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# SOME $n$-RECTANGLE NONCONFORMING ELEMENTS FOR FOURTH ORDER ELLIPTIC EQUATIONS *1)

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**Abstract**

In this paper, three $n$-rectangle nonconforming elements are proposed with $n \geq 3$. They are the extensions of well-known Morley element, Adini element and Bogner-Fox-Schmit element in two spatial dimensions to any higher dimensions respectively. These elements are all proved to be convergent for a model biharmonic equation in $n$ dimensions.

*Mathematics subject classification:* 65N30.
*Key words:* Nonconforming finite element, Forth order elliptic equation, Biharmonic.

## 1. Introduction

Motivated by both theoretical and practical interests, we will consider $n$-rectangle ($n \geq 2$) nonconforming finite elements for $n$-dimensional fourth order partial equations in this paper. In the two dimensional case, there are well-known nonconforming elements, such as the Morley element, the Zienkiewicz element and the Adini element, etc (see [1-4]). In a recent paper [10], we have discussed the motivation to construct nonconforming finite elements in three dimensions and proposed some tetrahedral nonconforming finite elements for 3-dimensional fourth order partial equations. As for the Morley element, we have extended it to any higher simplex case in another paper [11].

In this paper, we extend the Morley element, the Adini element and the Bogner-Fox-Schmit element to any higher dimensions, and obtain the following three types of $n$-rectangle nonconforming finite elements:

1. The $n$-rectangle Morley element, whose degrees of freedom are the value of the normal derivative at the centric point of each $(n-1)$-dimensional face and the function value at each vertex.

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2. The $n$-dimensional Adini element, whose degrees of freedom are the values of function and all first order derivatives at each vertex.

3. The $n$-dimensional BFS element, whose degrees of freedom are the values of function, all first order derivatives and all second order mixed derivatives at each vertex.

We will use the following standard notation. $\Omega$ denotes a general bounded polyhedral domain in $R^n$ $(n \geq 2)$, $\partial\Omega$ the boundary of $\Omega$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ the unit outer normal to $\partial\Omega$. For a nonnegative integer $s$, $H^s(\Omega)$, $H_0^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev spaces, its corresponding norm and semi-norm respectively, and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, set $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\forall x \in R^n$. For a subset $B \subset R^n$ and a nonnegative integer $r$, let $P_r(B)$ and $Q_r(B)$ be the spaces of polynomials on $B$ defined by

$$P_r(B) = \mathrm{span}\{\, x^\alpha \mid |\alpha| \leq r\}, \quad Q_r(B) = \mathrm{span}\{\, x^\alpha \mid \alpha_i \leq r\}.$$

The paper is organized as follows. The rest of this section gives some notation. Section 2 gives a detailed description of the $n$-rectangle Morley element, the $n$-dimensional Adini element and the BFS element. Section 3 and Section 4 show the convergence of these elements.

## 2. The $n$-Rectangle Elements

In this section, we will give our extensions of the Morley element, the Adini element and the Bogner-Fox-Schmit element to higher dimensions. For a finite element, it can be described by a triple $(T, P_T, \Phi_T)$ with $T$ the geometric shape, $P_T$ the shape function space and $\Phi_T$ the vector of degrees of freedom.

Given $a_0 = (a_{01}, a_{02}, \cdots, a_{0n})^\top \in R^n$ and positive numbers $h_1,\cdots,h_n$, an $n$-rectangle $T$ is given by

$$T = \{\, x \,|\, x_i = a_{0i} + h_i\xi_i, \ -1 \leq \xi_i \leq 1, \ 1 \leq i \leq n\}.$$

Let $\xi = (\xi_1, \cdots, \xi_n)^\top$, and let $a_i$, $1 \leq i \leq 2^n$, be the vertices of $T$. The vertices are written by

$$a_i = (a_{01} + \xi_{i1}h_1, \, a_{02} + \xi_{i2}h_2, \cdots, a_{0n} + \xi_{in}h_n)^\top, \quad 1 \leq i \leq 2^n,$$

and the barycenters of the $(n-1)$-dimensional faces of $T$ are written as

$$\begin{cases} b_{2k-1} = (a_{01}, \cdots, a_{0,k-1}, a_{0k} + h_k, a_{0,k+1}, \cdots, a_{0n})^\top, \\ b_{2k} \quad\; = (a_{01}, \cdots, a_{0,k-1}, a_{0k} - h_k, a_{0,k+1}, \cdots, a_{0n})^\top, \end{cases} \quad 1 \leq k \leq n.$$

Let $F_i$ $(1 \leq i \leq 2n)$ denote the $(n-1)$-dimensional face with $b_i$ as its barycenter. Define

$$\tilde{p}_i = \frac{1}{2^n} \prod_{j=1}^n (1 + \xi_{ij}\xi_j), \quad 1 \leq i \leq 2^n.$$

It is known that $\tilde{p}_i$, $1 \leq i \leq 2^n$, forms a basis of $Q_1(T)$. For a mesh size $h$, let $\mathcal{T}_h$ be a triangulation of $\Omega$ consisting of $n$-rectangles described above.

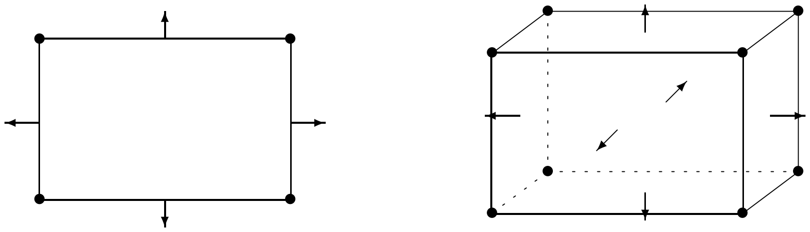


Fig. 2.1. Degrees of freedom of the $n$-rectangle Morley element.

### 2.1. The $n$-rectangle Morley element

Define

$$P_M(T) = Q_1(T) + \operatorname{span}\{x_1^2, x_2^2, \cdots, x_n^2, x_1^3, x_2^3, \cdots, x_n^3\}.$$

It can be verified that $P_2(T) \subset P_M(T)$. For the $n$-rectangle Morley element, $(T, P_T, \Phi_T)$ is given by (see Fig. 1).

- $T$ is an $n$-rectangle described above.
- $P_T = P_M(T)$.
- For $v \in C^1(T)$, the vector $\Phi_T(v)$ of degrees of freedom is

$$\Phi_T(v) = \Big(v(a_1), \cdots, v(a_{2^n}), \frac{\partial v}{\partial \nu}(b_1), \cdots, \frac{\partial v}{\partial \nu}(b_{2n})\Big)^\top.$$

Corresponding to $\Phi_T$, we define

$$\begin{cases} p_i = \dfrac{1}{2^{n+1}}\Big(2\displaystyle\prod_{j=1}^{n}(1+\xi_{ij}\xi_j) - \sum_{j=1}^{n}\xi_{ij}\xi_j(\xi_j^2-1)\Big), & 1 \le i \le 2^n, \\ q_{2k-1} = \dfrac{h_k}{4}(\xi_k+1)^2(\xi_k-1), & 1 \le k \le n, \\ q_{2k} = -\dfrac{h_k}{4}(\xi_k+1)(\xi_k-1)^2, & 1 \le k \le n. \end{cases} \tag{2.1}$$

Let $\delta_{ij}$ be the Kronecker delta. We can verify that

$$\begin{cases} p_i(a_j) = \delta_{ij}, & 1 \le i, j \le 2^n \\ \dfrac{\partial p_i}{\partial \nu}(b_j) = 0, & 1 \le j \le 2n,\ 1 \le i \le 2^n; \\ q_i(a_j) = 0, & 1 \le j \le 2^n,\ 1 \le i \le 2n, \\ \dfrac{\partial q_i}{\partial \nu}(b_j) = \delta_{ij}, & 1 \le i, j \le 2n. \end{cases} \tag{2.2}$$

That is, $p_i$ $(1 \le i \le 2^n)$ and $q_j$ $(1 \le j \le 2n)$ are basis functions. Consequently, $\Phi_T$ is $P_T$-unisolvent.

The corresponding interpolation operator $\Pi_T$ is given by

$$\Pi_T v = \sum_{i=1}^{2^n} p_i v(a_i) + \sum_{i=1}^{2n} q_i \frac{\partial v}{\partial \nu}(b_i), \quad \forall v \in C^1(T), \tag{2.3}$$

For the $n$-rectangle Morley element, we can define the corresponding finite element spaces $V_h$ and $V_{h0}$ as follows: $V_h$ consists of all functions $v_h$ such that for any $T \in T_h$, 1) $v_h|_T \in P_M(T)$, 2) $v_h$ is continuous at all vertices of $T$ and 3) the normal derivative of $v_h$ is continuous at the barycenters of all $(n-1)$-dimensional faces of $T$; $V_{h0}$ consists of all functions $v_h \in V_h$ such that for any $T \in T_h$, $v_h$ vanishes at the vertices of $T$ belonging to $\partial\Omega$ and the normal derivative of $v_h$ vanishes at the barycenters of all $(n-1)$-dimensional faces of $T$ on $\partial\Omega$.

**Lemma 2.1.** *Let $V_h$ and $V_{h0}$ be the finite element spaces of the $n$-rectangle Morley element. Then*

$$\int_F \nabla(v|_T) = \int_F \nabla(v|_{T'}), \quad \forall v \in V_h, \tag{2.4}$$

*where $T$ and $T' \in \mathcal{T}_h$ share a common $(n-1)$-dimensional face $F$. If an $(n-1)$-dimensional face $F$ of $T \in \mathcal{T}_h$ is on $\partial\Omega$, then*

$$\int_F \nabla(v|_T) = 0, \quad \forall v \in V_{h0}. \tag{2.5}$$

*Proof.* Let $v \in V_h$. Define $w \in L^2(\Omega)$ by

$$w|_T = \sum_{i=1}^{2^n} \tilde{p}_i v(a_i), \quad \forall T \in \mathcal{T}_h.$$

Then $w \in H^1(\Omega)$. For $T \in \mathcal{T}_h$ and $1 \le k \le n$, by (2.1) we have

$$\int_{F_j} \frac{\partial p_i}{\partial x_k} = \begin{cases} 0, & j = 2k-1, 2k, \\ \displaystyle\int_{F_j} \frac{\partial \tilde{p}_i}{\partial x_k}, & \text{otherwise}, \end{cases} \qquad 1 \le i \le 2^n, \tag{2.6}$$

$$\int_{F_j} \frac{\partial q_i}{\partial x_k} = \begin{cases} \displaystyle\prod_{\substack{1 \le m \le n \\ m \ne k}} 2h_m, & j = i,\ j \in \{2k-1, 2k\}, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \le i \le 2n. \tag{2.7}$$

Using (2.6) and (2.7), we obtain that for $1 \le k \le n$ and $1 \le j \le 2n$,

$$\int_{F_j} \frac{\partial v|_T}{\partial x_k} = \begin{cases} \dfrac{\partial v}{\partial x_k}(b_j) 2^{n-1} \displaystyle\prod_{\substack{1 \le m \le n \\ m \ne k}} h_m, & j = 2k-1, 2k, \\ \displaystyle\int_{F_j} \frac{\partial w}{\partial x_k}, & \text{otherwise}. \end{cases} \tag{2.8}$$

By (2.8), the definition of $V_h$ and the fact that $w \in H^1(\Omega)$, we obtain (2.4).

Using similar argument, we can obtain (2.5). This completes the proof of the lemma.

### 2.2. The $n$-dimensional Adini element

Define

$$P_A(T) = Q_1(T) + \text{span}\{\, x_i^2 x^\alpha \mid 1 \le i \le n,\ \alpha_j \le 1,\ 1 \le j \le n\}.$$

Obviously, $P_3(T) \subset P_A(T)$. For the $n$-dimensional Adini element, $(T, P_T, \Phi_T)$ is defined as follows (see Fig. 2).

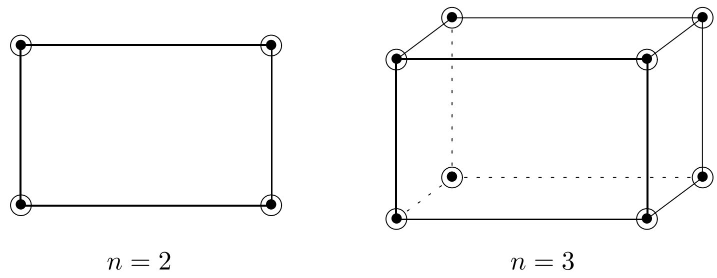


Fig. 2.2. Degrees of freedom of the $n$-dimensional Adini element.

- $T$ is an $n$-rectangle described above.
- $P_T = P_A(T)$.
- For $v \in C^1(T)$, the vector $\Phi_T(v)$ is given by

$$\Phi_T(v) = \Big(v(a_1), \nabla v(a_1)^\top, v(a_2), \nabla v(a_2)^\top, \cdots, v(a_{2^n}), \nabla v(a_{2^n})^\top\Big)^\top.$$

For $i \in \{1, 2, \cdots, 2^n\}$ and $j \in \{1, 2, \cdots, n\}$, we define

$$\begin{cases} p_{0i} = \dfrac{1}{2^{n+1}}\Big(2 + \displaystyle\sum_{k=1}^{n}(\xi_{ik}\xi_k - \xi_k^2)\Big)\prod_{k=1}^{n}(1 + \xi_{ik}\xi_k), \\ p_{ji} = \dfrac{h_j\xi_{ij}}{2^{n+1}}(\xi_j^2 - 1)\displaystyle\prod_{k=1}^{n}(1 + \xi_{ik}\xi_k). \end{cases} \tag{2.9}$$

It can be verified that $p_{ji}$, $0 \le j \le n$, $1 \le i \le 2^n$, are the basis functions with respect to the degrees of freedom. Consequently, $\Phi_T$ is $P_T$-unisolvent. The corresponding interpolation operator $\Pi_T$ is written by

$$\Pi_T v = \sum_{i=1}^{2^n} p_{0i} v(a_i) + \sum_{j=1}^{n}\sum_{i=1}^{2^n} p_{ji}\frac{\partial v}{\partial x_j}(a_i), \quad \forall v \in C^1(T). \tag{2.10}$$

For the $n$-dimensional Adini element, we can define the finite element spaces $V_h$ and $V_{h0}$ as follows: $V_h = \{\, v_h \in L^2(\Omega) \,|\, v_h|_T \in P_A(T),\ \forall T \in \mathcal{T}_h,\ v_h$ and $\nabla v_h$ are continuous at all vertices of elements in $\mathcal{T}_h\}$, $V_{h0} = \{\, v_h \in V_h \,|\ v_h$ and $\nabla v_h$ vanish at the vertices along $\partial\Omega\}$.

Given $v \in V_h$ and an $(n-1)$-dimensional face $F$ of $T \in \mathcal{T}_h$, the restriction $v|_F$ of $v$ on $F$ is a polynomial of $(n-1)$ variables in the shape function space $P_A(F)$. Then $v|_F$ is uniquely determined by the values of $v$ and $\nabla v$ at $2^{n-1}$ vertices of $F$. That is, $v$ is continuous through $F$. Thus, $v \in H^1(\Omega)$. If $v \in V_{h0}$ and $F \subset \partial\Omega$ in addition then $v|_F \equiv 0$, and this leads to $v \in H_0^1(\Omega)$.

Although $V_h \subset H^1(\Omega)$ and $V_{h0} \subset H_0^1(\Omega)$, the $n$-dimensional Adini element is still a nonconforming element for the fourth order problem.

### 2.3. The $n$-dimensional BFS element

Define

$$\begin{aligned} S_T &= \{\, x_1^2, x_2^2, \cdots, x_n^2\} + \{\, x_i^2 x_j^2 \,|\, 1 \le i < j \le n\}, \\ P_B(T) &= Q_1(T) + \mathrm{span}\{\, p x^\alpha \mid \forall p \in S_T,\ \alpha_i \le 1,\ 1 \le i \le n\}. \end{aligned}$$

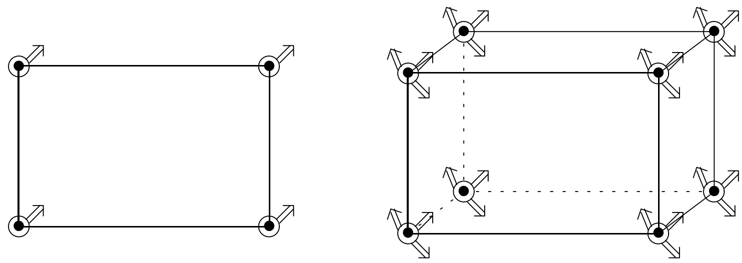


Fig. 2.3. Degrees of freedom of the $n$-dimensional BFS element.

We can verify that $P_3(T) \subset P_B(T)$. For the $n$-dimensional Bogner-Fox-Schmit (BFS) element, $(T, P_T, \Phi_T)$ is defined as follows (see Fig. 3).

- $T$ is an $n$-rectangle described above.
- $P_T = P_B(T)$.
- For $v \in C^2(T)$, all components of vector $\Phi_T(v)$ are:

$$v(a_i),\ \nabla v(a_i),\ \frac{\partial^2 v}{\partial x_j \partial x_k}(a_i), \quad 1 \le j < k \le n,\ 1 \le i \le 2^n.$$

**Lemma 2.2.** *For the $n$-dimensional BFS element, $\Phi_T$ is $P_T$-unisolvent.*

*Proof.* Since the dimension of $P_T$ and the number of degrees of freedom are all

$$\Big(\frac{n(n-1)}{2} + n + 1\Big)2^n,$$

it is enough to show that if $p \in P_B(T)$ and $\Phi_T(p) = 0$ then $p \equiv 0$. We show the conclusion by induction.

The 2-dimensional BFS element is just the Bogner-Fox-Schmit element in two dimensions. The conclusion is true when $n = 2$ (see [2]). Assume that the conclusion is true for $n = k,\ k \ge 2$.

Now let $n = k+1$. Write $p = p(\xi_1, \xi_2, \cdots, \xi_n)$. On the $k$-dimensional faces $F_\pm$ of $\xi_1 = \pm 1$, $p$ is a polynomial of $\xi_2, \cdots, \xi_n$ in $k$-dimensional shape function space $P_B(F_\pm)$. Since

$$p(\pm 1, \xi_2, \cdots, \xi_n), \quad \frac{\partial p}{\partial \xi_j}(\pm 1, \xi_2, \cdots, \xi_n), \quad 2 \le j \le n,$$
$$\frac{\partial^2 p}{\partial \xi_j \partial \xi_l}(\pm 1, \xi_2, \cdots, \xi_n), \quad 2 \le k < l \le n,$$

are all zero at each vertex of $F_\pm$, $p(\pm 1, \xi_2, \cdots, \xi_n) = 0$ for all $\xi_2, \cdots, \xi_n \in [-1, 1]$ by the inductive assumption. This leads that $\xi_1^2 - 1$ is a factor of $p$. Repeating the same argument for $\xi_2$ to $\xi_n$, we obtain that $(\xi_1^2 - 1) \cdots (\xi_n^2 - 1)$ is a factor of $p$. Consequently, $p \equiv 0$.

For the $n$-dimensional BFS element, we can define the finite element spaces $V_h$ and $V_{h0}$ as follows: $V_h = \{\, v_h \in L^2(\Omega) \,|\, v_h|_T \in P_B(T),\ \forall T \in \mathcal{T}_h,\ v_h,\ \nabla v_h$ and $\frac{\partial^2}{\partial x_j \partial x_k} v_h,\ 1 \le j < k \le n,$ are continuous at all vertices of elements in $\mathcal{T}_h\}$, $V_{h0} = \{\, v_h \in V_h \,|\ v_h,\ \nabla v_h$ and $\frac{\partial^2}{\partial x_j \partial x_k} v_h$, $1 \le j < k \le n$, vanish at the vertices along $\partial\Omega\}$.

Given $v \in V_h$ and an $(n-1)$-dimensional face $F$ of $T \in \mathcal{T}_h$, the restriction $v|_F$ of $v$ on $F$ is a polynomial of $(n-1)$ variables in the shape function space $P_B(F)$. Then it is uniquely determined by the values of $v$. $\nabla v$ and all second order mixed derivatives at all vertices of $F$. That is, $v$ is continuous through $F$. Consequently, $v \in H^1(\Omega)$. If $v \in V_{h0}$ and $F \subset \partial\Omega$ in addition then $v|_F \equiv 0$, and this leads to $v \in H_0^1(\Omega)$.

Although the 2-dimensional BFS element is a conforming element for the fourth order problem, one can verify that the general $n$-dimensional BSF element is not a $C^1$ element when $n > 2$.

## 3. Approximation Property

For nonconforming elements, the basic mathematical theory has been established (see [2,3,5-9]). We will use it to give the convergence analysis of our elements. In this section, we will consider the approximation properties.

For each element $T \in \mathcal{T}_h$, let $h_T$ be the diameter of the smallest ball containing $T$ and $\rho_T$ be the diameter of the largest ball contained in $T$. Let $\{\mathcal{T}_h\}$ be a family of triangulations described in previous section with $h \to 0$. We assume that $\{\mathcal{T}_h\}$ satisfied that $h_T \le h \le \eta\rho_T$, $\forall T \in \mathcal{T}_h$ for a positive constant $\eta$ independent of $h$.

We introduce the following mesh-dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2}$$

for a function $v$ with $v|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$.

For convenience, following [12], the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ will be used in this paper: $X_1 \lesssim Y_1$ and $X_2 \gtrsim Y_2$ mean that $X_1 \le c_1 Y_1$ and $c_2 X_2 \ge Y_2$ for some positive constants $c_1$ and $c_2$ that are independent of mesh size $h$. That $X_3 \eqsim Y_3$ means that $X_3 \lesssim Y_3$ and $X_3 \gtrsim Y_3$.

**Theorem 3.1.** *Let $\Pi_T$ be the interpolation operator of the $n$-rectangle Morley element, the $n$-dimensional Adini element or the $n$-dimensional BFS element. If $n < 4$ then for any $T \in \mathcal{T}_h$,*

$$|v - \Pi_T v|_{m,T} \lesssim h^{r-m}|v|_{r,T}, \quad 0 \le m \le r, \ \forall v \in H^r(T), \tag{3.1}$$

*where $r = 3$ for the $n$-rectangle Morley element, $r = 4$ for the other two elements.*

Theorem 3.1 can be obtained directly from the interpolation theory (see [2]). Although Theorem 3.1 is enough for practical situations, we would like to consider a result for all $n \ge 2$.

**Theorem 3.2.** *Let $V_h$ and $V_{h0}$ be the finite element spaces of the $n$-rectangle Morley element, the $n$-dimensional Adini element or the $n$-dimensional BFS element. Then*

$$\inf_{v_h\in V_h} \sum_{m=0}^{r} h^m |v - v_h|_{m,h} \lesssim h^r |v|_{r,\Omega}, \qquad \forall v \in H^r(\Omega), \tag{3.2}$$

$$\inf_{v_h\in V_{h0}} \sum_{m=0}^{r} h^m |v - v_h|_{m,h} \lesssim h^r |v|_{r,\Omega}, \qquad \forall v \in H^r(\Omega)\cap H_0^2(\Omega), \tag{3.3}$$

*where $r = 3$ for the $n$-rectangle Morley element, $r = 4$ for the other two elements.*

*Proof.* First, we consider the $n$-rectangle Morley element and inequality (3.3). For $v \in H^3(\Omega)\cap H_0^2(\Omega)$, let $w_h \in L^2(\Omega)$ such that for any $T \in \mathcal{T}_h$, $w_h|_T \in P_M(T)$ and

$$\int_T q w_h = \int_T q v, \quad \forall q \in P_M(T).$$

By the interpolation theory, we have

$$|v - w_h|_{m,h} \lesssim h^{3-m}|v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{3.4}$$

Given a set $B \subset R^n$, let $\mathcal{T}_h(B) = \{\, T \in \mathcal{T}_h \mid B \cap T \neq \emptyset \,\}$ and $N_h(B)$ be the number of the elements in $\mathcal{T}_h(B)$. For $w \in L^2(\Omega)$ and $T \in \mathcal{T}_h$, let $w^T$ denote the restriction of $w$ on $T$.

Now we define $v_h \in V_{h0}$ as follows: for any $T \in \mathcal{T}_h$,

- if the vertex $a_i$ $(1 \le i \le 2^n)$ of $T$ is in $\Omega$, then

$$v_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} w_h^{T'}(a_i),$$

- if $F_i$ $(1 \le i \le 2n)$ of $T$ is also a face of another element $T' \in \mathcal{T}_h$, then

$$\frac{\partial v_h}{\partial \nu}(b_i) = \frac{1}{2}\Big(\frac{\partial w_h^T}{\partial \nu}(b_i) + \frac{\partial w_h^{T'}}{\partial \nu}(b_i)\Big),$$

  where $\nu$ is the unit outer normal to $F_i$ respect to $T$.

Obviously, $v_h$ is well-defined. We will show

$$|v - v_h|_{m,h} \lesssim h^{3-m}|v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{3.5}$$

Let $T \in \mathcal{T}_h$. By a standard scaling argument, we obtain that

$$|p|_{m,T}^2 \lesssim h^{n-2m}\left(\sum_{i=1}^{2^n} |p(a_i)|^2 + h^2 \sum_{i=1}^{2n} \Big|\frac{\partial p}{\partial \nu}(b_i)\Big|^2\right), \quad 0 \le m \le 3,\ \forall p \in P_M(T). \tag{3.6}$$

Set $\phi_h = w_h - v_h$. Obviously, $\phi_h^T \in P_M(T)$. If the vertex $a_i$ of $T$ is in $\Omega$ then by the definition of $v_h$,

$$\phi_h^T(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \Big(w_h^T(a_i) - w_h^{T'}(a_i)\Big).$$

For $T' \in \mathcal{T}_h(a_i)$ there exist $T_1, \cdots, T_J \in \mathcal{T}_h(a_i)$ such that $T_1 = T$, $T_J = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common $(n-1)$-dimensional face of $T_j$ and $T_{j+1}$ and $a_i \in \tilde{F}_j$, $1 \le j < J$. By the inverse inequality, we have

$$\begin{aligned}
&\Big|w_h^T(a_i) - w_h^{T'}(a_i)\Big|^2 = \Big|\sum_{j=1}^{J-1}\big(w_h^{T_j}(a_i) - w_h^{T_{j+1}}(a_i)\big)\Big|^2 \\
\lesssim\ & \sum_{j=1}^{J-1}\Big|w_h^{T_j}(a_i) - w_h^{T_{j+1}}(a_i)\Big|^2 \le Ch^{1-n}\sum_{j=1}^{J-1}\Big|w_h^{T_j} - w_h^{T_{j+1}}\Big|_{0,\tilde{F}_j}^2 \\
\lesssim\ & h^{1-n}\sum_{j=1}^{J-1}\Big(\Big|v - w_h^{T_j}\Big|_{0,\tilde{F}_j}^2 + \Big|v - w_h^{T_{j+1}}\Big|_{0,\tilde{F}_j}^2\Big).
\end{aligned}$$

By the interpolation theory, we obtain

$$\Big|w_h^T(a_i) - w_h^{T'}(a_i)\Big|^2 \lesssim h^{6-n}\sum_{j=1}^{J}|v|_{3,T_j}^2.$$

Since $N_h(T)$ is bounded, it follows that

$$|\phi_h^T(a_i)|^2 \lesssim h^{6-n} \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2. \tag{3.7}$$

If the vertex $a_i$ of $T$ is on $\partial\Omega$ then there exists $T' \in \mathcal{T}_h(a_i)$ with an $(n-1)$-dimensional face $F$ of $T'$ belonging to $\partial\Omega$ and $a_i \in F$. By the definitions of $w_h$ and $v_h$,

$$|\phi_h^T(a_i)| = |w_h^T(a_i) - w_h^{T'}(a_i) + w_h^{T'}(a_i)| \leq |w_h^T(a_i) - w_h^{T'}(a_i)| + |w_h^{T'}(a_i)|.$$

By the inverse inequality and the interpolation theory, we have

$$|w_h^{T'}(a_i)|^2 \lesssim h^{1-n} |w_h^{T'}|_{0,F}^2 \eqsim h^{1-n} |v - w_h^{T'}|_{0,F}^2 \lesssim h^{6-n} |v|_{3,T'}^2.$$

By a similar analysis for $|w_h^T(a_i) - w_h^{T'}(a_i)|$, we conclude that (3.7) is also true in this case.

If the face $F_i$ of $T$ is also a face of another element $T' \in \mathcal{T}_h$ then

$$\begin{aligned} \Big|\frac{\partial \phi_h^T}{\partial \nu}(b_i)\Big|^2 &= \frac{1}{4}\Big|\frac{\partial (w_h^T - w_h^{T'})}{\partial \nu}(b_i)\Big|^2 \lesssim h^{1-n}\Big|\frac{\partial (w_h^T - w_h^{T'})}{\partial \nu}\Big|_{0,F_i}^2 \\ &\lesssim \ h^{1-n}\Big|\frac{\partial (w_h^T - v)}{\partial \nu}\Big|_{0,F_i}^2 + h^{1-n}\Big|\frac{\partial (v - w_h^{T'})}{\partial \nu}\Big|_{0,F_i}^2. \end{aligned}$$

By the interpolation theory, we have

$$\Big|\frac{\partial \phi_h^T}{\partial \nu}(b_i)\Big|^2 \lesssim h^{4-n} \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2. \tag{3.8}$$

If the face $F_i$ of $T$ is on $\partial\Omega$, then

$$\Big|\frac{\partial \phi_h^T}{\partial \nu}(b_i)\Big|^2 = \Big|\frac{\partial w_h^T}{\partial \nu}(b_i)\Big|^2 \lesssim h^{1-n}\Big|\frac{\partial w_h^T}{\partial \nu}\Big|_{0,F_i}^2 = h^{1-n}\Big|\frac{\partial (w_h^T - v)}{\partial \nu}\Big|_{0,F_i}^2.$$

Thus (3.8) is also true by the interpolation theory.

Combining (3.6), (3.7) and (3.8), we have

$$h^{2m}|\phi_h|_{m,T}^2 \lesssim h^6 \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2.$$

Summing the above inequality over all $T \in \mathcal{T}_h$, we obtain that

$$h^{2m}|\phi_h|_{m,h}^2 \lesssim h^6 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2.$$

Consequently,

$$h^{2m}|\phi_h|_{m,h}^2 \lesssim h^6 |v|_{3,\Omega}^2. \tag{3.9}$$

Inequality (3.5) follows from (3.9) and (3.4).

We have proved (3.3) for the $n$-rectangle Morley element. Using a similar argument, we can prove (3.3) for the other two elements as well as (3.2).

## 4. Convergence Analysis

In this section, we will give the convergence analysis of the elements given in Section 2 for the boundary value problem of fourth order partial differential equations.

For $f \in L^2(\Omega)$, we consider the following problem:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ is the unit outer normal to $\partial\Omega$ and $\Delta$ is the standard Laplacian operator.

Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega). \tag{4.2}$$

The weak form of problem (4.1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega). \tag{4.3}$$

For $v, w \in L^2(\Omega)$ that $v|_T, w|_T \in H^2(T)$, $\forall T \in \mathcal{T}_h$, we define

$$a_h(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \tag{4.4}$$

Corresponding to the $n$-rectangle Morley element, the $n$-dimensional Adini element or the $n$-dimensional BFS element, the finite element method for problem (4.3) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}. \tag{4.5}$$

Using Lemma 2.1 and the argument used in [5] for the Morley element, we can show the following lemma.

**Lemma 4.1.** *Let $V_{h0}$ be the finite element space of the $n$-rectangle Morley element. Then for $v \in H^3(\Omega) \cap H_0^2(\Omega)$ with $\Delta^2 v \in L^2(\Omega)$,*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \lesssim h\big(|v|_{3,\Omega} + h\|\Delta^2 v\|_{0,\Omega}\big)|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{4.6}$$

**Lemma 4.2.** *Let $V_{h0}$ be the finite element space of the $n$-dimensional Adini element or the $n$-dimensional BFS element. Then for $v \in H^3(\Omega)$*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \lesssim h|v|_{3,\Omega}|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{4.7}$$

*Proof.* First, we consider the $n$-dimensional Adini element. Given $T \in \mathcal{T}_h$, let $\Pi_T^1$ be the $n$-linear interpolation operator on $T$, that is,

$$\Pi_T^1 v = \sum_{i=1}^{2^n} \tilde{p}_i v(a_i), \quad \forall v \in C(T),$$

and let $\Pi_h^1$ be the one corresponding to $\mathcal{T}_h$. Let $P_T^0 : L^2(T) \to P_0(T)$ be the orthogonal projection.

Let $v_h \in V_{h0}$ and $\phi \in H^1(\Omega)$. For $i \in \{1, 2, \cdots, n\}$, we have that $\Pi_h^1 \frac{\partial v_h}{\partial x_i} \in H_0^1(\Omega)$. Using Green's formula gives

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i^2} + \frac{\partial \phi}{\partial x_i}\frac{\partial v_h}{\partial x_i}\Big) \\
&= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \phi \frac{\partial v_h}{\partial x_i}\nu_i = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \phi\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big)\nu_i \\
&= \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\phi - P_T^0\phi)\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big)\nu_i + \sum_{T\in\mathcal{T}_h} \int_{\partial T} P_T^0\phi\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big)\nu_i \\
&= \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\phi - P_T^0\phi)\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big)\nu_i + \sum_{T\in\mathcal{T}_h} \int_{T} P_T^0\phi\frac{\partial}{\partial x_i}\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big).
\end{aligned}$$

Using the Schwarz inequality and the interpolation theory, we have

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h} \int_{\partial T} (\phi - P_T^0\phi)\Big(\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big)\nu_i\Big| \\
&\le \sum_{T\in\mathcal{T}_h} \|\phi - P_T^0\phi\|_{0,\partial T}\Big\|\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i}\Big\|_{0,\partial T} \\
&\lesssim \sum_{T\in\mathcal{T}_h} h|\phi|_{1,T}|v_h|_{2,T} \lesssim h|\phi|_{1,\Omega}|v_h|_{2,h}.
\end{aligned}$$

For $T \in \mathcal{T}_h$, we define

$$G_i(T) = \mathrm{span}\{\, (\xi_j^2 - 1)\xi^\alpha \mid 1 \le j \le n;\ \alpha_i = 0,\ \alpha_j \le 1, j \ne i\},$$

and we have

$$\int_T \frac{\partial p}{\partial x_i} = 0, \quad \forall p \in G_i(T).$$

By the definition of $P_A(T)$,

$$\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i} \in Q_1(T) + G_i(T).$$

Because the left hand side above vanishes at the vertices of $T$,

$$\frac{\partial v_h}{\partial x_i} - \Pi_h^1 \frac{\partial v_h}{\partial x_i} \in G_i(T).$$

Consequently, we obtain that for any $\phi \in H^1(\Omega)$ and any $v_h \in V_{h0}$,

$$\Big|\sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big| \lesssim h|\phi|_{1,\Omega}|v_h|_{2,h} \tag{4.8}$$

is true when $1 \le i = j \le n$.

Now let $i, j \in \{1, 2, \cdots, n\}$ and $i \ne j$. On each $(n-1)$-dimensional face of $T \in \mathcal{T}_h$, $\nu_i\nu_j = 0$. It follows that $\frac{\partial}{\partial x_i} v_h$ is the tangent derivative along the faces on which $\nu_j$ is not zero. Since $v_h \in H_0^1(\Omega)$, it is follows that

$$\sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big) = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \phi\frac{\partial v_h}{\partial x_j}\nu_i = 0.$$

That is, (4.8) holds for all $i,j\in\{1,2,\cdots,n\}$.

By (4.8) and the following equality,

$$\begin{aligned}a_h(v,v_h)-(\Delta^2v,v_h)=&\sum_{i=1}^n\sum_{T\in\mathcal{T}_h}\int_T\Big(\Delta v\frac{\partial^2v_h}{\partial x_i^2}+\frac{\partial\Delta v}{\partial x_i}\frac{\partial v_h}{\partial x_i}\Big)\\&+\sum_{1\le i\ne j\le n}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2v}{\partial x_i\partial x_j}\frac{\partial^2v_h}{\partial x_i\partial x_j}+\frac{\partial^3v}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big)\\&-\sum_{1\le i\ne j\le n}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2v}{\partial x_i^2}\frac{\partial^2v_h}{\partial x_j^2}+\frac{\partial^3v}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big),\end{aligned}\tag{4.9}$$

we obtain the conclusion of the lemma for the $n$-dimensional Adini element.

Now we consider the $n$-dimensional BFS element. Let $1\le i\le n$, and let $F_i^\pm$ be the $(n-1)$-dimensional faces of $T$ with $\xi_i=\pm1$. Let $\bar G_i(T)$ be defined by

$$\bar G_i(T)=\mathrm{span}\{(\xi_j^2-1)(\xi_k^2-1)\xi^\alpha\,|\,1\le j<k\le n,\ j,k\ne i;\ \alpha_i=0,\ \alpha_l\le1,l\ne i\}.$$

It can be verified that for any $v_h\in V_{h0}$, $\frac{\partial}{\partial x_i}v_h$ can be divided into two parts:

$$\frac{\partial v_h}{\partial x_i}=\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)+\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big),$$

where for any $T\in\mathcal{T}_h$,

$$\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\Big|_{F_i^\pm}\in P_A(F_i^\pm),\quad \bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)|_T\in\bar G_i(T).$$

Using Green's formula gives

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2v_h}{\partial x_i^2}+\frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_i}\Big)=\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\frac{\partial v_h}{\partial x_i}\nu_i\\=&\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i\\=&\sum_{T\in\mathcal{T}_h}\Big(\int_{F_i^+}\phi\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)-\int_{F_i^-}\phi\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\Big)+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i.\end{aligned}$$

From the definition of $\bar G_i(T)$, we know that $\tilde Q_i(\frac{\partial v_h}{\partial x_i})|_{F_i^\pm}$ is just the Adini interpolation function of $\frac{\partial v_h}{\partial x_i}$ with respect to variable $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$, and we obtain that

$$\sum_{T\in\mathcal{T}_h}\Big(\int_{F_i^+}\phi\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)-\int_{F_i^-}\phi\tilde Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\Big)=0.\tag{4.10}$$

Since $\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)$ is independent of $\xi_i$ on each element $T\in\mathcal{T}_h$, we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i=\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\phi-P_T^0\phi)\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i.$$

Using the Schwarz inequality and the interpolation theory, we obtain

$$\Big|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\bar Q_i\Big(\frac{\partial v_h}{\partial x_i}\Big)\nu_i\Big|\lesssim h|v|_{1,\Omega}|v_h|_{2,h}.\tag{4.11}$$

It follows from (4.10) and (4.11) that (4.8) is true for $1 \le i = j \le n$.

Similarly, we can show that (4.8) is true for all $i, j \in \{1, 2, \cdots, n\}$. Then the conclusion of the lemma holds for the $n$-dimensional BFS element.

By a similar argument used in [11], we can obtain the following lemma.

**Lemma 4.3.** *Let $V_{h0}$ be the finite element space of the $n$-rectangle Morley element, the $n$-dimensional Adini element or the $n$-dimensional BFS element. Then*

$$|v_h|_{2,h} \le \|v_h\|_{2,h} \lesssim |v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{4.12}$$

Now let $u$ and $u_h$ be the solutions of problems (4.3) and (4.5) respectively. Combining Theorem 3.2, Lemmas 4.1-4.3 and the well-known Strang Lemma, we finally obtain the following convergence results.

**Theorem 4.1.** *Let $V_{h0}$ be the finite element space of the $n$-rectangle Morley element. Then*

$$\|u - u_h\|_{2,h} \lesssim h(|u|_{3,\Omega} + h\|f\|_{0,\Omega}) \tag{4.13}$$

*when $u \in H^3(\Omega)$.*

**Theorem 4.2.** *Let $V_{h0}$ be the finite element space of the $n$-dimensional Adini element or the $n$-dimensional BFS element. Then*

$$\|u - u_h\|_{2,h} \lesssim h(h|u|_{4,\Omega} + |u|_{3,\Omega}) \tag{4.14}$$

*when $u \in H^4(\Omega)$.*

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