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A FLEXIBLE PRECONDITIONED ARNOLDI METHOD FOR SHIFTED LINEAR SYSTEMS*

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Abstract

We are interested in the numerical solution of the large nonsymmetric shifted linear system, $(A + \alpha I)x = b$, for many different values of the shift α in a wide range. We apply the Saad's flexible preconditioning technique [14] to the solution of the shifted systems. Such flexible preconditioning with a few parameters could probably cover all the shifted systems with the shift in a wide range. Numerical experiments report the effectiveness of our approach on some problems.

Mathematics subject classification: 65F10, 65Y20.
Key words: Shifted linear systems, Arnoldi method, Flexible preconditioning.

1. Introduction

We are interested in the numerical solution of the following large nonsymmetric shifted linear system,

$$(A + \alpha_j I)x(\alpha_j)b, \qquad j = 1, \cdots, s, \tag{1.1}$$

for many, possibly a few hundreds, different values of the shift α_j in a wide range, all available simultaneously. This problem arises in many engineering applications like in quantum chromodynamics [8], electromagnetics [12], structural dynamics [5,17], wave propagation [15] and control theory [4]. The traditional approach to this problem is to factorize $A + \alpha_j I$ and solve (1.1) by backtransformation for each α_j. This can be quite expensive when s is large. Now the Krylov subspace methods is a popular approach to solve (1.1); see e.g. [4,7,9,10,17],since these are invariant with respect to shift α_j. More precisely, the Krylov subspace satisfies $\mathcal{K}_m(A, b) = \mathcal{K}_m(A + \alpha_j I, b)$, for any α_j. Hence, all approximation solutions can be sought in a single subspace generated by the constant coefficient matrix A.

However, convergence may be slow if the coefficient shifted matrix $A + \alpha_j I$ has unfavorable spectral properties. Applying an efficient preconditioner for the shifted systems (1.1) is necessary and important. Some attempts have been made in the past, e.g. polynomial preconditioning, which preserves the shifted form [6], and approximate inverse preconditioner for each shifted matrix $A + \alpha_j I$ by cheaply modifying an existing sparse approximate inverse preconditioner for A [3].

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In [13], the preconditioning matrix $(A+\sigma I)^{-1}$ with a fixed reference σ is used. This leads to solve the preconditioned shifted systems

$$(A+\sigma I)^{-1}(A+\alpha_j I)x(\alpha_j) = (A+\sigma I)^{-1}b, \qquad j = 1, \cdots, s. \tag{1.2}$$

This preconditioning approach maintains the shift-invariance properties of the Krylov subspace, since

$$(A+\sigma I)^{-1}(A+\alpha_j I) = I + (\alpha_j - \sigma)(A+\sigma I)^{-1}. \tag{1.3}$$

Thus, all the approximation solutions can be sought in one Krylov subspace generated by the matrix $(A+\sigma I)^{-1}$. Meerbergen [13] analyzed the spectrum of the preconditioned matrix $(A+\sigma I)^{-1}(A+\alpha_j I)$. The nice features are that the preconditioner is well suited for values of α_j near the reference value σ. However, it is difficult for only one reference value σ to cover many different values α_j in a wide range.

Based on Saad's flexible preconditioning idea, the FOM/GMRES method with a variable preconditioning for solving the shifted systems (1.1) is presented in this paper. The method allows us to incorporate the different preconditioner, e.g. $(A+\sigma_i I)^{-1}$ with different σ_i in our problem, into the Arnoldi procedure when constructing a projective subspace. It is possible for such single projective subspace with a few different preconditioning matrices $(A+\sigma_i I)^{-1}$ to cover all the different values α_j in a wide range. Although such projective subspace is not a Krylov subspace, it is still invariant with respect to the shift α_j, or say that the subspace is independent on the shift α_j. Hence, all the approximation solutions can still be sought in the single projective subspace generated by the preconditioned matrices $(A+\sigma_i I)^{-1}$. Numerical experiments report the effectiveness of our approach on some problems.

The remainder of the paper is organized as follows. In Section 2, we first change the left version of Meerbergen's preconditioner to the right version. We then present the Arnoldi method with a flexible preconditioning. In Section 3, we discuss some implementation issues of the algorithm. Numerical experiments are shown in Section 4.

2. Projective Subspace with Preconditioning

In this section, we first present the right version of Meerbergen's preconditioning to construct a flexible preconditioning, and then based on it, we establish FOM/GMRES method with a flexible preconditioning for solving the shifted linear systems (1.1).

2.1. Right Preconditioning

We employ the right preconditioner $(A+\sigma I)^{-1}$ to the shifted system (1.1),

$$(A+\alpha_j I)(A+\sigma I)^{-1}\tilde{x}(\alpha_j) = b, \qquad j = 1, \cdots, s, \tag{2.1}$$

where $\tilde{x}(\alpha_j) = (A+\sigma I)x(\alpha_j)$. Since

$$(A+\alpha_j I)(A+\sigma I)^{-1} = I + (\alpha_j - \sigma)(A+\sigma I)^{-1}, \tag{2.2}$$

Krylov subspace $\mathcal{K}_m((A+\sigma I)^{-1}, b)$ generated by $(A+\sigma I)^{-1}$ is invariant with respect to shift α_j, *i.e.*, $\mathcal{K}_m((A+\sigma I)^{-1}, b) = \mathcal{K}_m((A+\alpha_j I)(A+\sigma I)^{-1}, b)$, for any α_j. Hence, all preconditioned shifted systems(2.1) can be projected onto a single approximation subspace $\mathcal{K}_m((A+\sigma I)^{-1}, b)$. The following Arnoldi procedure builds an orthogonal basis of the Krylov subspace $\mathcal{K}_m((A+\sigma I)^{-1}, b)$.

Algorithm 2.1. BUILD $\mathcal{K}_m((A+\sigma I)^{-1}, b)$ BY THE M-ARNOLDI PROCEDURE.

1. Choose σ and factorize $A+\sigma I$.
2. Set $v_1 = b/\beta$, where $\beta = \|b\|_2$.
3. For $k = 1, \cdots, m$. Do
   1) Solve $(A+\sigma I)w_k = v_k$.
   2) $\tilde{v}_{k+1} = w_k - \sum_{i=1}^{k} h_{ik}v_i$, where $h_{ik} = (v_i, w_k)$.
   3) $h_{k+1k} = \|\tilde{v}_{k+1}\|_2$.
   4) $v_{k+1} = \tilde{v}_{k+1}/h_{k+1k}$.

Set $W_m = [w_1, \cdots, w_m], V_m = [v_1, \cdots, v_m], and\ \bar{H}_m = \{h_{ij}\}$. By 1) of step 3 and the Arnoldi procedure, we have

$$W_m = (A+\sigma I)^{-1}V_m, \tag{2.3}$$

and

$$W_m = V_{m+1}\bar{H}_m, \tag{2.4}$$

where $V_m^T V_m = I$. Thus by (2.2), the following holds,

$$(A+\alpha_i I)W_m = (A+\alpha_i I)(A+\sigma I)^{-1}V_m V_{m+1}\bar{H}_m(\alpha_i),$$

where $\bar{H}_m(\alpha_i) = \begin{bmatrix} I \\ 0 \end{bmatrix} + (\alpha_i - \sigma)\bar{H}_m$. We seek the approximation solutions $\tilde{x}_m(\alpha_j)$ in the subspace $\mathcal{K}_m((A+\sigma I)^{-1}, b)$ by setting zero initial, *i.e.,* $\tilde{x}_m(\alpha_j) = V_m u_m(\alpha_j)$. For the original problem(1.1), the approximation solutions are $x_m(\alpha_j) = (A+\sigma I)^{-1}\tilde{x}_m(\alpha_j) = W_m u_m(\alpha_j)$, which means that all the approximation solutions $x_m(\alpha_j)$ are sought in the subspace $span\{W_m\} = span\{(A+\sigma I)^{-1}V_m\}$. The residuals have the expression,

$$\begin{aligned} r_m(\alpha_j) &= b - (A+\alpha_j I)x_m(\alpha_j) = V_{m+1}(\beta e_1 - \bar{H}_m(\alpha_j)u_m(\alpha_j)) \\ &= V_m(\beta e_1 - H_m(\alpha_j)u_m(\alpha_j)) - v_{m+1}h_{m+1m}e_m^T u_m(\alpha_j), \end{aligned} \tag{2.5}$$

where $H_m(\alpha_j)$ is the matrix by deleting the last row of $\bar{H}_m(\alpha_j)$ and $b = \beta v_1$. By imposing the Galerkin condition $r_m(\alpha_j) \perp \mathcal{K}_m((A+\sigma I)^{-1}, b)$, or minimizing the residual norm over all vectors in $\mathcal{K}_m((A+\sigma I)^{-1}, b)$, we can obtain the following algorithms, *i.e.*, FOM method and GMRES method with a right preconditioning $(A+\sigma I)^{-1}$, for solving the shifted systems (1.1).

Algorithm 2.2. FOM/GMRES METHOD WITH A RIGHT PRECONDITIONING.

1. Choose $x_0(\alpha_j) = 0, j = 1, \cdots, s$.
2. By Algorithm 2.1, $V_m, W_m, \bar{H}_m$, are produced.
3. (i) FOM: $u_m(\alpha_j) = H_m^{-1}(\alpha_j)(\beta e_1)$.
   (ii) GMRES: $u_m(\alpha_j) arg\min_u \|\beta e_1 - \bar{H}_m(\alpha_j)u\|_2$.
4. $x_m(\alpha_j) = W_m u_m(\alpha_j)$.

In the above algorithm, $\{V_m\}$ is an orthogonal basis of the Krylov subspace $\mathcal{K}_m((A+\sigma I)^{-1}, b)$. However, all the approximation solutions $x_m(\alpha_j)$ are sought in the subspace $\{W_m\}$, which is the subspace spanned by the vectors $w_k = (A+\sigma I)^{-1}v_k$. In the context of iterative methods, $(A+\sigma I)^{-1}$ can be regarded as a preconditioner for the shifted systems $(A+\alpha_j I)x(\alpha_j) = b$.

In [13], the spectral properties of the preconditioned matrix $(A+\sigma I)^{-1}(A+\alpha_j I)$ are analyzed and it is shown that the preconditioner $(A+\sigma I)^{-1}$ is well suited only for values of α_j near the reference value σ. In the application problems, the shifts α_j are selected in a wide range. In this case, only one preconditioner could not reach a good efficiency. In the implementation, another (restarted) Krylov subspace with a new select preconditioner $(A+\bar{\sigma}I)^{-1}$ and new initial value $\bar{r}_0 = r_{m+1}$, which is the last residual, for solving unconverged shifted systems could be generated, after some systems converge. However, some information obtained in the previous subspace could be lost.

2.2. Flexible preconditioning

Based on Saad's idea [14], a projective subspace with a flexible preconditioning could be built for seeking all the approximation solutions, and it could probably cover the whole shifted systems with many different values of α_j in a wide range, although it is not Krylov subspace. We note that in 1) of step 3, for each k, $w_k = (A+\sigma I)^{-1}v_k$ for a fixed σ. We now replace it by

$$w_k = (A+\sigma_k I)^{-1}v_k. \tag{2.6}$$

Again, we set $W_m = [w_1, \cdots, w_m]$, $V_m = [v_1, \cdots, v_m]$, $\bar{H}_m = \{h_{ij}\}$. The relation(2.3) can be rewritten in the following matrix form by setting $\Sigma_m = diag\{\sigma_1, \cdots, \sigma_m\}$:

$$AW_m + W_m\Sigma_m = V_m. \tag{2.7}$$

In addition, the Arnoldi factorization(2.4) still satisfies

$$W_m = V_{m+1}\bar{H}_m, \tag{2.8}$$

and $V_m^T V_m = I$. In this case, we have

$$(A+\alpha_j I)W_m = AW_m + \alpha_j W_m = V_m - W_m\Sigma_m + \alpha_j W_m$$

$$= V_m + V_{m+1}\bar{H}_m(\alpha_j I - \Sigma_m) = V_{m+1}(\begin{bmatrix} I \\ 0 \end{bmatrix} + \bar{H}_m(\alpha_j I - \Sigma_m)).$$

Setting $\bar{H}_m(\alpha_j, \Sigma_m) = \begin{bmatrix} I \\ 0 \end{bmatrix} + \bar{H}_m(\alpha_j I - \Sigma_m)$, we have the following relation,

$$(A+\alpha_j I)W_m = V_{m+1}\bar{H}_m(\alpha_j, \Sigma_m). \tag{2.9}$$

We seek the approximation solutions $x_m(\alpha_j)$ still in the subspace $span\{W_m\} \equiv span\{w_1, \cdots, w_m\}$ by setting zero initial, where w_k is given by (2.6). The approximation solutions can be written as $x_m(\alpha_j) = W_m u_m(\alpha_j)$. Their residuals have the expression

$$r_m(\alpha_j) = b - (A+\alpha_j I)x_m(\alpha_j) = V_{m+1}(\beta e_1 - \bar{H}_m(\alpha_j, \Sigma_m)u_m(\alpha_j)).$$

$$= V_m(\beta e_1 - H_m(\alpha_j, \Sigma_m)u_m(\alpha_j)) - v_{m+1}h_{m+1m}e_m^T(\alpha_j - \sigma_m)u_m(\alpha_j), \tag{2.10}$$

where $H_m(\alpha_j, \Sigma_m)$ is the matrix by deleting the last row of the matrix $\bar{H}_m(\alpha_j, \Sigma_m)$. By imposing the Galerkin condition $r_m(\alpha_j) \perp span\{v_1, \cdots, v_m\}$, or minimizing the residual norm over all vectors in $span\{w_1, \cdots, w_m\}$, we can establish the following FOM method and GMRES method with flexible preconditioning for solving the shifted systems (1.1).

Algorithm 2.3. FOM/GMRES WITH FLEXIBLE PRECONDITIONING.

1. Choose $\sigma_k, k = 1, \cdots, m$.
2. Choose $x_0(\alpha_j) = 0, j = 1, \cdots, s$.
3. Set $v_1 = b/\beta$, where $\beta = \|b\|_2$.
4. For $k = 1, \cdots, m$. Do
   (i) Solve $(A + \sigma_k I)w_k = v_k$.
   (ii) $\tilde{v}_{k+1} = w_k - \sum_{i=1}^{k} h_{ik} v_i$, where $h_{ik} = (v_i, Aw_k)$.
   (iii) $h_{k+1k} = \|\tilde{v}_{k+1}\|_2$.
   (iv) $v_{k+1} = \tilde{v}_{k+1}/h_{k+1k}$.
5. (i) FOM: $u_m(\alpha_j) = H_m^{-1}(\alpha_j, \Sigma_m)(\beta e_1)$.
   (ii) GMRES: $u_m(\alpha_j) arg \min_u \|\beta e_1 - \bar{H}_m(\alpha_j, \Sigma_m)u\|_2$.
6. $x_m(\alpha_j) = W_m u_m(\alpha_j)$.

2.3. Some Propositions

It is important that the projective subspace $\{W_m\}$ is independent of the shift α_j. In fact, we have the following propositions about the subspace $\{W_m\}$ and about Algorithm 2.3. These propositions are easily obtained by using the formulas (2.8) and (2.9).

Proposition 2.1. *Let $W_m = [w_1, \cdots, w_m]$ and $V_m = [v_1, \cdots, v_m]$ be defined by Algorithm* 2.1. *Then for any α_j,*

$$span\{W_m\} \subseteq span\{V_{m+1}\} \quad and \quad span\{(A + \alpha_j I)W_m\} \subseteq span\{V_{m+1}\}.$$

Proposition 2.2. *If $h_{m+1m} = 0$ and H_m is nonsingular, then for any α_j,*

$$span\{W_m\} = span\{V_m\} = span\{(A + \alpha_j I)W_m\}.$$

Proposition 2.3. *Assume that Algorithm* 2.3 *does not break down. Then $rank\{W_m\} = rank\{\bar{H}_m\}$.*

3. Implementation Issues

3.1. Restarting

With m increasing, the computational cost and the memory will increase significantly in Algorithm 2.3. So restarting is necessary.

Note the expression of the residual (2.10). For FOM method, the residual of the approximation $x_m(\alpha_j)$ is

$$r_m(\alpha_j) = b - (A + \alpha_j I)x_m(\alpha_j) = -v_{m+1} h_{m+1m} (\alpha_j - \sigma_m)[u_m(\alpha_j)]_m,$$

where $[u_m(\alpha_j)]_m$ is the last element of vector $u_m(\alpha_j)$. This means that the residuals for all α_j are collinear with v_{m+1}. Thus we can restart the algorithm with new initial vector $\bar{v}_1 = v_{m+1}, \bar{x}_0(\alpha_j) = x_m(\alpha_j)$ for solving unconverged systems.In this case, $\bar{x}_m(\alpha_j) = \bar{x}_0(\alpha_j) + W_m u_m(\alpha_j)$.

For GMRES method, Frommer's approach [9] can be applied to the preconditioned shifted system with a flexible preconditioning. We assume that the approximation solution x_m of $Ax = b$, called seed system, in the affine subspace $\{x_0\}+\{W_m\}$ has been obtained by Algorithm 2.3. We now want to obtain the other approximation solutions $x_m(\alpha_j)$ of $(A+\alpha_j I)xb, \alpha_j \neq 0$ in $\{x_0(\alpha_j)\} + \{W_m\}$, i.e., $x_m(\alpha_j) = x_0(\alpha_j) + W_m z_m$, such that the residuals $r_m(\alpha_j)$ are collinear with the residual r_m of x_m, i.e., $r_m(\alpha_j) = \beta_m(\alpha_j) r_m$. We first assume that initial residuals are collinear with $r_0 : r_0(\alpha_j) = \beta_0(\alpha_j) r_0$. From (2.10), the residual r_m can be set as $r_m = V_{m+1} y_m$, where $y_m = \beta e_1 - \bar{H}_m(0, \Sigma_m) u_m(0)$. Thus the collinear condition can be written as

$$b - (A + \alpha_j I)(x_0(\alpha_j) + W_m z_m(\alpha_j))\beta_m V_{m+1} y_m,$$

$$\Longleftrightarrow r_0(\alpha_j) - (A + \alpha_j I) W_m z_m(\alpha_j) = \beta_m V_{m+1} y_m,$$

$$\Longleftrightarrow \beta_0 r_0 - V_{m+1} \bar{H}_m(\alpha_j, \Sigma_m) z_m(\alpha_j) = \beta_m V_{m+1} y_m.$$

Note $r_0 = \beta V_{m+1} e_1$. Multiplying the above formula from the left by V_{m+1}^T, we have

$$\beta\beta_0 e_1 - \bar{H}_m(\alpha_j, \Sigma_m) z_m(\alpha_j) = \beta_m y_m.$$

So we can obtain the approximation solutions $x_m(\alpha_j) x_0(\alpha_j) + W_m z_m(\alpha_j)$ by solving the following $(m+1) \times (m+1)$ system,

$$\begin{pmatrix} \bar{H}_m(\alpha_j, \Sigma_m) & y_m \end{pmatrix} \begin{pmatrix} z_m(\alpha_j) \\ \beta_m \end{pmatrix} = \beta\beta_0 e_1. \tag{3.1}$$

Frommer shows that if the matrix A is positive (i.e., the eigenvalue has a positive real part) and the shift $\alpha_j > 0$, the approximation solution $x_m(\alpha_j)$ from the above system converges to the solution of $(A + \alpha_j I)x = b$, if the residual r_m converges to zero.

3.2. Unnecessary to Select Each Different σ_k for Each k

Since m inner systems $(A + \sigma_k I)w = v$ must be solved in Algorithm 2.3, the computation cost is expensive if m is large. In application problems, we may select only very few different σ_k in all inner iteration(In our numerical experiments, two or three σ_k are selected.) according to the clustered distributing of the parameter α_j.

For the inner systems $(A + \sigma_k I)w = v$, it is generally practical to use a direct linear solver [11,13]. Since very few σ_k is selected for a large number of shifts α, only very few large-scale sparse factorizations of $A+\sigma_k I$, *e.g.*, LU factorization, are required and such factorized matrices could be stored if the memory is not limited. Thus even if m is large and s is very large, the computation cost will be improved greatly.

4. Numerical Experiments

In this section, we report the results of our numerical experiments with a Fortran 77 implementation of the algorithm described in Section 2. We compare the effect of the flexible preconditioning with a fixed right preconditioning and unpreconditioning, based on GMRES method

(restart version), denoted by FGMRES(m) (the GMRES method with a flexible preconditioning), PGMRES(m) (the GMRES method with a fixed right preconditioning) and GMRES(m) (the GMRES method without preconditioning) separately; here m stands for the size of the approximation subspace. In our all experiments, $m = 14$ is set. Also we carry out the FOM method for Example 1 to show the effectiveness of flexible preconditioning. The algorithms were executed on the basis of the iteration number (restart runs) necessary to achieve the residual norm, $\|r_m(\alpha_j)\|_2 < \epsilon \equiv 10^{-6}$. The maximum restart run is set to 30. In all methods the zero initial vector is chosen for all shifted systems and each new initial vector $\bar{x}_0^{(j)}$ of the j-th system is set to the last iteration solution $x_m^{(j)}$. We consider the different distributing of the parameter α_j in a wide range, and the choice of σ_k.

Three sets of values for the shift parameters are:

1) Set one Π_1 is formed by 80 shift parameters: $\alpha_j = 0.001 * j$, $1 \le j \le 40$, and $\alpha_j = 1.0 + 0.001 * j$, $41 \le j \le 80$, which are gathered in two intervals [0.001, 0.04] and [1.041, 1.08].

2) Set two Π_2 is formed by 80 shift parameters: $\alpha_j = 0.001 * j$, $1 \le j \le 30$, $\alpha_j = 0.5 + 0.001 * j$, $31 \le j \le 50$, and $\alpha_j = 5.0 + 0.001 * j$, $51 \le j \le 80$, which are gathered in three intervals [0.001, 0.03], [0.531, 0.55] and [5.051, 5.08].

3) Set three Π_3 is formed by 200 shift parameters: $\alpha_j = 0.01 + 0.002 * j$, $1 \le j \le 200$, which are gathered in the interval [0.012, 0.41].

The experiments are based on the following shifted linear systems:

$$(A + \alpha_j I)x(\alpha_j) = b, \qquad j = 1, \cdots, s. \tag{4.1}$$

Example 1. We first consider the problems arising from the centered difference discretization of problems of the form

$$-\Delta u + 2\gamma_1 u_x + 2\gamma_2 u_y + \beta u = f$$

on square regions with zero Dirichlet boundary condition. The grid consists of a square of 50 internal mesh points in each direction, leading to a matrix A of size $n = 2500$. The right-hand side b is formed by $b = (A + \alpha_1 I)e$, where e is a vector with all entries 1. The parameters $\gamma_1 = 5$ and $\gamma_2\beta = 0$ are selected.

In FGMRES(m) and FFOM(m), one reference value σ_k in each separated interval can be selected.

For the set Π_1, only two references are selected in all inner iteration ($m = 14$). More precisely, $\sigma_k = 0.006$, $1 \le k \le 10$, and $\sigma_k = 1.0$, $11 \le k \le 14$. This means in the first ten inner iterative steps, σ_k=0.006 is fixed, and in the last four inner iterative steps, σ_k=1.0 is fixed.

Similarly, for the set Π_2, only three references are selected by $\sigma_k = 0.0054$, $1 \le k \le 8$, $\sigma_k = 0.5$, $9 \le k \le 11$ and $\sigma_k = 5.0$, $12 \le k \le 14$; and for the set Π_3, only two references are selected by $\sigma_k = 0.018$, $1 \le k \le 8$ and $\sigma_k = 0.31$, $9 \le k \le 14$.

In PGMRES(m) and PFOM(m), a reference σ is first selected around the first gathered interval of shifts to form a preconditioner $(A + \sigma I)^{-1}$, *e.g.*, in our experiments, σ is set to be the reference σ_1 selected in the above flexible preconditioner. After one restart run, we delete the converged shifted systems and solve the unconverged shifted systems. We then form a new preconditioner $(A + \bar{\sigma} I)^{-1}$ by selecting a new reference $\bar{\sigma}$ around the next gathered interval of shifts, *e.g.*, $\bar{\sigma}$ is set to be the reference σ_1 selected in the above flexible preconditioner.

The results of the iteration number (restart runs) for the convergence are reported in Table 4.1.

shift set	FGMRES	PGMRES	GMRES	FFOM	PFOM	FOM
Π_1	1	2	15	1	2	18
Π_2	1	3	15	1	3	18
Π_3	1	2	13	1	2	14

Table 4.1: Example 1. Restart runs for the convergence of three methods for solving the shifted systems with three sets of shifted parameters.

shift set	FGMRES	PGMRES	GMRES
Π_1	1	2	29
Π_2	1	3	29
Π_3	1	2	25

Table 4.2: Example 2. Restart runs for the convergence of three methods for solving the shifted systems with three sets of shifted parameters.

The methods without preconditioning take much more iterative steps to reach the convergence. The effectiveness of preconditioning is apparent. The experiments show that the GMRES/FOM methods with a flexible preconditioner have a better convergence history than the methods with a fixed preconditioner. Since only one fixed reference σ can be selected at one restart run in the PGMRES/PFOM methods, the effectiveness of preconditioning is mainly on the shifted systems with the shift parameters α near the reference value σ, and it is difficult for only one reference value σ to cover many different values α_j in the different intervals or in a wide range.

However, more interesting is the convergence behavior of the FGMRES/FFOM methods. Since each one reference value σ_k in each separated interval can be selected at one restart run in these methods, the experimental results show that the effectiveness of flexible preconditioning could probably be on the all shifted systems with the shift parameters α near each reference value σ, even located in the different separated intervals or in a wide range.

Example 2. We next consider the matrix A, pde2961, from the Matrix Market [19]. The matrix is real unsymmetric matrix with order 2961 and 14585 nonzero entries. We carry out the GMRES method to show the effectiveness of the flexible preconditioning.

In FGMRES(m), the strategy to select the reference is similar to the one in Example 1, *i.e.*, one reference value σ_k in each separated interval is selected.

For the set Π_1, two references are selected by $\sigma_k = 0.009$, $1 \le k \le 9$ and $\sigma_k = 1.0$, $10 \le k \le 14$; for the set Π_2, three references are $\sigma_k = 0.006$, $1 \le k \le 8$, $\sigma_k = 0.53$, $9 \le k \le 12$ and $\sigma_k = 5.065$, $k = 13, 14$; for the set Π_3, two references are $\sigma_k = 0.046$, $1 \le k \le 9$ and $\sigma_k = 0.32$ and $10 \le k \le 14$.

In PGMRES(m), the strategy to select reference σ is the same as the one in Example 1. The results of the iteration number (restart runs) for the convergence are reported in Table 4.2.

The experimental results show that the projective subspace with a flexible preconditioner $(A + \sigma_k I)^{-1}$ by selecting a few suitable references σ_k could be efficient to all shifted systems with the shift in a wide range. So the methods with a flexible preconditioning have a better convergence history than the methods with a fixed preconditioning.

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