psi(x - x^{(1)}), \cdots, \psi(x - x^{(N)}), \cdots$ is still a basis of $H^1(\Omega)$, and by Céa's Theorem again, as $h \to 0$ $(N \to +\infty)$, we have

$$\|u - u_N\|_{H^1(\Omega)} \le C \inf_{v_N \in V_N} \|u - v_N\|_{H^1(\Omega)} \to 0. \tag{5.11}$$

Hence, the approximate solution given by the meshless method above is also convergent to the exact solution.

Example 3. We consider the model problem (5.1) with $\Omega = [-1, 1] \times [-1, 1]$, $f = (4 - 4x^2 - 4y^2)e^{-x^2-y^2}$, $\alpha = 2$. The exact solution is $u = e^{-x^2-y^2}$. Using

$$\psi_1(r) = e^{-r}(3 + 3r + r^2) \quad (strictly\ positive\ definite) \tag{5.12}$$

$$\psi_2(r) = r^3 \quad (strictly\ conditionally\ positive\ definite\ of\ order\ 2) \tag{5.13}$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$, to construct a basis of V_N, respectively.

Table 5.1: The maximum relative errors

N	16	25	36	81
ψ_1	9.05%	1.15%	1.21%	0.56%
ψ_2	11.75%	2.36%	2.08%	0.91%

Table 5.1 presents the maximum relative errors between the numerical solutions and the exact solutions. It is observed that the numerical errors become smaller as the node number N increases. In particular, when $N = 81$, the maximum relative error is less than 1% in both cases.

Thus, the meshless Galerkin method provides a satisfactory approximation, which is found particularly useful for the scattered data and higher dimensional problems.

6. Concluding Remarks

It is known X' satisfying condition (1.7) is also called the correct lattice with respect to $\mathbb{P}_q$. For more details, readers may refer to [3].

Remark 6.1. In Theorem 3.1 and Theorem 4.1, the restriction on the domain Ω can be weaken to $\Omega \subset \mathbb{R}^d$ *is a bounded domain with Lipschitz boundary and satisfies a uniform interior cone condition.*

Remark 6.2. If the condition (1.9) is replaced by

$$c_1\|\xi\|^{-2m} \le \widehat{\psi}(\xi) \le c_2\|\xi\|^{-2m}, \quad c_1, c_2 > 0,$$

then Theorem 3.1 and Theorem 4.1 still hold. This allows us to use the thin plate spline which is a strictly conditionally positive definite function in computations.

7. Appendix

In the appendix, we will provide a proof to Theorem 1.1. The proof relies on two lemmas below. To begin with, define

$$G \triangleq \begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}_{(N+Q)\times(N+Q)},$$

where $P = \big(p_j(x^{(i)})\big)_{N\times Q}$ and $A = \big(\psi(x^{(i)} - x^{(j)})\big)_{N\times N}$.

Lemma 7.1. *Assume that $N \ge Q$. Then G is nonsingular if the rank $R(P)$ of P is Q.*

Proof. To show that G is nonsingular, it suffices to prove that the following linear system has only the zero solution:

$$A\lambda + P\mu = 0, \tag{7.1}$$

$$P^T\lambda = 0. \tag{7.2}$$

If $\lambda \ne 0$, we get from (7.1) that

$$0 = \lambda^T(A\lambda + P\mu) = \lambda^T A\lambda + \lambda^T P\mu = \lambda^T A\lambda$$

which contradicts with the conditionally positive definiteness of ϕ. If $\lambda = 0$, (7.1) implies that $P\mu = 0$. Then it follows from $R(P) = Q$ $(N \ge Q)$ that $\mu = 0$.

Lemma 7.2. *Assume that $N \ge Q$. Then $R(P) = Q$ if and only if there exists a subset $X' \subset X$ containing Q pairwise distinct points such that*

$$p|_{X'} = 0 \ \ \textit{for } p \in \mathbb{P}_q \qquad \textit{implies} \quad p \equiv 0. \tag{7.3}$$

Proof. Since $N \ge Q$, $R(P) = Q$ is equivalent to nonsingularity of the submatrix of order Q

$$P' = \begin{pmatrix} p_1(x^{(i_1)}) & \cdots & p_Q(x^{(i_Q)}) \\ \vdots & \ddots & \vdots \\ p_1(x^{(i_Q)}) & \cdots & p_Q(x^{(i_Q)}) \end{pmatrix}_{Q\times Q}.$$

Without loss of generality, we may choose $i_j = i, 1 \le j \le Q$. Assume that $R(P') = Q$ and $p|_{X'} = 0$ for $p \in \mathbb{P}_q$. Since $p_1(x), \cdots, p_Q(x)$ is a basis of $\mathbb{P}_q$, there exist $c_1, \cdots, c_Q$ such that $p(x) = \sum_{j=1}^{Q} c_j p_j(x)$. However, under this assumption, Cramer's rule implies $c_j = 0$ $(j = 1, \cdots, Q)$. Consequently, $p(x) = 0$.

On the other hand, suppose (7.3) holds. If $\det P' = 0$, then the column vectors of P' are linearly dependent, i.e., there exists $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_Q)^T \neq 0$ such that

$$\sum_{j=1}^{Q} \lambda_j p_j(x^{(i)}) = 0, \quad i = 1, \cdots, Q.$$

However, this contradicts with (7.3). This completes the proof of Lemma 7.2.

Theorem 1.1 follows immediately from Lemmas 7.1 and 7.2.

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