Journal of Computational Mathematics, Vol.25, No.5, 2007, 553–560.

STRUCTURES OF CIRCULANT INVERSE M-MATRICES *

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Abstract

In this paper, we present a useful result on the structures of circulant inverse M-matrices. It is shown that if the $n\times n$ nonnegative circulant matrix $A = Circ[c_0, c_1, \cdots, c_{n-1}]$ is not a positive matrix and not equal to $c_0 I$, then A is an inverse M-matrix if and only if there exists a positive integer k, which is a proper factor of n, such that $c_{jk} > 0$ for $j = 0, 1, \cdots, [\frac{n-k}{k}]$, the other c_i are zero and $Circ[c_0, c_k, \cdots, c_{n-k}]$ is an inverse M-matrix. The result is then extended to the so-called generalized circulant inverse M-matrices.

Mathematics subject classification: 15A48, 15A29, 15A57.
Key words: Nonnegative matrices, Circulant matrix, Inverse M-matrices.

1. Introduction

A real matrix A is called positive (nonnegative), denoted by $A > 0$ ($A \geq 0$), if every entry $a_{i,j}$ is positive (nonnegative). A real matrix is called a Z-matrix if all its off-diagonal entries are nonpositive. A nonnegative square matrix is called an inverse M-matrix if it is invertible and its inverse is a Z-matrix.

A square matrix A is called reducible if there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are non-empty square matrices. A matrix is irreducible if it is not reducible.

The following lemmas, which will be used later, involve zero and nonzero pattern or structures of inverse M-matrices.

Lemma 1.1. (Corollary 2.2 in [10]) *If A is an irreducible inverse M-matrix, then A is positive.*

Lemma 1.2. ([6]) *Suppose that A is an inverse M-matrix, let k be a positive integer. Then the (i,j) entry of A^k is zero if and only if the (i,j) entry of A is zero.*

Lemma 1.3. *Let A be a partitioned inverse M-matrix:*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,r} \\ A_{2,1} & A_{2,2} & \dots & A_{2,r} \\ \dots & \dots & \dots & \dots \\ A_{r,1} & A_{r,2} & \dots & A_{r,r} \end{bmatrix}.$$

* Received December 15, 2005; final revised September 1, 2006; accepted October 1, 2006.

Assume that $A_{i,i}$ $(i = 1, 2, \ldots, r)$ are positive square matrices. Then $A_{i,j}$ also is positive if $A_{i,j} \neq 0$ when $i \neq j$.

Proof. Let A^k have the same partition as A and denote the (i,j) block of A^k by $A^{(k)}_{i,j}$. If $A_{i,j} \neq 0$ for some $i \neq j$, then

$$A^{(2)}_{i,j} = \sum_{l=1}^{r} A_{i,l}A_{l,j} \geq A_{i,i}A_{i,j} + A_{i,j}A_{j,j}.$$

Since $A_{i,i}$, $A_{j,j}$ are positive and $A_{i,j}$ is nonnegative, we know from the inequality that $A^{(2)}_{i,j}$ has at least one positive row and one positive column. Thus

$$A^{(3)}_{i,j} = \sum_{l=1}^{r} A^{(2)}_{i,l}A_{l,j} \geq A^{(2)}_{i,i}A_{i,j} + A^{(2)}_{i,j}A_{j,j}$$

must be positive. By Lemma 1.2, $A_{i,j}$ is positive.

A matrix C is called a circulant matrix if it is of the form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix} \tag{1.1}$$

We will denote the circulant matrix C in (1.1) by $Circ[c_0, c_1, \cdots, c_{n-1}]$ for notational convenience.

Inverse M-matrices and circulant matrices are two classes of important matrices. Inverse M-matrices often occur in systems of linear or non-linear equations or eigenvalues problems in a wide variety of areas including finite difference methods for partial differential equations, input-output production and growth models in economics, iterative methods in numerical analysis, and Markov processes in probability and statistics. A number of properties of inverse M-matrices have been given in [1], [6]-[9]. Circulant matrices are often used as preconditioner for Toeplitz linear systems since they can be easily inverted and super-fast computed [2, 3].

In this paper, we present an interesting result on the structures of circulant inverse M-matrices. We show that a nonnegative but not positive circulant matrix $Circ[c_0, c_1, \cdots, c_{n-1}](\neq c_0I)$ is an inverse M-matrix if and only if there exists a positive integer k, which is a proper factor of n, such that $c_{jk} > 0$ for $j = 0, 1, \ldots, [\frac{n-k}{k}]$, the other c_i $(i.e., i \neq jk)$ are zero and $Circ[c_0, c_k, \cdots, c_{n-k}]$ is an inverse M-matrix. The result is then extended to so-called generalized circulant inverse M-matrices.

In the next section, we review some definitions and basic properties of digraphs and introduce a new digraph we will use in this paper. Section 3 presents our main result. The result then is extended to so-called generalized circulant matrices in the last section.

2. Preliminaries

Let $< n >= \{1, 2, \ldots, n\}$. The digraph $G = (N, E)$ consists of the vertex set N, conveniently labeled from 1 to n, and the set of directed edges (arcs) $E = \{(i, j)|i, j \in N\}$. A path in a digraph (N, E) is a sequence of vertices $v_1, v_2, \ldots, v_k, v_{k+1}$ in N such that for $i = 1, 2, \ldots, k$, $(v_i, v_{i+1}) \in E$ and all vertices are distinct except possibly $v_1 = v_{k+1}$. If $v_1 = v_{k+1}$ in the path formed by the vertices $v_1, v_2, \ldots, v_k, v_{k+1}$, then the path is called a cycle, which will be denoted by $\{v_1, v_2, \ldots, v_{k+1}(= v_1)\}$, its length is k. A graph is connected if there is a path from any vertex to any other vertex; otherwise it is disconnected. It is easy to know that a cycle of length n is connected. A subgraph of the digraph $G = (V_G, E_G)$ is a digraph $H = (V_H, E_H)$, where $V_H \subseteq V_G$ and $E_H \subseteq E_G$ and that $(u, v) \in E_H$ requires $u, v \in V_H$ since H is a digraph.

The digraph of a matrix $A = (a_{i,j}) \in R^{n\times n}$ is denoted by $D(A) = (N, E)$ with the vertex set $N =< n >$ and the arc set $E = \{(i, j)|a_{i,j} \neq 0\}$. Relabeling the vertices of the digraph of a matrix corresponds to performing a permutation similarity transformation. Since the class of inverse M-matrices is closed under permutation similarity, we are free to relabel the digraph of an inverse M-matrix as desired. It is well known that a matrix A is irreducible if and only if its digraph $D(A)$ is connected.

In the following, in order to study the structure of the circulant matrices, we have to introduce a new digraph and some notations. Let $gcd(n, k)$ denote the greatest common divisor of the two positive integers n and k. Let

$$\overline{x} = x\ (mod\, n,\ but\ \ \overline{n} = n), \quad d = gcd(n,\ k), \quad t = \frac{n}{d}. \tag{2.1}$$

Definition 2.1. *A digraph is called a C_n^k digraph if its vertex set is $< n >$ and its edge set is*

$$(1, \overline{k+1}), (2, \overline{k+2}), \ldots, (n, \overline{k+n}),$$

where $1 \leq k \leq n - 1$.

According to the definition, we have

a) If $A \neq \alpha I$ (I is the unit matrix, α is a number) is a circulant matrix, then $D(A)$ must consist of some C_n^k as its subgraph;

b) The digraph C_n^1 is a cycle of length n, so if $C_n^1 \subseteq D(A)$, then A is irreducible;

c) If $C_n^k \in D(A)$, then $C_n^{n-k} \in D(A^T)$.

The property c) suggests that we can restrict our discussion for C_n^k on $1 \leq k \leq [\frac{n}{2}]$. Now we present a most important property of the digraph C_n^k used in this paper.

Lemma 2.1. ([5]) *The digraph C_n^k is composed of d independent cycles of length t, where d and t are defined in* (2.1).

Proof. Let $G = (N, E)$ be the C_n^k digraph. Using the notation $\overline{x}$ of (2.1), the edge set of G can be written as

$$E = \{(\overline{i + (j-1)k},\ \overline{i + jk})|1 \leq i \leq d; 1 \leq j \leq t\}.$$

By noting that $\overline{i + tk} = i$, it is not difficult to prove that the digraph G is composed by the following d independent cycles of length t:

$$\begin{array}{cccccc}
\{1, & \overline{1+k}, & \overline{1+2k}, & \cdots, & \overline{1+(t-1)k}, & \overline{1+tk}\} \\
\{2, & \overline{2+k}, & \overline{2+2k}, & \cdots, & \overline{2+(t-1)k}, & \overline{2+tk}\} \\
\{3, & \overline{3+k}, & \overline{3+2k}, & \cdots, & \overline{3+(t-1)k}, & \overline{3+tk}\} \\
 & \cdots, & \ldots, & \cdots, & \cdots, & \cdots \\
\{d, & \overline{d+k}, & \overline{d+2k}, & \cdots, & \overline{d+(t-1)k}, & \overline{d+tk}\}.
\end{array} \tag{2.2}$$

The proof of Lemma 2.1 is complete.

Remark 2.1. The above lemma shows that the digraph C_n^k is a cycle graph, that is, it is composed of the cycles in (2.2). But only one cycle also is possible even for $k > 1$ if n and k have no common divisor larger than and equal to 2 or $d = 1$. In this case, $t = n$.

Remark 2.2. By using the above lemma, for any $n \times n$ matrix A, relabel the vertices of $D(A)$ according to (2.2), we can get a matrix that is a permutation similar to A. For example, let

$$A = \begin{pmatrix} 1 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & c_2 & 0 \\ 0 & 0 & 1 & 0 & c_3 \\ c_4 & 0 & 0 & 1 & 0 \\ 0 & c_5 & 0 & 0 & 1 \end{pmatrix}.$$

Since C_5^2 is a cycle of length 5, by relabeling the vertices $1, 3, 5, 2, 4$ as $1, 2, 3, 4, 5$, we know that A is a permutation similar to

$$B = \begin{pmatrix} 1 & c_1 & 0 & 0 & 0 \\ 0 & 1 & c_3 & 0 & 0 \\ 0 & 0 & 1 & c_5 & 0 \\ 0 & 0 & 0 & 1 & c_2 \\ c_4 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Structures of Circulant Inverse M-matrices

In this section, we present the structures of the circulant inverse M-matrices. In following discussion, we first have to get rid of the two trivial cases: the circulant inverse M-matrix $Circ[c_0, c_1, \ldots, c_{n-1}]$ is positive or equal to $c_0 I$.

Lemma 3.1. *If $A = Circ[c_0, c_1, \ldots, c_{n-1}]$ is an inverse M-matrix and there is a positive integer k such $c_k > 0$ and $d = gcd(k, n) = 1$, then A is positive.*

Proof. By assumption, $C_n^k \subset D(A)$. Since $d = 1$, we know that C_n^k is a cycle of length n. Thus A is irreducible and so positive by Lemma 1.1.

We remark that $d = gcd(k, n) = 1$ contains three cases: $k = 1$, n is a prime number and $k > 1$, but k and n have no common divisor larger than or equal to 2.

Lemma 3.2. *Let A be an $n \times n$ circulant matrix: $Circ[c_0, c_1, \ldots, c_{n-1}]$ and n not a prime number. Then there is a positive integer $k \geq 2$ such that A is permutation similar to a block-Toeplitz with circulant-block matrix:*

$$B = \begin{bmatrix} B_0 & B_1 & B_2 & \ldots & B_{d-1} \\ B_{-1} & B_0 & B_1 & \ldots & B_{d-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B_{-d+2} & \ddots & \ddots & \ddots & B_1 \\ B_{-d+1} & B_{-d+2} & \ldots & B_{-1} & B_0 \end{bmatrix}, \tag{3.1}$$

where all B_i are circulant matrices,

$$\begin{cases} B_j &= Circ[c_j, c_{\overline{j+k}}, \ldots, c_{\overline{j+(t-1)k}}], \quad j = 0, 1, \ldots, d-1, \\ B_{-d+j} &= Circ[c_{\overline{s}}, c_{\overline{s+k}}, \ldots, c_{\overline{s+(t-1)k}}], \quad j = 1, 2, \ldots, d-1, \end{cases} \tag{3.2}$$

where $s = n - d + j$. Furthermore, if A is an inverse M-matrix and k is the least integer such that $c_k \neq 0$, then B_0 is a positive inverse M-matrix.

Proof. Since n is not a prime number, there is an integer $k \geq 2$ such that $d = gcd(n,k) \geq 2$. By relabelling the vertices according to (2.2) and applying Lemma 2.1, we know that A is a permutation similar to a $d \times d$ block matrix $B = (B_{i,j})$, where $B_{i,j}$ are all $t \times t$ matrices. Now we show that B is of the form (3.1).

Let $D(B_{i,i}) = (V_i, E_i)$, $i = 1, 2, \ldots, d$, then $V_i = \{i, \overline{i+k}, \ldots, \overline{i+(t-1)k}\}$. Note that $A = (a_{i,j})$ is a circulant matrix with

$$a_{i,j} = \begin{cases} c_{j-i} & if \;\; j \geq i \\ c_{n+j-i} & if \;\; j < i \end{cases};$$

by careful manipulation, it is not difficult to verify that for $i, j = 1, 2, \ldots, d-1$, $B_{i,j}$ can be written as B_{j-i} and B_l is of the form (3.2).

If A is an inverse M-matrix, then B_0 is also an inverse M-matrix since B is permutation similar to A and B_0 (or $B_{i,i}$) is a principal submatrix of B. Since $c_k \neq 0$ means that $C_t^1 \subseteq D(B_0)$, we know that B_0 is irreducible and so positive by Lemma 1.1. This lemma is proved.

Remark 3.1. In (3.2), there is $s \geq 1$ such that $B_{-d+j} = B_j J^s$ for $j = 1, 2, \ldots, d-1$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

is a shift matrix. In particular, $s = 1$ if $d = k$.

Example 1. Let $n = 15$ and $k = 6$ in Lemma 3.2, we get $d = 3$, $t = 5$ from (2.1). Thus $A = Circ(c_0, c_1, \ldots, c_{14})$ is permutation similar to the matrix (3.1) with

$$\begin{cases} B_i &=& Circ[c_i, c_{i+6}, c_{i+12}, c_{i+3}, c_{i+9}], & i = 0, 1, 2; \\ B_{-i} &=& Circ[c_{15-i}, c_{6-i}, c_{12-i}, c_{3-i}, c_{9-i}], & i = 1, 2. \end{cases}$$

It is easy to verify that $B_{-3+i} = B_i J^3$ for $i = 1, 2$.

We easily deduce the following corollary from Lemma 3.2.

Corollary 3.1. *Let $A = Circ[c_0, c_1, \ldots, c_{n-1}]$ be an inverse M-matrix and n have proper factorization: $n = pq$, $(p \geq 2,\ q \geq 2)$. Then the circulant matrices*

$$Circ[c_0, c_p, \ldots, c_{(q-1)p}], \quad Circ[c_0, c_q, \ldots, c_{(p-1)q}]$$

are inverse M-matrices.

Now we readily give the main result of this paper.

Theorem 3.1. *Let $A = Circ[c_0, c_1, \ldots, c_{n-1}] \geq 0$ be not positive and not $c_0 I$. Then A is an inverse M-matrix if and only if there is a positive integer k, which is a proper factor of n, such that*

$$\begin{cases} c_i > 0 & if \;\; i = jk \\ c_i = 0 & if \;\; i \neq jk \end{cases}, \quad j = 1, 2, \ldots, [\frac{n-k}{k}], \tag{3.3}$$

and the circulant matrix

$$C = Circ[c_0, c_k, \ldots, c_{(t-1)k}] \tag{3.4}$$

is a positive inverse M-matrix.

Proof. Necessity. We know from Lemma 3.1 that A is an inverse M-matrix but not positive means $c_1 = 0$. Since $A \neq c_0 I$, we let $k \geq 2$ be the least integer such that $c_k \neq 0$ or $C_n^k \subseteq D(A)$. We show that k must be a proper factor of n since A is an inverse M-matrix.

Firstly, if k and n have no common divisor then since C_n^k is a cycle of length n, A is irreducible and so positive. This contradicts $c_1 = 0$. So $d = gcd(n, k) \geq 2$. By Lemma 3.2, A is permutation similar to the block-Toeplitz with circulant-block matrix B in (3.1) with a positive inverse M-matrix B_0 in (3.2).

Secondly, assume that $k = sd$. Then $s \leq t$ and $gcd(s, t) = 1$ since $k \leq n$, $d = gcd(n, k)$ and $n = td$. Let $u = min\{j | t < js\}$, then it is obvious that $u \geq 2$. We want to show that $s = 1$ by using contradiction. If $s \geq 2$, then it is easy to verify that $u \leq t - 1$ and $\overline{usd} = \overline{uk} < k$. Since B_0 is positive, we have $c_{\overline{uk}} > 0$. This contradicts the assumption of $c_l = 0$ for $1 \leq l < k$. So $s = 1$ or $d = k$ or B_0 in (3.2) is equal to C in (3.4). It follows from $B_0 > 0$ that $c_{jk} > 0$, $j = 0, 1, \ldots, t-1$. The first equation of (3.3) is proved.

For the second equation of (3.3), since $c_1 = c_2 = \ldots = c_{k-1} = 0$, we have $B_{\pm 1} = B_{\pm 2} = \ldots = B_{\pm(d-1)} = 0$ by Lemma 1.3. Thus from (3.2) we see that $c_i = 0$, if $i \neq jk$, $j = 1, 2, \ldots, d-1$.

Sufficiency. In terms of the assumption, it is obvious that n is not a prime number. So by using Lemma 3.2 with $d = k$, A is a permutation similar to the block-Toeplitz matrix (3.1) with the circulant-blocks (3.2).

If (3.3) holds, then the matrix B in (3.1) becomes a block diagonal matrix $diag(B_0, B_0, \ldots, B_0)$, which is an inverse M-matrix since B_0 is. Thus it is easy to see that A is an inverse M-matrix since it is a permutation similar to B.

The result of this theorem is very interesting: it shows that only when the subscripts of the positive c's is an arithmetic sequence it is possible that the nonnegative circulant matrix $A = Circ[c_0, c_1, \ldots, c_{n-1}] \neq c_0 I$ is an inverse M-matrix. This result also is very useful, as it can be applied to judge more conveniently whether a nonnegative but not positive circulant matrix is an inverse M-matrix.

4. Generalized Circulant Inverse M-matrices

In this section, we extend the result of Theorem 3.1 to more general matrices.

Definition 4.1. *An $n \times n$ matrix A is called a generalized circulant matrix if the digraph $D(A)$ has the property: whenever $D(A)$ contains an edge of some digraph C_n^k, $D(A)$ contains all the edges of the digraph C_n^k.*

For notational convenience, denote the vector $(c_{i,1}, c_{i,2}, \ldots, c_{i,n})$ by $\vec{c}_i$ so that any $n \times n$ matrix can be denoted by

$$Circ[\vec{c}_0, \vec{c}_1, \cdots, \vec{c}_{n-1}] \begin{pmatrix} c_{0,1} & c_{1,1} & c_{2,1} & \cdots & c_{n-1,1} \\ c_{n-1,2} & c_{0,2} & c_{1,2} & \cdots & c_{n-2,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_{2,n-1} & \cdots & \cdots & c_{0,n-1} & c_{1,n-1} \\ c_{1,n} & c_{2,n} & \cdots & c_{n-1,n} & c_{0,n} \end{pmatrix}.$$

If $A = Circ[\vec{c}_0, \vec{c}_1, \cdots, \vec{c}_{n-1}]$ is a generalized circulant matrix, then each $\vec{c}_i \neq 0$ means that every entry of $\vec{c}_i$ is not zero by the definition.

For example, the matrix

$$\begin{pmatrix} 1 & 0 & c_{2,1} & 0 & c_{4,1} \\ c_{4,2} & 1 & 0 & c_{2,2} & 0 \\ 0 & c_{4,3} & 1 & 0 & c_{2,3} \\ c_{2,4} & 0 & c_{4,4} & 1 & 0 \\ 0 & c_{2,5} & 0 & c_{4,5} & 1 \end{pmatrix} \tag{4.1}$$

is a generalized circulant matrix if all $c_{2,i}$ are nonzero and is denoted by $Circ(\vec{e}, 0, \vec{c}_2, 0, \vec{c}_4)$, where $\vec{e} = (1, 1, \ldots, 1)$.

Theorem 4.1. *Let $A = Circ[\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{n-1}] \geq 0$ be not positive and not a diagonal matrix. Then A is an inverse M-matrix if and only if there is a positive integer k, which is a proper factor of n, such that*

$$\begin{cases} \vec{c}_i > 0 & if \;\; i = jk \\ \vec{c}_i = 0 & if \;\; i \neq jk \end{cases}, \quad j = 1, 2, \ldots, [\frac{n-k}{k}], \tag{4.2}$$

and the generalized circulant matrices

$$C_i = Circ[\vec{c}_0^{(i)}, \vec{c}_k^{(i)}, \ldots, \vec{c}_{(t-1)k}^{(i)}], \quad i = 1, 2, \ldots, d, \tag{4.3}$$

are positive inverse M-matrices, where $\vec{c}_j^{(i)} = [c_{j,i}, c_{j,i+2k}, \ldots, c_{j,i+(t-1)k}]$.

The proof of this theorem is similar to that of Theorem 3.1 and is omitted here. We give an example to illustrate it.

Example 2. Applying Theorem 4.1, we know that the matrix in (4.1) is by no means an inverse M-matrix. The generalized circulant matrix:

$$Circ[\vec{e}, 0, \vec{c}_2, 0, \vec{c}_4, 0, \vec{c}_6, 0] = \begin{pmatrix} 1 & 0 & c_{2,1} & 0 & c_{4,1} & 0 & c_{6,1} & 0 \\ 0 & 1 & 0 & c_{2,2} & 0 & c_{4,2} & 0 & c_{6,2} \\ c_{6,3} & 0 & 1 & 0 & c_{2,3} & 0 & c_{4,3} & 0 \\ 0 & c_{6,4} & 0 & 1 & 0 & c_{2,4} & 0 & c_{4,4} \\ c_{4,5} & 0 & c_{6,5} & 0 & 1 & 0 & c_{2,5} & 0 \\ 0 & c_{4,6} & 0 & c_{6,6} & 0 & 1 & 0 & c_{2,6} \\ c_{2,7} & 0 & c_{4,7} & 0 & c_{6,7} & 0 & 1 & c_{2,6} \\ 0 & c_{2,8} & 0 & c_{4,6} & 0 & c_{6,8} & 0 & 1 \end{pmatrix}$$

is an inverse M-matrix if and only if the two matrices:

$$C_i = Circ[\vec{e}, \vec{c}_2^{(i)}, \vec{c}_4^{(i)}, \vec{c}_6^{(i)}] = \begin{pmatrix} 1 & c_{2,i} & c_{4,i} & c_{6,i} \\ c_{6,i+2} & 1 & c_{2,i+2} & c_{4,i+2} \\ c_{4,i+4} & c_{6,i+4} & 1 & c_{2,i+4} \\ c_{2,i+6} & c_{4,i+6} & c_{6,i+6} & 1 \end{pmatrix}, \quad i = 1, 2$$

are inverse M-matrices.

Corollary 4.1. *Let* $A = Circ[\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{n-1}]$ *be an inverse M-matrix and* n *have proper factorization:* $n = pq$, $(p \geq 2,\ q \geq 2)$. *Then the circulant matrices:*

$$Circ[\vec{a}_0^{(i)}, \vec{a}_p^{(i)}, \ldots, \vec{a}_{(q-1)p}^{(i)}], \quad i = 1, 2, \ldots, q$$

and

$$Circ[\vec{b}_0^{(j)}, \vec{b}_q^{(j)}, \ldots, \vec{b}_{(p-1)q}^{(j)}], \quad j = 1, 2, \ldots, p$$

are inverse M-matrices, where

$$\vec{a}_j^{(i)} = [c_{j,i}, c_{j,i+2p}, \ldots, c_{j,i+(q-1)p}], \quad \vec{b}_j^{(i)} = [c_{j,i}, c_{j,i+2q}, \ldots, c_{j,i+(p-1)q}].$$

Acknowledgments. This work is supported by National Natural Science Foundation of China (No. 10531080).

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