Journal of Computational Mathematics, Vol. 25, No. 3, 2007, 244–251.

SENSITIVITY ANALYSIS WITH RESPECT TO THE ELECTRICAL CONDUCTIVITY *1)

Habib Ammari
(*Centre de Mathématiques Appliquées, CNRS UMR 7641 and Ecole Polytechnique, 91128 Palaiseau Cedex, France*
Email: ammari@cmapx.polytechnique.fr)
Hyeonbae Kang
(*School of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea*
Email: hkang@math.snu.ac.kr)
Habib Zribi
(*Centre de Mathématiques Appliquées, CNRS UMR 7641 and Ecole Polytechnique, 91128 Palaiseau Cedex, France*
Email: zribi@cmapx.polytechnique.fr)

Abstract

In this paper, we consider conductivity inclusions inside a homogeneous background conductor. We provide a complete asymptotic expansion of the solution of such problems in terms of small variations in the electrical conductivity of the inclusion. Our method is based on a boundary integral perturbation theory. Our results are valid for both high and low contrast inclusions.

Mathematics subject classification: 35B30.
Key words: High contrast material, Full-asymptotic expansions, Boundary integral equation.

1. Introduction

An interesting problem arising in the study of photonic band gap structures concerns the calculation of electrostatic properties of systems made by high contrast materials. By high contrast, we mean that the electrical conductivity ratio is high. When the material contrast is high, standard numerical procedures can become ill-conditioned. We refer to Tausch, White, and Wang [10,11] and Greengard and Lee [6] for effective algorithms for this class of problems. The Tausch-White-Wang approach is based on a perturbation theory while the method of Greengard and Lee is a modification of the classical integral equation.

In this paper, we derive a complete asymptotic expansion of the solution of the conductivity problem due to small variations in the conductivity ratio by a boundary integral perturbation method. We provide error estimates for the approximation. Our results are valid for inclusions with extreme conductivities (zero or infinite conductivity). In particular, our method may be viewed as a different approach which can potentially simplify calculations for problems involving highly conducting inclusions.

Consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected Lipschitz boundary $\partial\Omega$. We assume, for the sake of simplicity, that its conductivity is

* Received November 7, 2006; Final revised January 18, 2007.
1) The work of the second author was partly supported by Korea Science and Engineering Foundation grant R02-2003-000-10012-0.

equal to 1. Let D with Lipschitz boundary be a conductivity inclusion inside Ω of conductivity equal to some positive constant $k \neq 1$. Let u_k be the solution of

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_D)\nabla u_k = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu}\Big|_{\partial\Omega} = g \in L^2_0(\partial\Omega), \quad \displaystyle\int_{\partial\Omega} u_k = 0, \end{cases} \tag{1.1}$$

where χ_D is the indicator function of D. We allow k to be 0 or $+\infty$. If $k = 0$, the inclusion D is insulated, and the equation in (1.1) is replaced with

$$\begin{cases} \Delta u_0 = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial D} = 0, \quad \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} u_0 = 0, \end{cases}$$

and if $k = +\infty$, then D is a perfect conductor and the equation in (1.1) is replaced with

$$\begin{cases} \Delta u_\infty = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \nabla u_\infty = 0 \quad \text{in } D, \\ \dfrac{\partial u_\infty}{\partial \nu}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} u_\infty = 0. \end{cases} \tag{1.2}$$

It was proved in [4, 7] that u_k converges in $W^{1,2}(\Omega \setminus \overline{D})$ to u_0 or u_∞ as $k \to 0$ or $k \to +\infty$. Here the space $W^{1,2}(\Omega \setminus \overline{D})$ is the set of functions $f \in L^2(\Omega \setminus \overline{D})$ such that $\nabla f \in L^2(\Omega \setminus \overline{D})$. The main result of this paper is a rigorous derivation, based on layer potential techniques, of a complete asymptotic expansion of $u_k|_{\partial\Omega}$ as $k \to +\infty$ or 0. In fact we will derive an asymptotic formula of $u_k|_{\partial\Omega}$ when $k \to k_0$.

This paper is organized as follows. In the next section we give an explicit asymptotic formula of u_k as $k \to +\infty$ or 0 when Ω is a disk and D is a concentric disk. In Section 3, we derive a complete asymptotic formula for $u_k - u_{k_0}$ on $\partial\Omega$ when $k \to k_0$. The formula is valid even when $k_0 = 0$ or $+\infty$.

2. Explicit Formula

In this section, Ω is assumed to be the unit disk centered at the origin, and D to be the concentric disk centered at the origin with radius α. Set

$$g(1,\theta) = \sum_{n \in \mathbb{Z}\setminus\{0\}} g_n e^{in\theta}.$$

Write

$$u_k = \begin{cases} a_0 + b_0 \ln(r) + \displaystyle\sum_{n \in \mathbb{Z}\setminus\{0\}} (a_n r^{|n|} + b_n r^{-|n|}) e^{in\theta} \quad \text{in } \Omega \setminus \overline{D}, \\ \displaystyle\sum_{n \in \mathbb{Z}} \frac{c_n}{\alpha^{|n|}} r^{|n|} e^{in\theta} \quad \text{in } D, \end{cases}$$

where the Fourier coefficients a_n, b_n and c_n are to be found.

Since $g \in L^2_0(\partial\Omega)$ and $\int_{\partial\Omega} u_k = 0$, we have that $a_0 = b_0 = 0$. Using the continuity of u_k across the interface ∂D, we get $c_0 = 0$. Then, for $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{cases} |n| a_n - |n| b_n = g_n, \\ a_n \alpha^{|n|} + b_n \alpha^{-|n|} - c_n = 0, \\ a_n \alpha^{|n|} - b_n \alpha^{-|n|} - k c_n = 0, \end{cases}$$

which yields

$$a_n = \frac{g_n}{|n|}\frac{(k+1)\alpha^{-|n|}}{(k+1)\alpha^{-|n|}+(k-1)\alpha^{|n|}},$$
$$b_n = -\frac{g_n}{|n|}\frac{(k-1)\alpha^{|n|}}{(k+1)\alpha^{-|n|}+(k-1)\alpha^{|n|}},$$
$$c_n = 2\frac{g_n}{|n|}\frac{1}{\alpha^{-|n|}(k+1)+\alpha^{|n|}(k-1)}.$$

Therefore, we have

$$u_k(1,\theta) = \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n+b_n)e^{in\theta} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{g_n}{|n|}\frac{(k+1)\alpha^{-|n|}-(k-1)\alpha^{|n|}}{(k+1)\alpha^{-|n|}+(k-1)\alpha^{|n|}}e^{in\theta}.$$

In similar fashion we get

$$u_\infty(1,\theta) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{g_n}{|n|}\frac{\alpha^{-|n|}-\alpha^{|n|}}{\alpha^{|n|}+\alpha^{-|n|}}e^{in\theta}.$$

Then the following asymptotic expansion holds as k goes to $+\infty$:

$$u_k(1,\theta) = u_\infty(1,\theta) + \sum_{l=1}^{+\infty}\frac{1}{(k-1)^l}v_\infty^{(l)}(\theta),$$

where

$$v_\infty^{(l)}(\theta) = 2^{l+1}(-1)^{l+1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha^{-(l-1)|n|}}{(\alpha^{|n|}+\alpha^{-|n|})^{l+1}}\frac{g_n}{|n|}e^{in\theta}. \tag{2.1}$$

Similarly, we get the following asymptotic formula when $k\to 0$:

$$u_k(1,\theta) = u_0(1,\theta) + \sum_{l=1}^{+\infty}\frac{k^l}{(k-1)^l}v_0^{(l)}(\theta),$$

where

$$v_0^{(l)}(\theta) = 2^{l+1}(-1)^{l+1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha^{-(l-1)|n|}}{(\alpha^{|n|}-\alpha^{-|n|})^{l+1}}\frac{g_n}{|n|}e^{in\theta}. \tag{2.2}$$

3. The General Case

3.1. Representation formula

Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ in $\mathbb{R}^2$: $\Gamma(x) = 1/(2\pi)\ln|x|$. The single and double layer potentials of the density function ϕ on D are defined by

$$\mathcal{S}_D\phi(x) := \int_{\partial D}\Gamma(x-y)\phi(y)d\sigma(y), \quad x\in\mathbb{R}^2, \tag{3.1}$$

$$\mathcal{D}_D\phi(x) := \int_{\partial D}\frac{\partial}{\partial\nu_y}\Gamma(x-y)\phi(y)d\sigma(y), \quad x\in\mathbb{R}^2\setminus\partial D. \tag{3.2}$$

For a function u defined on $\mathbb{R}^2 \setminus \partial D$, we denote

$$\frac{\partial}{\partial \nu^\pm} u(x) := \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \partial D,$$

if the limit exists.

The proof of the following trace formula can be found in [5]:

$$\frac{\partial}{\partial \nu^\pm} \mathcal{S}_D \phi(x) = \left(\pm \frac{1}{2} I + \mathcal{K}_D^* \right) \phi(x), \tag{3.3}$$

$$(\mathcal{D}_D \phi)|_\pm = \left(\mp \frac{1}{2} I + \mathcal{K}_D \right) \phi(x), \quad x \in \partial D, \tag{3.4}$$

where

$$\mathcal{K}_D \phi(x) = \frac{1}{2\pi} \int_{\partial D} \frac{\langle y - x, \nu_y \rangle}{|x - y|^2} \phi(y) d\sigma(y)$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$. Let $L_0^2(\partial D) := \{ f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0 \}$. The following results are of importance to us. For proofs see [5] or [2, p. 17].

Lemma 3.1. *The operator $\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, $\lambda I - \mathcal{K}_D^*$ is invertible on $L^2(\partial D)$.*

Denote by $\mathcal{S}_\Omega, \mathcal{D}_\Omega, \mathcal{K}_\Omega$, and $\mathcal{K}_\Omega^*$ the layer potentials on Ω. Define the functions $H_k(x)$, for $x \in \mathbb{R}^2 \setminus \partial \Omega$, by

$$H_k(x) := \mathcal{D}_\Omega(u_k|_{\partial \Omega})(x) - \mathcal{S}_\Omega g(x), \tag{3.5}$$

and introduce $N(\cdot, y)$ to be the Neumann function for Δ in Ω corresponding to a Dirac mass at y, that is, N is the solution to

$$\begin{cases} \Delta_x N(x, y) = -\delta_y \quad \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial \Omega} = -\dfrac{1}{|\partial \Omega|}, \\ \displaystyle\int_{\partial \Omega} N(x, y) d\sigma(x) = 0 \quad \text{for } y \in \Omega. \end{cases}$$

Define the background voltage potential, U, to be the unique solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial \Omega} = g, \displaystyle\int_{\partial \Omega} U = 0. \end{cases} \tag{3.6}$$

The following representation was proved in [1]:

$$u_k(x) = U(x) - \int_{\partial D} N(x, y)(\lambda I - \mathcal{K}_D^*)^{-1} \left(\frac{\partial H_k}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y), \quad x \in \partial \Omega, \tag{3.7}$$

where $\lambda = (k+1)/(2(k-1))$.

Lemma 3.2. *Let $k_0 \neq 1$ and $\lambda_0 = (k_0 + 1)/(2(k_0 - 1))$. Let $v_k = u_k - u_{k_0}$. Then, for any $x \in \partial \Omega$, we have*

$$v_k(x) + \int_{\partial D} N(x, y)(\lambda I - \mathcal{K}_D^*)^{-1} \left(\frac{\partial}{\partial \nu} \mathcal{D}_\Omega(v_k)\Big|_{\partial D} \right)(y) d\sigma(y)$$

$$= \int_{\partial D} N(x, y) \Big[-(\lambda I - \mathcal{K}_D^*)^{-1} + (\lambda_0 I - \mathcal{K}_D^*)^{-1} \Big] \left(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y). \tag{3.8}$$

Proof. It follows from (3.7) that, for $x \in \partial\Omega$,

$$u_k(x) - u_{k_0}(x) + \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1} \left(\frac{\partial(H_k - H_{k_0})}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y)$$
$$= \int_{\partial D} N(x,y) \Big[-(\lambda I - \mathcal{K}_D^*)^{-1} + (\lambda_0 I - \mathcal{K}_D^*)^{-1} \Big] \left(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y).$$

Thanks to (3.5), we get

$$H_k(x) - H_{k_0}(x) = \mathcal{D}_\Omega(u_k|_{\partial\Omega} - u_{k_0}|_{\partial\Omega})(x), \quad x \in \Omega,$$

and hence the proof is complete.

3.2. Derivation of the asymptotic expansion

Now, we expand $(\lambda I - \mathcal{K}_D^*)^{-1}$ as k goes to k_0, *i.e.*, with respect to $\lambda - \lambda_0$:

$$(\lambda I - \mathcal{K}_D^*)^{-1} = \sum_{n=0}^{+\infty} (-1)^n (\lambda - \lambda_0)^n (\lambda_0 I - \mathcal{K}_D^*)^{-n-1}. \tag{3.9}$$

Note that the series on the right-hand side of (3.9) converges absolutely as an operator on $L_0^2(\partial D)$ as long as $\lambda - \lambda_0$ is small enough. Thus (3.8) reads, for any $x \in \partial\Omega$,

$$v_k(x) + \sum_{n=0}^{+\infty} (-1)^n (\lambda - \lambda_0)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1} (\nabla \mathcal{D}_\Omega(v_k)|_{\partial D} \cdot \nu)(y) d\sigma(y)$$
$$= \sum_{n=1}^{+\infty} (-1)^{n+1} (\lambda - \lambda_0)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1} \left(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y),$$

or equivalently,

$$\left(I + \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n T_n \right)(v_k) = \sum_{n=1}^{+\infty} (\lambda - \lambda_0)^n F_n, \tag{3.10}$$

where

$$T_n(v)(x) = (-1)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1} (\nabla \mathcal{D}_\Omega(v)|_{\partial D} \cdot \nu)(y) d\sigma(y), \quad x \in \partial\Omega,$$

and

$$F_n(x) = (-1)^{n+1} \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1} \left(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y).$$

Note that, since ∂D is away from $\partial\Omega$, we have

$$\|T_n v\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C \|(\lambda_0 I - \mathcal{K}_D^*)^{-n-1} (\nabla \mathcal{D}_\Omega(v)|_{\partial D} \cdot \nu)\|_{L^2(\partial D)}$$
$$\le C C_0^{n+1} \|\nabla \mathcal{D}_\Omega(v)\|_{L^2(\partial D)} \le C_1 C_0^{n+1} \|v\|_{L^2(\partial\Omega)}, \tag{3.11}$$

where C_0 is the operator norm of $(\lambda_0 I - \mathcal{K}_D^*)^{-1}$ on $L_0^2(\partial\Omega)$ and C and C_1 are positive constants independent of n. Here $W^2_{\frac{1}{2}}(\partial\Omega)$ is the set of functions $f \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^2} \, d\sigma(x) \, d\sigma(y) < +\infty.$$

Likewise, we have

$$\|F_n\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C C_0^{n+1} \left\| \frac{\partial H_{k_0}}{\partial \nu} \right\|_{L^2(\partial D)}.$$

Note that

$$\left\| \frac{\partial H_{k_0}}{\partial \nu} \right\|_{L^2(\partial D)} \le C(\|u_{k_0}\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}) \le C'\|g\|_{L^2(\partial\Omega)},$$

for some C', and hence we get

$$\|F_n\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C C_0^{n+1} \|g\|_{L^2(\partial\Omega)}, \tag{3.12}$$

for some constant C independent of n. If $\partial\Omega$ is $\mathcal{C}^{1,\beta}$, $\beta > 0$, then we get in the same way

$$\|T_n v\|_{\mathcal{C}^1(\partial\Omega)} \le C C_0^{n+1} \|v\|_{L^2(\partial\Omega)}, \tag{3.13}$$

$$\|F_n\|_{\mathcal{C}^1(\partial\Omega)} \le C C_0^{n+1} \|g\|_{L^2(\partial\Omega)}. \tag{3.14}$$

We need the following lemma, which was proved in [3].

Lemma 3.3. *If $\partial\Omega$ is Lipschitz, then the operator $I + T_0$ is invertible on $L^2_0(\partial\Omega)$. If $\partial\Omega$ is $\mathcal{C}^{1,\beta}$ for some $\beta > 0$, then it is invertible on $\mathcal{C}^1_0(\partial\Omega)$, where $\mathcal{C}^1_0(\partial\Omega)$ denotes the collection of $f \in \mathcal{C}^1(\partial\Omega)$ with $\int_{\partial\Omega} f = 0$.*

We seek a solution v_k to (3.10) in the form

$$v_k(x) = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n v_{k_0}^{(n)}(x).$$

Substituting the above expansion of v_k into (3.10), we obtain

$$\sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n v_{k_0}^{(n)}(x) + \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \left(\sum_{p=0}^{n} T_p v_{k_0}^{(n-p)} \right)(x)$$
$$= \sum_{n=1}^{+\infty} (\lambda - \lambda_0)^n F_n(x), \quad x \in \partial\Omega. \tag{3.15}$$

By equating powers of $\lambda - \lambda_0$, we find that $v_{k_0}^{(0)} = 0$ and, for any $n \ge 1$,

$$(I + T_0) v_{k_0}^{(n)} + \sum_{p=1}^{n} T_p v_{k_0}^{(n-p)} = F_n.$$

Using Lemma 3.3, it follows that

$$v_{k_0}^{(n)} = (I + T_0)^{-1} \Big(- \sum_{p=1}^{n} T_p v_{k_0}^{(n-p)} + F_n \Big). \tag{3.16}$$

Using (3.11) and (3.12), one can show inductively that

$$\|v_{k_0}^{(n)}\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C_2 C_0^{n+1} n \|g\|_{L^2(\partial\Omega)}, \quad n = 1, 2, \ldots,$$

for some constant C_2 independent of n. The same estimates with the $W^2_{\frac{1}{2}}$-norm replaced with the $\mathcal{C}^1(\partial\Omega)$-norm holds if $\partial\Omega$ is $\mathcal{C}^{1,\beta}$. Since

$$\lambda - \lambda_0 = \frac{k_0 - k}{(k-1)(k_0 - 1)},$$

we finally arrive at the following theorem.

Theorm 3.1. *Let C_0 be the operator norm of $(\lambda_0 I-\mathcal{K}_D^*)^{-1}$ on $L_0^2(\partial D)$. Let $0 \leq k_0 \neq 1 \leq +\infty$. The following asymptotic expansion holds uniformly and absolutely if $|k-k_0| \leq C < 1/2C_0$ on $\partial\Omega$:*

$$u_k(x) = u_{k_0}(x) + \sum_{n=1}^{+\infty} \left[\frac{k_0 - k}{(k-1)(k_0-1)}\right]^n v_{k_0}^{(n)}(x), \tag{3.17}$$

where the functions $v_{k_0}^{(n)}$ are defined by the recursive formula (3.16). The convergence of the series is in $W_{\frac{1}{2}}^2(\partial\Omega)$ if $\partial\Omega$ is Lipschitz, and in $\mathcal{C}^1(\partial\Omega)$ if $\partial\Omega$ is $\mathcal{C}^{1,\beta}$.

In the most significant case, $k_0 = 0$ or $+\infty$, the formula takes the following form:

$$u_k(x) = u_0(x) + \sum_{n=1}^{+\infty} \frac{k^n}{(k-1)^n} v_0^{(n)}(x), \tag{3.18}$$

and

$$u_k(x) = u_\infty(x) + \sum_{n=1}^{+\infty} \frac{1}{(k-1)^n} v_\infty^{(n)}(x). \tag{3.19}$$

Moreover, if we interchange the conductivities of $\Omega \setminus D$ and D, the boundary perturbations in the voltage potentials are given by

$$\sum_{n=1}^{+\infty} \frac{1}{(k-1)^n} v_0^{(n)} \quad \text{if } k \to +\infty \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{k^n}{(k-1)^n} v_\infty^{(n)} \quad \text{if } k \to 0,$$

where $v_0^{(n)}$ and $v_\infty^{(n)}$ are defined by (3.16). This is related to the Keller-Mendelson inversion theorem [8, 9].

Now, if we consider the case when Ω is the unit disk centered at the origin, and D is the concentric disk centered at the origin with radius α then, using

$$\mathcal{K}_D^*\phi(x) = \frac{1}{4\pi\alpha}\int_{\partial D} \phi(y)\, d\sigma(y), \quad \mathcal{K}_\Omega\psi(x) = \frac{1}{4\pi}\int_{\partial\Omega} \psi(y)\, d\sigma(y),$$

and

$$N(x,y) = -2\Gamma(x-y) \text{ modulo constants}, \forall\, x \in \partial\Omega, y \in \partial D,$$

we easily obtain from Theorem 3.1 the explicit formulae (2.1) and (2.2) for $v_\infty^{(n)}$ and $v_0^{(n)}, n \geq 1$.

The formula (3.17) holds for all $k_0 \neq 1$. In low contrast case, *i.e.*, $k_0 = 1$, we can get the asymptotic formula trivially. In fact, if $k_0 = 1$, then $H_{k_0} = U$, the background potential. Define

$$\widetilde{T}_n(v)(x) := \int_{\partial D} N(x,y)(\mathcal{K}_D^*)^n \left(\frac{\partial \mathcal{D}_\Omega v}{\partial \nu}\Big|_{\partial D}\right)(y) d\sigma(y), \quad x \in \partial\Omega,$$
$$\widetilde{F}_n(x) := -\int_{\partial D} N(x,y)(\mathcal{K}_D^*)^n \left(\frac{\partial U}{\partial \nu}\Big|_{\partial D}\right)(y) d\sigma(y), \quad x \in \partial\Omega,$$

and let the functions $\widetilde{v}^{(n)}$ on $\partial\Omega$, for $n \in \mathbb{N}$, be given by

$$\widetilde{v}^{(0)}(x) = 0, \widetilde{v}^{(n)}(x) = -\sum_{p=0}^{n-1} \widetilde{T}_p \widetilde{v}^{(n-p-1)}(x) + \widetilde{F}_{n-1}(x), \quad n \geq 1.$$

Then we easily get that

$$u_k(x) = U(x) + \sum_{n=1}^{+\infty} \frac{2^n (k-1)^n}{(k+1)^n} \widetilde{v}^{(n)}(x), \quad x \in \partial\Omega.$$

The above asymptotic expansion holds uniformly and absolutely on $\partial\Omega$ if $|k-1| \leq C < 1/2\times$ the operator norm of $\mathcal{K}_D^*$ on $L_0^2(\partial D)$. The convergence of the series is in $W_{\frac{1}{2}}^2(\partial\Omega)$ if $\partial\Omega$ is Lipschitz, and in $\mathcal{C}^1(\partial\Omega)$ if $\partial\Omega$ is $\mathcal{C}^{1,\beta}, \beta > 0$.

References

[1] H. Ammari and H. Kang, High-order terms in the asymptotic expansions of the steady-state voltage potentials in the presence of conductivity inhomogeneities of small diameter, *SIAM J. Math. Anal.*, **34** (2003), 1152-1166.

[2] H. Ammari and H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Mathematics, Vol. 1846, Springer-Verlag, Berlin, 2004.

[3] H. Ammari, H. Kang, M. Lim and H. Zribi, Conductivity interface problems. Part I: Small perturbations of an interface, preprint.

[4] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, *Arch. Ration. Mech. Anal.*, **105** (1989), 563-579.

[5] G.B. Folland, Introduction to Partial Differential Equations, Princeton University Press, Princeton, New Jersey, 1976.

[6] L. Greengard and J.-Y. Lee, Electrostatics and heat conduction in high contrast composite materials, *J. Comput. Phys.*, **211** (2006), 64-76.

[7] H. Kang and J.K. Seo, Identification of domains with near-extreme conductivity: global stability and error estimates, *Inverse Probl.*, **15** (1999), 851-867.

[8] J.B. Keller, A theorem on the conductivity of a composite medium, *J. Math. Phys.*, **5** (1964), 548-549.

[9] K.S. Mendelson, Effective conductivity of two-phase material with cylindrical phase boundaries, *J. Appl. Phys.*, **46** (1975), 917-918.

[10] J. Tausch and J. White, Capacitance extraction of 3-D conductor systems in dielectric media with high permittivity ratios, *IEEE T. Microw. Theory Tech.*, **47** (1999), 18-26.

[11] J. Tausch, J. Wang and J. White, Improved integral formulations for fast 3-D method of moment solvers, *IEEE T. Comput. Aid. Design*, **20** (2001), 1398-1405.